Wigner’s little group as a generator of gauge transformations

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Abstract
The role of Wigner’s little group, as an abelian gauge generator in different contexts, is studied.

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In his paper On unitary representations of the inhomogeneous Lorentz group, published in 1932 [1], Eugene Paul Wigner introduced the concept of little group and used it to classify the elementary particles on the basis of their helicity/spin quantum numbers. The topic of little group was a personal favorite of Prof. Wigner. During the last years of his life, he wrote a series of seven papers in collaboration with Y. S. Kim elaborating the geometrical meaning of the little group [2]. Meanwhile there were also studies regarding gauge generating aspects of the little group [3, 4]. Recently, we have found some more interesting and hitherto unknown facets of the little group in generating the gauge transformations in various abelian gauge theories, including topologically massive ones [5, 6, 7, 8, 9]. In this talk, I describe our work in this direction, highlighting the major results. Further details can be found from the references given.

Wigner’s little group is defined as the subgroup of homogeneous Lorentz group that leaves the energy-momentum vector of a particle invariant: \( W^\mu_{\nu}k^\nu = k^\mu \). In 3+1 dimensions, the little group for a massive particle is the rotation group \( SO(3) \). On the other hand, for a massless particle, the little group is the Euclidean group \( E(2) \) which is a semi-direct product of \( SO(2) \) and \( T(2) \) - the group of translations in the 2-dimensional plane. As is well known, both the rotational groups \( SO(3) \) and \( SO(2) \) determine the classification of particles on the basis of

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2Notation: Greek alphabets \( \mu, \nu \) etc denote the space-time indices in 3+1 dimensions, letters \( a, b, c \) etc stands for 2+1 dimensions and those from the middle of the alphabet, i.e. \( i, j, k \) etc stands for 4+1 dimensions.
their spin quantum numbers. One can obtain the little group $E(2)$ as a particular limit of the rotation group $SO(3)$ by Inonu-Wigner group contraction. However, while the significance of rotational groups was evident, the role of translational group remained a mystery for a long time. Weinberg and Han et. al. [3, 4] noticed that the translational group acts as a gauge generator in Maxwell theory. Following Weinberg [3], one can find the explicit representation of Wigner’s little group which leaves invariant the 4-momentum $k^\mu = (\omega, 0, 0, \omega)^T$ of a photon of energy $\omega$ moving in the $z$-direction, to be $W_4(p, q; \phi) = W(p, q)R(\phi)$, where

$$W(p, q) = W_4(p, q; 0) = \begin{pmatrix}
1 + \frac{\omega^2 + q^2}{2} & p & q & -\frac{\omega^2 + q^2}{2} \\
p & 1 & 0 & -p \\
q & 0 & 1 & -q \\
\frac{p^2 + q^2}{2} & p & q & 1 - \frac{p^2 + q^2}{2}
\end{pmatrix}$$

is a particular representation of the translational subgroup $T(2)$ of the little group and $R(\phi)$ represents a $SO(2)$ rotation about the $z$-axis. Note that the representations $W(p, q)$ and $R(\phi)$ of the translation and rotation groups satisfy the relations $W(p, q)W(\bar{p}, \bar{q}) = W(p + \bar{p}, q + \bar{q})$ and $R(\phi)R(\bar{\phi}) = R(\phi + \bar{\phi})$.

We begin the discussion by showing that (1) acts as a gauge generator for the Maxwell theory. The free Maxwell theory has the equation of motion $\partial_\mu F^{\mu\nu} = 0$ which follows from the Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gauge field $A^\mu(x)$ for a single mode can be written as $A^\mu(x) = \varepsilon^\mu(k)e^{ik\cdot x}$ suppressing the positive frequency part without any loss of generality. In terms of the polarization vector $\varepsilon^\mu$, the gauge transformation $A_\mu(x) \rightarrow A'_\mu = A_\mu + \partial_\mu f$ (where $f(x)$ is an arbitrary scalar function) is expressed as $\varepsilon^\mu(k) \rightarrow \varepsilon^\mu(k) + if(k)k_\mu$ where $f(x)$ has been written as $f(x) = f(k)e^{ik\cdot x}$. The equation of motion, in terms of the polarization vector, will now be given by $k^2\varepsilon^\mu - k^\mu k_\nu \varepsilon^{\nu} = 0$. The massive excitation corresponding to $k^2 \neq 0$ leads to the solution $\varepsilon^\mu \propto k^\mu$ which can be gauged away. For massless excitations ($k^2 = 0$), the Lorentz condition $k_\mu\varepsilon^\mu = 0$ follows immediately from the momentum space equation of motion. Taking $k^\mu = (\omega, 0, 0, \omega)^T$, corresponding to a photon of energy $\omega$ propagating in the $z$ direction, and using the Lorentz condition, one can easily show that $\varepsilon^\mu(k)$ is gauge equivalent to the maximally reduced form $^3\varepsilon^\mu(k) = (0, \varepsilon^1, \varepsilon^2, 0)^T$ displaying the two transverse degrees of freedom corresponding to $\varepsilon^1$ and $\varepsilon^2$. Under the action of the translational group $T(2)$ in (1), this polarization vector transforms as follows: $\varepsilon^\mu \rightarrow \varepsilon'^\mu = W^\mu_\nu (p, q)\varepsilon^\nu = \varepsilon^\mu + \left(\frac{\varepsilon^1 - \varepsilon^2}{\omega}k^\mu\right)$. It is now obvious that, this can be identified as a gauge transformation by choosing $f(k)$ suitably. This shows that the translational subgroup $T(2)$ of Wigner’s little group for massless particles acts as gauge generator in free Maxwell theory [3, 4].

Similar conclusions hold also for Kalb-Ramond theory in 3+1 dimensions [5].

Next, consider the $B \wedge F$ theory, which is a topologically massive gauge theory described by the Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{m^2}{2} \varepsilon^{\mu\nu\rho\lambda} H_{\mu\nu\lambda} A_\rho$ based on the vector field $A_\mu$, from which $F_{\mu\nu}$ is constructed and the antisymmetric tensor field $B_{\mu\nu}$, which defines $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$.

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3This method of obtaining the maximally reduced form of polarization vector/tensor of a theory is henceforth called the ‘plane wave method’.
It is invariant under the combined gauge transformations $A_\mu(x) \to A_\mu + \partial_\mu f$ and $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu F_\nu(x) - \partial_\nu F_\mu(x)$ where $f(x)$ and $F_\mu(x)$ are arbitrary functions. Employing the plane wave method one can show that the massless excitations of $B \wedge F$ theory are gauge artifacts and the maximally reduced form of the polarization vector and tensor corresponding to the massive physical excitations are given by

$$\{\varepsilon^{\mu\nu}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & c & -b \\ 0 & -c & 0 & a \\ 0 & b & -a & 0 \end{pmatrix}, \quad \varepsilon^\mu = -i \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}$$

with the duality relation $\varepsilon^{\mu\nu} \varepsilon_\mu = 0$ between them [5]. Correspondingly, the momentum vector is $p^\mu = (m, 0, 0, 0)^T$. Now, unlike in the case of Maxwell and KR theories, here the representation $W(p, q)$ (1) fails to be the generator of gauge transformations. However one can arrive at the representation of the group that generates gauge transformations in $B \wedge F$ theory by considering the action of the matrix

$$D(p, q, r) = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3)

where $p, q, r$ are real parameters) on the polarization vector and tensor in (2):

$$\varepsilon^\mu \to \varepsilon'^\mu = D^\mu_\nu(p, q, r) \varepsilon^\nu = \varepsilon^\mu - \frac{i}{m}(pa + qb + rc)p^\mu$$

$$\{\varepsilon_{\mu\nu}\} \to \{\varepsilon'^{\mu\nu}\} = D(p, q, r) \{\varepsilon_{\mu\nu}\} D^T(p, q, r) = \{\varepsilon_{\mu\nu}\} + (\Delta \varepsilon)_{\mu\nu}$$

where $(\Delta \varepsilon)_{\mu\nu} = k_\mu F_\nu(k) - k_\nu F_\mu(k)$ and $F_\mu(k)$ is an arbitrary function. It is obvious that these reproduce the gauge transformations in the $B \wedge F$ theory [5].

The group, of which $D(p, q, r)$ is a representation, can be found by noticing that $D(p, q, r) \times D(p', q', r') = D(p + p', q + q', r + r')$ which is the composition rule for the 3-dimensional translational group $T(3)$. Moreover, the generators $P_1 = \frac{\partial D(p, q, 0)}{\partial p}, P_2 = \frac{\partial D(0, q, 0)}{\partial q}$ and $P_3 = \frac{\partial D(0, 0, r)}{\partial r}$ of $D(p, q, r)$ are the same as those of $T(3)$ as can be seen from their Lie algebra $[P_1, P_2] = [P_2, P_3] = [P_3, P_1] = 0$. Thus the gauge generator for $B \wedge F$ theory is the representation (3) of the translational group $T(3)$. One may also notice that there are three different embeddings of $T(2)$ within $T(3)$ and they preserve the 4-momentum of massless particles moving in the three different spatial directions.

Now we describe a method by which one can derive the gauge generating representation of translational group for topologically massive theories from the corresponding representation for ordinary gauge theories living in one higher space-time dimensions. The starting point of this dimensional descent method is to note that one can interpret a massive particle in $d$-dimensions as a massless particle in $d + 1$-dimensions, with the mass being considered as the momentum
component along the additional dimension [10]. In its content, dimensional descent is related to the Inonu-Wigner group contraction.

For example the momentum and polarization vectors of Maxwell theory in 5-dimensions are given by \( p^i = (\omega, 0, 0, 0, \omega)^T \) and \( \varepsilon^i = (0, a_1, a_2, a_3, 0)^T \). With the identification of \( \omega \) with the mass \( m \) and the deletion of the last columns (by the applying the projection operator \( \mathcal{P} = \text{diagonal}(1, 1, 1, 1, 0) \)), these vectors respectively becomes the rest frame momentum and polarization vectors of Proca model in 4-dimensions which is equivalent to \( B \wedge F \) theory [5, 7]. The 5-dimensional analogue \( W_5(p, q, r) \) of (1) generate gauge transformations in Maxwell theory in that space-time dimension [7]. That is, \( \varepsilon'^i = W_5(p, q, r)^i_j \varepsilon^j = \varepsilon^i + \frac{p a_1 + q a_2 + r a_3}{\omega} p^i \). Using the projection operator \( \mathcal{P} \) one can project out the extra 5th dimension: \( \delta \bar{\varepsilon}^\mu = \mathcal{P} \delta \varepsilon^i = \frac{p a_1 + q a_2 + r a_3}{\omega} \bar{p}^\mu \). This is precisely the gauge transformation of the polarization vector in \( B \wedge F \) theory in 4-dimensions. From the form of this transformation, one can readily read off the matrix representation of the group that generate the gauge transformation of the \( B \wedge F \) theory in 4-dimensions to be (3). Dimensional descent from 4 to 3 dimensions yields analogous results for 2+1 dimensional Maxwell-Chern-Simons theory [6, 7].

The above considerations are also valid for linearized gravity theories, both the usual as well as the topologically massive ones [8]. The role of Wigner’s little group in generating the star gauge invariance in noncommutative gauge theories can be a possible extension of the present work. It is also of interest to construct and study the equivalent of the little group in AdS space where there is a novel gauge transformation connected to partially massless theories, as reported in [11].

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**References**


