\[ (x)^{\theta} \phi' = (x)^{\theta} (\mathcal{L}(\phi)) H \]

and

\[ (\mathcal{L}(\phi) H - \mathcal{M}) = (\mathcal{L}(\phi)) H - \mathcal{M} \]

As symmetrical expression under the transformation (\(\mathcal{L}(\phi)\)) and a coordinate set of singlet-gauge fields as the primary symmetry, the condition (\(\mathcal{L}(\phi)\)) H can be introduced the one particle Dirac Hamiltonian.

\[ \text{with the condition (\(\mathcal{L}(\phi)\)) H} \]

\[ \text{for the case of by} \quad \int \left[ x_{\mu} x_{\mu}^{\prime} \left( x_{\mu} x_{\mu}^{\prime} \right) \right]^{\frac{1}{2}} \delta^{4}(x_{\mu} x_{\mu}^{\prime}) = 1 \]

\[ \text{where} \quad \int x_{\mu} x_{\mu}^{\prime} \left( x_{\mu} x_{\mu}^{\prime} \right) \delta^{4}(x_{\mu} x_{\mu}^{\prime}) = 1 \]

\[ \text{if} \quad \mathcal{L}(\phi) \text{ H is introduced the one particle Dirac Hamiltonian.} \]

\[ \text{for the case of by} \quad \text{the condition (\(\mathcal{L}(\phi)\)) H} \]

\[ \int \left[ x_{\mu} x_{\mu}^{\prime} \left( x_{\mu} x_{\mu}^{\prime} \right) \right]^{\frac{1}{2}} \delta^{4}(x_{\mu} x_{\mu}^{\prime}) = 1 \]
Using these eigenvalues, the total energy of the system can be written as
\[
E_{\text{static}}[U] = N_c E^{(1)}_v[U] + N_c E^{(2)}_v[U] + E_{\text{field}}[U] - E_{\text{field}}[U = 1],
\]
where
\[
E^{(i)}_v = g^{(i)} E^{(i)}_v,
\]
\[
E_{\text{field}} = N_c \sum_{\nu} \left( N_{\nu} E_{\nu} + \frac{A}{\sqrt{4\pi}} \exp \left[ - \left( \frac{E_{\nu}}{A} \right)^2 \right] \right) \tag{8}
\]
with
\[
N_{\nu} = \frac{1}{\sqrt{4\pi}} \int \left( \frac{E_{\nu}}{A} \right)^2 . \tag{9}
\]
The $E^{(i)}_v$ and $E_{\text{field}}$ stand for the valence quark contribution to the energy for the ith baryon and the sea quark contribution to the total energy, respectively. Here, $n_0$ is the occupation number of the valence quark; that is, $n_0$ is 0 or 1. $E_{\text{field}}$ is evaluated by the familiar proper-time regularization scheme\[20]. $A$ is the cutoff parameter.

The Dirac hamiltonian $H$ with axially symmetric meson fields $U$ commutes with the operator $K_3 = L_3 + \frac{1}{2} \sigma_3 + \frac{1}{2} m \tau_3$. For $m = 2$, $H$ also commutes with the operator $P = \beta \cdot \tau$. Due to the symmetries of the meson field configuration, above operator $\beta \cdot \tau$ works as the parity operator. (For $m = 1$, the parity operator is given by a conventional form $P = \beta$.) $L_3$, $\sigma_3$ and $\tau_3$ are the third-component of the orbital angular momentum, the spin angular momentum, and the isospin operator of the quark, respectively. $K_3$ is often called the third-component of the grand spin operator. Consequently, the eigenstates of $H$ are specified by the magnitude of $K_3$ and the parity $\pi = \pm$. As $L_3$ is integer and $\sigma_3$ and $\tau_3$ are $\pm 1$, so the possible values of $K_3$ are $\pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \cdots$ for $m = 2$. $H$ also commutes with the “time-reversal” operator $T = i \gamma_1 \gamma_2 \gamma_3 \gamma_5 \gamma_7 C$, where $C$ is the charge conjugation operator. By virtue of this invariant, we see that the states of $+K_3$ and $-K_3$ are degenerate in energy\[21, 22]. According to the Kahana and Ripka\[23], we begin with investigating the spectrum of quark orbits as a function of the “soliton size” $X$ (see Fig. 1). We find that only the $K_3 = \pm \frac{1}{2}$ states dive into the negative-energy region as $X$ increases. Therefore one conclude that the lowest-lying axially symmetric $B = 2$ configuration is obtained by putting three valence quarks each in the first two positive energy states $K_3 = \pm \frac{1}{2}$. In that case, one immediately find that six valence quarks are all degenerate in energy. As a result, in our $B = 2$ system each baryon has equal classical mass. The degeneracy of the baryon in our system is a distinct feature of choosing axial symmetry as the symmetry of the meson fields.

On the other hand, if we adopt the hedgehog ansatz for the $B = 2$ system, one find that the resultant hamiltonian commutes with the grand spin operator $K$ and the parity operator $P$. The magnitude of the grand spin operator $K$ has the values of $0, 1, 2, 3, \cdots$, then if we first put three valence quarks on the state $K = 0^+$, the next three quarks must be placed in higher energy states. If the second strong level $K = 0^-$ is occupied by next three valence quarks, the total energy is about 4260 MeV\[9]. If we do not restrict the problem to “Skyrme-like configuration”\[24], one can place the quarks into $K = 1^+$ or $K = 1^-$. Unfortunately, there are not thorough analyses which adopt these configurations within the framework of the $\chi$QSM. In the $\sigma$-model calculation, their total energy is about 2620 MeV for the $K = 0^+, 1^+$ configuration\[24]. In any case, the resulting total energy is much larger than two times of the mass with isolated baryon. As a result, we confirm that the energy of the axially symmetric $K_3 = \pm \frac{1}{2}$ configuration is lower than that of all possible $B = 2$ hedgehog configuration. Finally we conclude that the lowest-lying $B = 2$ state has axial symmetry, and is obtained by putting six valence quarks in the degenerated states $K_3 = \pm \frac{1}{2}$.

The meson field configuration that minimizes the total energy $E_{\text{static}}[U]$ is determined by the following extremum conditions:
\[
\frac{\delta}{\delta F(\rho, \omega)} E_{\text{static}}[U] = 0, \quad \frac{\delta}{\delta \Theta(\rho, \omega)} E_{\text{static}}[U] = 0. \tag{10}
\]
By using the explicit form of $E_{\text{static}}[U]$, this yields the following equations of motion for the profile functions $F(\rho, \omega)$ and $\Theta(\rho, \omega)$,
\[
R_{12}(\rho, \omega) \cos \Theta(\rho, \omega) = R_3(\rho, \omega) \sin \Theta(\rho, \omega), \tag{11}
\]
\[
S(\rho, \omega) \cos F(\rho, \omega) = P(\rho, \omega) \sin F(\rho, \omega), \tag{12}
\]
and
\[
P(\rho, \omega) = R_{12}(\rho, \omega) \sin \Theta(\rho, \omega) + R_3(\rho, \omega) \cos \Theta(\rho, \omega), \tag{13}
\]
where
\[
R_{12}(\rho, \omega) = 2 R_{12a}(\rho, \omega) + R_{12b}(\rho, \omega), \tag{14}
\]
\[
R_3(\rho, \omega) = 2 R_{3a}(\rho, \omega) + R_{3b}(\rho, \omega), \tag{15}
\]
\[
P(\rho, \omega) = 2 P_a(\rho, \omega) + P_b(\rho, \omega), \tag{16}
\]
where subscripts $a$ and $b$ denote the contributions from the valence quarks and the sea quarks, respectively. The
explicit forms of $R_{12}$, $R_3$ and $S$ are

$$R_{12}(\rho, z) = n_0 \times \int \, d\varphi \frac{\partial \phi_0 (\rho, \varphi, z)}{\partial \varphi} \hat{\gamma}_5 (\tau_1 \cos \varphi + \tau_2 \sin \varphi) \phi_0 (\rho, \varphi, z),$$

$$R_{12}(\rho, z) = \sum_\nu N_\nu \, \text{sign}(E_\nu) \times \int \, d\varphi \frac{\partial \phi_0 (\rho, \varphi, z)}{\partial \varphi} \hat{\gamma}_5 (\tau_1 \cos \varphi + \tau_2 \sin \varphi) \phi_0 (\rho, \varphi, z),$$

$$R_3(\rho, z) = n_0 \int \, d\varphi \frac{\partial \phi_0 (\rho, \varphi, z)}{\partial \varphi} \hat{\gamma}_5 \gamma_3 \phi_0 (\rho, \varphi, z),$$

$$R_3(\rho, z) = \sum_\nu N_\nu \, \text{sign}(E_\nu) \times \int \, d\varphi \frac{\partial \phi_0 (\rho, \varphi, z)}{\partial \varphi} \hat{\gamma}_5 \gamma_3 \phi_0 (\rho, \varphi, z),$$

$$S_0(\rho, z) = n_0 \int \, d\varphi \frac{\partial \phi_0 (\rho, \varphi, z)}{\partial \varphi} \phi_0 (\rho, \varphi, z),$$

$$S_0(\rho, z) = \sum_\nu N_\nu \, \text{sign}(E_\nu) \times \int \, d\varphi \frac{\partial \phi_0 (\rho, \varphi, z)}{\partial \varphi} \phi_0 (\rho, \varphi, z).$$

In order to evaluate Eqs. (11) and (12) numerically, the following procedures were employed. Firstly, we start from the initial functions $F_0(\rho, z)$ and $\Theta_0(\rho, z)$ that satisfy the boundary conditions given by Braaten and Carson[19]:

$$F(\rho, z) \to 0$$

$$F(0, 0) = -\pi,$$

$$F(0, 0) = \begin{cases} \frac{\sqrt{\rho^2 + z^2}}{R}, & \rho > 0 \\ \frac{\sqrt{\rho^2 + z^2}}{R}, & \rho < 0 \end{cases}.$$  

We solve the one particle Dirac equation using the above functions $F_0(\rho, z)$ and $\Theta_0(\rho, z)$. Secondly, $R_{12}(\rho, z)$ and $R_3(\rho, z)$ and $S_0(\rho, z)$ are calculated from the resultant eigenvalues and eigenfunctions; $\Theta_0(\rho, z)$ is given by Eq. (11). Thirdly, the function $F(\rho, z)$ is obtained on the basis of Eqs. (12) and (13). Then, new iterates of $F(\rho, z)$ and $\Theta_0(\rho, z)$ are obtained by solving the Dirac equation. This procedure is continued until self-consistency is attained.

Before reporting our results, we provide some comments on our numerical calculations. (i) Numerical calculations were performed for several values for the constituent quark mass $M$, from 350 MeV to 1000 MeV. The proper-time cut-off parameter $\Lambda$ was not a free parameter, but was determined so as to reproduce the pion decay constant $f_\pi = 93$ MeV [23]. (ii) We chose the initial functions $F_0(\rho, z) = -\pi + \pi \sqrt{\rho^2 + z^2}/R$, with $R = 1.0$, and $\Theta_0(\rho, z) = \tan^{-1} (\rho/z)$. We tried several forms of the initial functions and confirmed that the final result was independent of these choices. (iii) Diagonalization of the Dirac hamiltonian was done following the method of Kahana and Ripka[23]. They used a discretized plane wave basis in a spherical box with radius $D$, which were defined.
by their grand spin and the parity. As previously stated, since our Dirac Hamiltonian had axially symmetric property, the grand spin $K$ was no longer a good quantum number of the eigenstates. In our case, since the third component of the grand spin $K_3$ and the parity $\pi = \pm$ were only the good quantum number of the states, then we modified the Kahan-Ripka basis into those which were defined by the $K_3$ and the parity $\pi = \pm$. The new basis are the eigenstates of the free Hamiltonian in a cylindrical box with height $2 \times D_z$ and radius $D_x$. These basis enable us to diagonalize our Dirac Hamiltonian.

In Figs. 2-3, we present results for the profile functions $F(\rho, z)$, $\Theta(\rho, z)$ with $M = 400$ MeV. In Figs. 4-5 we display the baryon number density. Fig. 4 shows the contribution from the valence quark and Fig. 5 from the sea quark. From Figs. 4-5 it is found that the baryon number density has a toroidal shape. This result is consistent with other chiral invariant models using axially symmetric meson fields, such as the Skyrme model[19] and a naive quark meson model which involved six valence quark and a pion cloud[22]. The classical soliton energies corresponding to various values of $M$ are given in Table I. As $M$ increases, the valence quark contribution rapidly decreases, while that of the sea quark grows rapidly. Around $M \sim 650$ MeV, the valence level crosses zero energy and dives into the negative-energy region. For $M > 650$ MeV, the systems are dominated by the sea quark contribution. As for the total energy, which is sum of the valence quark and the sea quark contribution, there is essentially no noticeable change as $M$ increases. This is a characteristic behaviour of our solution. The increase in the total energy value for large $M$ is perhaps due to the omission of the higher wave numbers from our basis.

Here, one could consider the classical “binding energy” $\mathcal{E}_{\text{class}}$, which is given by

$$\mathcal{E}_{\text{class}} = \mathcal{E}_{\text{static}}(U) - 2M_{B=1, \text{hedgehog}}. \quad (25)$$

The mean radius of the toroid for the quark distribu-
tion is estimated by

\[\langle \rho \rangle_v = \frac{1}{2} \int \rho d\rho dz d\varphi \rho \delta v((\rho, \varphi, z)\delta \rho(\rho, \varphi, z)), \quad (26)\]

\[\langle \rho \rangle_0 = \frac{1}{2} \sum_N N \text{sign}(E_N)\]

\[\times \int \rho d\rho dz d\varphi \rho \delta v((\rho, \varphi, z)\delta \rho(\rho, \varphi, z)), \quad (27)\]

\[\langle \rho \rangle = \langle \rho \rangle_v + \langle \rho \rangle_0. \quad (28)\]

These values are also given in Table I and show a rapid decrease with increasing \( M \). This is reasonable, because here \( M \) is regarded as the coupling-constant between the quark and pion, so the larger \( M \) means a stronger quark-pion interaction. The stronger interaction may produce more compact solitons. At \( M = 400 \) MeV, which may be a suitable choice, we obtained \( \langle \rho \rangle \approx 0.672 \) fm which is in qualitative agreement with the Skyrme model value 0.78 fm[19].

In summary, we have obtained the axially symmetric \( B = 2 \) soliton solution of the SU(2) \( \chi \)QSM. The solution was obtained in a self-consistent manner. The results are in qualitative agreement with those from the Skyrme model and other quark meson models. This suggests that these features are independent of the particular choice of chirally invariant model. Individualities of each model will become clearer after thorough investigations for various physical observables[26]. The most striking difference between our \( \chi \)QSM and the Skyrme model is the existence of quark degrees of freedom. From consideration of the single quark energy level, we confirm that the minimum energy configuration of \( B = 2 \) is axially symmetric, while in the Skyrme model it is a conjecture. Since the \( \chi \)QSM includes the valence and the sea quark degrees of freedom, we can give theoretical support for nuclear medium effects such as the EMC effect in deep inelastic scattering experiments.

The solutions obtained here were classical ones which have no definite spin, isospin quantum numbers corre-
sponding to physical particles. Therefore the solutions should be quantized by projecting onto good spin, isospin states in order to estimate the energies, the mean square radius, and other static properties of the physical \( B = 2 \) system. The quantization of our solution using the well-known cranking procedure in SU(2) is now in progress.

One of us (N.S.) is very grateful to Dr. S. Akiyama for many valuable discussions and comments from the beginning of this work.