Quantum gates on hybrid qubits

1. INTRODUCTION

Although quantum computing (quantum computers) can be treated as a general system, the qubit model [1] is widely used in quantum computing. We introduce hybrid qubits (hybrid qubits) that act on qubits of different dimensions. In particular, we describe a wider range of applications, for example, a hybrid qubit, to introduce an idealized qubit. We generalize the quantum computing model to hybrid qubits and define the hybrid qubits in this way.

II. HYBRID QUANTUM GATES

A. Generalized Pauli Group

A basis for operators on $\mathbb{C}^d$ is given by the following:

$$\{ |0\rangle - \exp(i \theta |1\rangle |1\rangle, \qquad |0\rangle - \exp(i \theta |1\rangle |1\rangle, \qquad |0\rangle, \qquad |1\rangle \}$$

for $\theta \in \mathbb{R}$.

The generalization is illuminating because it differs subtly from standard non-locality models (see e.g., [3]).

We introduce quantum qubits that act on qubits of different dimensions. In particular, we describe a wider range of applications, for example, a hybrid qubit, to introduce an idealized qubit. We generalize the quantum computing model to hybrid qubits and define the hybrid qubits in this way.
where \( X \) and \( Z \) are defined by their action on the computational basis

\[
X|s\rangle = |s + 1 \pmod{d}\rangle, \quad Z|s\rangle = \exp(2\pi i s/d)|s\rangle = \zeta_d^s|s\rangle,
\]

where

\[
\zeta_d \equiv \exp(i2\pi/d).
\]

In the following we shall write for simplicity \( \zeta \) instead of \( \zeta_d \), if the dimension is easily understood from the context.

The unitary operators \( X \) and \( Z \) generate the generalized Pauli group \( \mathcal{P}_d \). Note that \( X \) and \( Z \) do not commute; they obey

\[
Z^j X^k = \zeta^{jk} X^k Z^j,
\]

and \( X^d = Z^d = 1 \).

### B. One-qudit gates

Before we consider two-qudit gates, we review some of the properties of the useful one-qudit ‘Fourier gate’ \( \hat{F} \), which transfers the qudit computational basis \( |\delta\rangle \) to the dual state

\[
|s\rangle \equiv \hat{F}|s\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \zeta_d^{sk}|k\rangle \quad \text{for} \quad s \in \mathbb{Z}_d
\]

such that \( \langle s'|s\rangle = 1/\sqrt{d} \zeta_d^{s's} \). These dual states are related to the computational basis by a discrete Fourier transform, and distinguished by a rounded bracket notation. As an example, if the computational basis corresponds to the number states for the harmonic oscillator, the dual basis corresponds to the Gell-Mann phase states [17]. Similarly the SU(2) phase states are dual to angular momentum eigenstates [18].

The \( \hat{F} \) gate is a qudit version of the one-qudit Hadamard gate \( H \). However, and in contrast to \( H \), the \( \hat{F} \) operator for \( d \geq 3 \) is not Hermitian and its order is 4 instead of 2, as [19]

\[
\hat{F}^2|s\rangle = |-s\rangle, \quad \hat{F}^4 = 1.
\]

Similarly, the unitary operator \( X \) can be considered as the qudit version of the NOT gate, and \( Z \) is the qudit version of the phase gate for qudits.

### C. Two-qudit gates

1. **Hybrid SUM gate**

Two representative quantum gates on qubits are the controlled-NOT (CNOT) and SWAP gate. A generalised CNOT gate for qudits [2, 3, 20] has been called the displacement gate, or SUM gate [20]. As a compromise, we refer to the hybrid version of this ‘controlled-SHIFT’ operator as the ‘SUM gate’, but use the notation \( \mathcal{D} \) to emphasize its displacement nature. To achieve unity in notation, we shall use calligraphic letters to denote two- and multi-qudit gates. In particular, we shall use \( \mathcal{S}, \mathcal{T} \) and \( \mathcal{F} \) to denote the SWAP, the hybrid Toffoli, and Fredkin gates, respectively.

We now define the hybrid version of the SUM or displacement gate \( \mathcal{D} \) on \( \mathcal{H}_d \otimes \mathcal{H}_d \) for arbitrary \( d_c \) and \( d_t \) (the subscript \( c \) refers to “control” and \( t \) to “target”) by

\[
\mathcal{D} := \sum_{n=0}^{d_c-1} P_n \otimes X^n \quad \text{for} \quad d_c, d_t \in \mathbb{N},
\]

where

\[
P_n \equiv |n\rangle\langle n|, \quad n \in \mathbb{Z}_{d_c}
\]

is a primitive projection operator on a computational basis state of the control space \( \mathcal{H}_d \).

It is important to note the following subtle difference between hybrid and non-hybrid qudit systems: although the states \( |i\rangle \otimes |j\rangle \) and \( |i + d_c\rangle \otimes |j\rangle \) are formally equivalent, the operators \( P_i \otimes X^j = |i\rangle\langle i| \otimes X^j \) and \( P_{i + d_c} \otimes X^{i + d_c} = |i + d_c\rangle\langle i + d_c| \otimes X^{i + d_c} \) are not equal in general, if \( d_c \neq d_t \). Hence, in order to obtain a unique definition, we insist that the summation in (9) is restricted to \( 0 \leq n < d_c \). This subtle difference has interesting consequences when we try to define a SWAP gate for hybrid systems.

For \( d_c > d_t \), we can combine together all the projection operators \( P_n \), which yield the same \( X^s \), and obtain

\[
\mathcal{D} = \sum_{s=0}^{d_c-1} \Pi_s \otimes X^s, \quad \text{for} \quad d_c > d_t,
\]

where

\[
\Pi_s = \sum_{n = 0}^{d_t-1} P_n, \quad \text{for} \quad s \in \mathbb{Z}_{d_c}
\]

For example, the SUM gate for \( d_c = 3 \) and \( d_t = 2 \) is given by

\[
\mathcal{D} = \sum_{s=0}^{1} \Pi_s \otimes X^s = \Pi_0 \otimes I + \Pi_1 \otimes X,
\]

where \( \Pi_0 = P_0 + P_1 \) and \( \Pi_1 = P_1 \).

We can extend expression (11), also for \( d_c \leq d_t \), by defining

\[
\mathcal{D} := \sum_{s=0}^{d_{\text{min}}-1} \Pi_s \otimes X^s,
\]

where \( d_{\text{min}} := \min(d_c, d_t) \). Note that \( \sum_{s=0}^{d_{\text{min}}-1} \Pi_s = I_{d_c \times d_c} \).
We introduce another interesting hybrid gate:
\[
\mathcal{D}_{12}^I(m) \otimes |n\rangle := |m\rangle \otimes |m-n\rangle, \quad \text{for } m \in \mathbb{Z}_{d_c} \text{ and } n \in \mathbb{Z}_{d_t}.
\] (14)

This operator is unitary and Hermitian, as \( (\mathcal{D}_{12}^I)^2 = I \). It is related to the SUM gate by
\[
\mathcal{D}_{12}^I = \mathcal{D}_{12} (I \otimes F^2).
\]

For \( d_c = d_t \), our hybrid \( \mathcal{D}_{12}^I \) reduces to the generalized CNOT gate given by Alber et al. [4].

2. The SWAP gate

The SWAP operation on \( \mathcal{H}_d \times \mathcal{H}_d \) systems, i.e. for \( d_c = d_t = d \), is defined by
\[
\mathcal{S} |i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle, \quad \text{for } i, j \in \mathbb{Z}_d.
\] (15)

hence \( \mathcal{S} = \sum_{i=0}^{d-1} |j\rangle \langle i| \otimes |i\rangle \langle j| \). Clearly, the definition cannot be used for hybrid systems. Instead, for \( d_c \neq d_t \) (and also for \( d_c = d_t \)) we define partial-SWAP operators by
\[
\mathcal{S}_P |i\rangle \otimes |j\rangle = \begin{cases} |j\rangle \otimes |i\rangle, & \text{for } i,j \leq d_{\text{min}}, \\ |i\rangle \otimes |j\rangle, & \text{otherwise} \end{cases}
\] (16)

where \( d_P \leq d_{\text{min}} = \min(d_c, d_t) \). Obviously, \( \mathcal{S}_P \) in (16) is unitary and Hermitian, as \( \mathcal{S}_P \mathcal{S}_P^\dagger = I \). This partial SWAP gate only acts as a SWAP operation on a subspace of the original Hilbert space.

3. Relation between SWAP and SUM operators

It is easy to check that \( \mathcal{S} \) can be written in terms of three SUM gates as follows
\[
\mathcal{S} = (F^2 \otimes I) \mathcal{D}_{12} \mathcal{D}_{21}^{-1} \mathcal{D}_{12}.
\] (17)

Another possibility is to use expressions (17) formally to define a swap-like gate for hybrid system. However, contrary to what one might expect, this operator does not yield a swap operation, even for \( 0 \leq i, j \leq d_{\text{min}} \).

We illustrate this claim by a simple example, where \( d_1 = 3 \) and \( d_2 = 2 \). By applying expression (17) to the state \( |0\rangle \otimes |1\rangle \). We obtain successively
\[
|0\rangle \otimes |1\rangle \rightarrow |0\rangle \otimes |1\rangle \rightarrow |2\rangle \otimes |1\rangle
\rightarrow |2\rangle \otimes |1\rangle \rightarrow |1\rangle \otimes |0\rangle
\]

Recently, Fujiw constructed a swap gate, as follows [21]
\[
\mathcal{S} = \mathcal{D}_{12} (F^2 \otimes I) \mathcal{D}_{21} (F^2 \otimes I) \mathcal{D}_{12} (I \otimes F^2),
\] (19)

expressed in our notations. Note that both constructions of SWAP gates actually require three SUM gates and three local \( F^2 \) gates. This is because
\[
\mathcal{D}^{1}_{21} = (F^2 \otimes I) \mathcal{D}_{12} (I \otimes F^2),
\] (20)
so that our SWAP gate (17) can be written as
\[
\mathcal{S} = (F^2 \otimes I) \mathcal{D}_{12} (I \otimes F^2) \mathcal{D}_{21} (I \otimes F^2) \mathcal{D}_{12}.
\] (21)

We also note that the SWAP gate on continuous variables can be constructed by three generalized controlled-NOT gates on continuous variables [22].

D. Higher order quantum hybrid gates

Representative higher-order hybrid gates include the quantum versions of the Toffoli gate [5, 6, 7, 8, 9] and of the Fredkin gate [10, 11, 12, 13, 14]; these three-bit gates are important primitives for logically reversible classical computation, for which universal reversible two-bit gates do not exist. The Toffoli gate is effectively a controlled-controlled-NOT (CCNOT), and the Fredkin gate is another universal three-bit gate.

As a controlled-controlled-NOT, the quantum Toffoli gate has two qubits as control and one qubit as target, and the target qubit flips if and only the two control qubits are in the state \( |1\rangle \otimes |1\rangle \). The Fredkin gate has one qubit as control and two qubits as target, and the states of two target qubits swap if and only if the control qubit is in the state \( |1\rangle \). Here we give the hybrid version of these two higher-order gates.

1. The hybrid Toffoli gate

A general controlled unitary gate acting on Hilbert spaces \( \mathcal{H}_d \otimes \mathcal{H}_d \) can be written as
\[
\mathcal{C} \mathcal{U} = \sum_{s=0}^{d_s-1} P_s \otimes U_s = \sum_{s=0}^{d_s-1} |s\rangle \langle s| \otimes U_s,
\] (22)

where \( U_s \) are arbitrary unitary operators on the target space \( \mathcal{H}_d \).

Note that \( \{U_s\} \) may be unitary operators on single or multiple qudits, and may include the case of qudit-controlled operators on other qudits. The latter case allows unitary operators on qudits that can be jointly controlled by two or more qudits. An example is provided by the following ‘natural’ generalization of the Toffoli gate [5, 6, 7, 8, 9]
\[
\mathcal{T} := \sum_{s=0}^{d_s-1} P_s \otimes \mathcal{D}_s^s,
\] (23)

where the \( U_s \) in (22) are replaced by \( \mathcal{D}_s^s \), which are powers of the generalized displacement operator (9). The hybrid
Tofoli-type gate is thus a ‘triple gate’

\[ T = \sum_{s=0}^{d_t-1} \sum_{r=0}^{d_t-1} P_r \otimes P_s \otimes X^s = \sum_{m=0}^{d_t-1} \Pi_m \otimes X^m , \quad (24) \]

where \( \Pi_m \) are compound projection operators, given by

\[ \Pi_m = \sum_{r=0}^{d_t-1} \sum_{s=0}^{d_t-1} \delta_{m,r,s} \space P_r \otimes P_s , \quad m \in \mathbb{Z}_{d_t} . \quad (25) \]

where the products \( rs \) of the delta in (25) are defined modulo \( d_t \). Hence, the order of the Toffoli gate is equal to \( d_t \).

2. The hybrid Fredkin gate

Another type of multi-qudit gate is the quantum Fredkin gate [10]-[14]. We define the hybrid Fredkin gate on \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_3} \) by

\[ F := \sum_{m=0}^{d_1-1} P_m \otimes S_P^m = \Pi_+ \otimes I + \Pi_- \otimes S_P \quad (26) \]

where \( \Pi_\pm \) are the following projection operators

\[ \Pi_+ := \sum_{m \text{ odd}} P_m \quad \text{and} \quad \Pi_- := \sum_{m \text{ even}} P_m \quad (27) \]

where we have used the property \( S_P^2 = I \).

The hybrid Fredkin gate executes a swap for purely odd state \( |\psi_-\rangle \), i.e. for \( \Pi_- |\psi_-\rangle = |\psi_-\rangle \), and does nothing for the even states. However, for mixed odd and even states, one obtains a mixed result. For instance, if we choose a input state as \( (|0\rangle + |1\rangle) \otimes |\alpha\rangle \otimes |\beta\rangle \), the output state after the gate is \( |0\rangle \otimes |\alpha\rangle \otimes |\beta\rangle + |1\rangle \otimes |\beta\rangle \otimes |\alpha\rangle \), which is in general an entangled state.

III. ENTANGLEMENT PRODUCED BY QUANTUM GATES

Hybrid two- and multi-qudit gates can enhance entanglement, i.e. the entanglement of the output state can be greater than that of the input state. In this case we regard the hybrid gates as entangling gates. Different methods exist for characterizing the enhancement of entanglement. In this section, we discuss entanglement enhancement by the hybrid SUM gate.

A. Entanglement measures for states and operators

There are various measures of entanglement for a normalized state \( |\psi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \). Here, we shall use the von Neumann entropy

\[ E(|\psi\rangle) = - \sum_{n=0}^{N_{s}-1} p_n \log p_n . \quad (28) \]

where \( \{p_n\} \) is defined in terms of the Schmidt decomposition of \( |\psi\rangle \):

\[ |\psi\rangle = \sum_{n=0}^{N_{s}-1} \sqrt{p_n} \phi_n \otimes |\chi_n\rangle , \quad p_n > 0 \forall n, \quad (29) \]

and log is always taken to be base 2. Definition (28) was adapted [23, 24] to define operator entanglement as follows. Let \( \mathcal{Q} \) be an operator acting on a hybrid space \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \), with the following Schmidt decomposition [24]

\[ \mathcal{Q} = \sum_{n=0}^{N_{s}-1} s_n A_n \otimes B_n , \quad (30) \]

where \( s_n > 0 \forall n \), and the two operators \( A_n \) and \( B_n \) are orthonormal with respect to the Hilbert-Schmidt scalar product defined by \( (A, B) := \text{tr} (A^\dagger B) \) for \( A \) and \( B \) two arbitrary operators. In particular, \( \|A\| := \sqrt{\text{tr} (A^\dagger A)} \) is the Hilbert-Schmidt norm of the operator \( A \), and \( A := A/\|A\| \) if \( \|A\| \neq 0 \).

Since linear operators over a finite-dimensional vector space \( \mathcal{H} \) can be regarded as \( d^2 \)-dimensional vectors, we may think of \( \mathcal{Q} \equiv \mathcal{Q}/\|\mathcal{Q}\| \) as a normalized state, which we denote by \( |\mathcal{Q}\rangle \), so that (30) becomes

\[ |\hat{\mathcal{Q}}\rangle = \sum_{n=0}^{N_{s}-1} \sqrt{p_n} |A_n\rangle \otimes |B_n\rangle , \quad (31) \]

where \( \sqrt{p_n} = s_n/\|\mathcal{Q}\| \). In particular, if \( \mathcal{Q} \) is unitary, then \( \|\mathcal{Q}\| = \sqrt{d_1 d_2} \). The operator entanglement [24]

\[ E_{op}(\mathcal{Q}) = - \sum_n \frac{s_n^2}{d_1 d_2} \log \left( \frac{s_n^2}{d_1 d_2} \right) . \quad (32) \]

B. Operator entanglement of the SUM gate

In Sec. II, we essentially obtained in Eq. (13) the Schmidt decomposition of the operator \( \mathcal{D} \) because the projection operators \( \Pi_r \) and the unitary operators \( X^s \) are mutually orthogonal, i.e.

\[ \langle \Pi_r, \Pi_s \rangle = \|(\Pi_r)\|^2 \delta_{r,s} , \quad r, s \in \mathbb{Z}_{d_{min}} \quad (33) \]

\[ \langle X^s, X^t \rangle = \|(X^s)\|^2 \delta_{r,s} = d \delta_{r,s} , \quad r, s \in \mathbb{Z}_{d_1} , \quad (34) \]

where we used \( \|(X^s)\| = \text{tr} (X^s X^s \dagger) = \text{tr} I = d_1 \), because \( X^s \) is unitary. Hence, by dividing the operators \( \Pi_r \) and \( X^s \) in (13) by their norms, we immediately obtain the following Schmidt decomposition of \( \mathcal{D} \):

\[ \mathcal{D} := \sum_{s=0}^{d_{min}-1} (\Pi_s \otimes X^s) , \quad (35) \]

where for \( d_c = K d_r + r \) we have

\[ \|\Pi_s\| = \left\{ \begin{array}{ll} \sqrt{K + 1} & (0 \leq s \leq r - 1), \\
\sqrt{K} & (r \leq s \leq d_c - 1). \end{array} \right. \quad (36) \]
From Eqs. (35) and (36) expression (31) yields immediately
\[ E_{\text{op}}(\mathcal{D}) = e_{\mathcal{D}}(d_c, d_1), \]
(37)
where \( (d_c = K d_1 + r) \)
\[ e_{\mathcal{D}}(d_c, d_1) = -\frac{K + 1}{d_c} \log \frac{K + 1}{d_c} - (d_1 - \frac{K}{d_c} \log \frac{K}{d_c}). \]
(38)
Note that for \( d_c < d_1 \) the general expression (38) reduces simply to
\[ e_{\mathcal{D}}(d_c, d_1) = \log d_c, \quad \text{for} \quad d_c < d_1, \]
(39)
by substituting \( K = 0 \) and \( r = d_c \).

C. Entanglement produced by the SUM gate

We prove the following lemma:

Lemma 1: The entanglement generated by the hybrid SUM gate \( \mathcal{D} \) on the following three product states (one without and two with ancillae)
\[ |\Psi_1 \rangle \equiv |\gamma \rangle \otimes |t \rangle = \left( \frac{1}{\sqrt{d_c}} \sum_{m=0}^{d_c-1} |m \rangle \otimes |t \rangle, \right), \]
(40)
\[ |\Psi_2 \rangle \equiv |\alpha \rangle \otimes |t \rangle = \left( \frac{1}{\sqrt{d_c}} \sum_{m=0}^{d_c-1} |\alpha \rangle \otimes |m \rangle \otimes |t \rangle \right), \]
(41)
\[ |\Psi_3 \rangle \equiv |\alpha \rangle \otimes |\beta \rangle = |\alpha \rangle \otimes \left( \frac{1}{\sqrt{d_c}} \sum_{n=0}^{d_c-1} |n \rangle \otimes |\beta \rangle \right), \]
(42)
where \( |t \rangle \) is any of the computational states of the target space, are equal to the operator entanglement (38) of \( \mathcal{D} \).
\[ E(\mathcal{D} |\Psi_3 \rangle) = E_{\text{op}}(\mathcal{D}) = e_{\mathcal{D}}(d_c, d_1). \]
(43)

Proof: The three initial states have zero entanglement, since they were chosen to be product states. Therefore, the increase of entanglement due to \( \mathcal{D} \) is equal to \( E(\mathcal{D} |\Psi_3 \rangle) \).

We shall now apply \( \mathcal{D} \) to (40):
\[ |\Psi_1 \rangle \equiv |\gamma \rangle \otimes |t \rangle = \left( \frac{1}{\sqrt{d_c}} \sum_{i=0}^{d_c-1} \Pi_s |m \rangle \otimes X_s |t \rangle \right). \]
(44)
Let \( d_c = K d_1 + r \) (Note that \( K = 0 \) and \( r = d_c \) if \( d_c < d_1 \)). Hence,
\[ \sum_{i=0}^{d_c-1} \Pi_s |m \rangle = \begin{cases} |s \rangle + |s + d_1 \rangle + \ldots + |s + Kd_1 \rangle = \sqrt{K+1} |\psi_s \rangle, & \text{for} \quad 0 \leq s < r - 1, \\ |s \rangle + \ldots + |s + (K - 1)d_1 \rangle = \sqrt{K} |\psi_s \rangle, & \text{for} \quad r \leq s < d_1 - 1, \end{cases} \]
(45)
where the \( |\psi_s \rangle \), \( s \in \mathbb{Z}, d_m \) are orthonormal states which, for \( d_s < d_m \), span a \( d_s \)-dimensional subspace of \( \mathcal{H}_{d_s} \).
By substituting (45) into (44), we obtain the following Schmidt decomposition of the final state
\[ |\Psi_1 \rangle = \mathcal{D} |\gamma \rangle \otimes |t \rangle = \sum_{s=0}^{d_c-1} \sqrt{d_c} |\psi_s \rangle \otimes |t + s \rangle \]
(46)
where
\[ p_s = \begin{cases} \frac{1}{d_c} & \text{for} \quad 0 \leq s \leq r - 1, \\ 0 & \text{for} \quad r \leq s \leq d_c - 1, \end{cases} \]
(47)
By substituting the above equation into (29) we obtain exactly the same expression (38). Similarly, we can prove that the entanglement of \( E(\mathcal{D} |\alpha \rangle \otimes |t \rangle) \) is also given by (38).

Finally, since the states \( \{ X_s |\beta \rangle \} \) are orthonormal for different \( s \), we get essentially the same Schmidt decomposition for \( \mathcal{D} |\alpha \rangle \otimes |t \rangle \) as in (46), and hence the same final entanglement. This result also follows from lemma 5 of Ref. [24].

The entanglement function (38) is plotted in Fig. 1. As the generated entanglement equals the operator entanglement according to Eq. (43), Fig. 1 presents \( E \) as the ordinate axis. We observe in Fig. 1 that the entanglement approaches \( \log d_c \) as \( d_c \) becomes large. We can see this asymptotic result in Eq. (38) by noting that
\[ \frac{K + 1}{d_c} = \frac{d_c + d_1 - r}{d_c d_1} = \frac{1}{d_c} \]
so the entanglement asymptotically approaches \( \log d_c \) as observed in Fig. 1.

IV. PHYSICAL REALIZATION OF HYBRID GATES

One can encode a qudit in physical systems such as spin systems and harmonic oscillators [2]. The Hilbert space associated with a spin-\( j \) system is spanned by the basis \( \{|j, m \rangle; m = -j, \ldots, j \} \), and the su(2) algebra is generated by \( \{ J_x, J_y, J_z \} \), with \( [J_x, J_y] = iJ_z \), etc., and \( (J_x^2 + J_y^2 + J_z^2)|j, m \rangle = j(j + 1)|j, m \rangle \). It is natural to define a number operator \( \hat{N} \) and number states as follows
\[ \hat{N} : = J_z + j, \]
\[ |n \rangle_j : = |n - j \rangle \quad (n = 0, \ldots, 2j). \]
(48)
(49)
Then we have \( \hat{N} |n \rangle_j = n |n \rangle_j \). In the spin system the operators \( X \) and \( Z \) are realized as
\[ X = \sum_{n=0}^{2j} |n + 1 \rangle_j \langle n |, \]
\[ Z = \exp \left[ \frac{2 \pi N}{2j + 1} \right]. \]
A. Controlled-phase and SUM gates

We consider interaction between spin–\( j_1 \) and spin–\( j_2 \) systems, via the Hamiltonian \( H = -g J_{zz} J_{zz} \). Up to local unitary operators, the evolution operator \( \exp(i g J_{zz} J_{zz}) \) is equivalent to \( U(t) = \exp(i g N_x N_z) \). By choosing \( t = \frac{2 \pi}{d_2} \), we obtain the unitary operator

\[
V = \exp \left[ \frac{2 \pi}{d_2} N_x N_z \right] = \mathcal{C}^{N_x N_z}_t,
\]

which is just the controlled-phase gate [3]. On the other hand, we know that the SUM gate can be obtained from the controlled-phase gate as follows [25]

\[
D = (I \otimes F) \mathcal{C}^{N_x N_z}_t (I \otimes F).
\]

Therefore, with the aid of \( F \) gate we realized the hybrid SUM gate.

B. Toffoli gate

Now let us see how to physically create a hybrid Toffoli gate. Refs. [9, 26] show that the interaction Hamiltonian \( N_1 N_2 N_3 \) (\( N_i \) correspond to spin–\( j_i \) and one \( j_i \) is equal to \( 1/2 \)) arises in ion-trap systems when coupling these operators \( N_i \) to a common continuous variable. The dimension of a spin–\( j_i \) system is given by \( d_i = 2j_i + 1 \).

Therefore, we have the three-body controlled-phase gate

\[
W(\theta) = e^{i \theta N_1 N_2 N_3} \tag{54}
\]

By choosing, say, \( \theta = 2 \pi / d_2 \), we make \( \mathcal{H}_{d_2} \) the target space while \( \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_2} \) becomes the control space. Then, by appending the appropriate \( F \) gate on the target system, we can realize the Toffoli gate acting on the systems \( \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_2} \).

C. Fredkin gate

As a final remark we point out that we can construct a control-SWAP gate \( \mathcal{S}_{12} \) acting on \( \mathcal{H}_d \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty \) as a generalization of the controlled-SWAP gate acting on \( \mathcal{H}_2 \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty \) system [22].

The SWAP gate between two bosonic modes \( a_1 \) and \( a_2 \) is given by [22]

\[
\mathcal{S}_{12} = e^{2 \pi i a_1 a_2 \frac{1}{2}} e^{2 \pi i (a_1^a a_2 - a_1^d a_2)} \tag{55}
\]

In an ion-trap system we can couple the spin–\( j \) system to two bosonic modes \( a_i \) (\( i = 1, 2 \)) as [27, 28]

\[
H_i = \chi N a_i^a a_i \tag{56}
\]

Since operators \( H_i \) commute with each other, we can simulate the following Hamiltonian

\[
H = H_1 - H_2 = \chi N (a_1^a a_1 - a_2^a a_2) = 2 \chi N J_z, \tag{57}
\]

where \( J_z = \frac{1}{2} (a_1^a a_1 - a_2^a a_2) \). The operators \( J_z = a_1^a a_2 - a_2^a a_1 \) form the \( su(2) \) Lie algebra. The evolution operator of the Hamiltonian \( H \) at time \( t = -\pi / 2 \chi \) is given by

\[
U = U(-\pi / 2 \chi) = e^{i \pi J_x N} \tag{58}
\]

The evolution operator \( U \) can be transformed to \( U' \) as

\[
U' = e^{i \pi J_x N} U e^{i \pi J_x N} = e^{i \pi J_x N} e^{2 \pi \frac{N (a_1^a a_1 - a_2^a a_2)}{2}} \tag{59}
\]

where \( J_{xx} = (J_+ + J_-) / 2 \) and \( J_{xx} = (J_+ - J_-) / \sqrt{2} \).

From Eqs. (55), (56), and (59), we construct the controlled-SWAP gate (hybrid Fredkin gate) as

\[
\mathcal{F} = e^{i \pi a_1^a a_2 N} e^{i \pi a_1^a a_2^d a_2} \times e^{i \pi a_2^d a_2 N} e^{i \pi a_1^d a_2^a a_2} = \mathcal{S}^N \tag{60}
\]

Therefore we have provided a controlled-SWAP gate on \( \mathcal{H}_d \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty \) systems in terms of five two-body operators.
V. CONJUGATION BY THE SUM GATE

A conjugation by the SUM gate $D$ is described by the following lemma:

**Lemma 2:** The hybrid SUM gate $D$ yields, by conjugation, an automorphism of the Pauli group $\mathcal{P}_{d_c} \odot \mathcal{P}_{d_t}$, iff $d_c/d_t$ is an integer $K$. More explicitly,

\[
\begin{align*}
D(X \otimes I)D^\dagger &= X \otimes X \quad \text{(61)} \\
D(I \otimes X)D^\dagger &= I \otimes X \quad \text{(62)} \\
D(Z \otimes I)D^\dagger &= Z \otimes X \quad \text{(63)} \\
D(I \otimes Z)D^\dagger &= \left( \sum_{s=0}^{d_c-1} \zeta_{d_c}^{-s} P_s \right) \otimes Z \quad \text{(64)} \\
&= Z^{-K} \otimes Z \quad \text{for } \frac{d_c}{d_t} = K. \quad \text{(65)}
\end{align*}
\]

**Proof:** By noting that

\[
P_rX_P_s = P_s|s + 1\rangle\langle s| = |s + 1\rangle\langle s| \delta_{r,s+1}, \quad \text{(66)}
\]

we obtain

\[
D(X \otimes X^k)D^\dagger = \sum_{s=0}^{d_c-1} P_s X P_s \otimes X^{r-k-s} = X \otimes X^{k+1}. \quad \text{(67)}
\]

This proves both (61) and (62) simultaneously. By noting that $Z^2 = \sum_{s=0}^{d_t-1} \zeta_{d_t}^{s} P_s$, we get

\[
D(Z \otimes I)D^\dagger = \sum_{r,s,t=0}^{d_c-1} \zeta_{d_c}^t P_r P_t P_s \otimes X^{r-t-s} = Z \otimes I. \quad \text{(68)}
\]

Finally, by using the commutation relation (6) and $\zeta_{d_t} = (\zeta_{d_c})^{d_c/d_t}$, we obtain

\[
D(I \otimes Z)D^\dagger = \sum_{s=0}^{d_c-1} P_s \otimes X^s \otimes X^{-s} = \sum_{s=0}^{d_t-1} P_s \otimes X_1^{-s} \otimes Z \quad \text{(69)}
\]

\[
= \sum_{s=0}^{d_c-1} \zeta_{d_c}^{-s} P_s \otimes Z
\]

\[
= Z^{-K} \otimes Z, \quad \text{for } \frac{d_c}{d_t} = K. \quad \text{(70)}
\]

\[\square\]

Note that even if $d_c/d_t = K \geq 2$ is an integer, then $D_{12}$ but not $D_{21}$ will belong to the Clifford algebra of the hybrid Pauli group.

VI. SUMMARY

We considered quantum hybrid gates which act on tensor products of qudits of different dimensions. In particular, we constructed two-body hybrid SUM and partial-SWAP gates, and also many-body hybrid Toffoli and Fredkin gates. We have calculated the entanglement generated by the SUM gate. We describe a physical realization of these hybrid gates for spin systems. We also proved two lemmas, one related to entanglement generation with and without ancillas, and the other involving conjugation by the SUM gate.

Acknowledgments

Jamil Dabouei thanks Macquarie University for its hospitality. We appreciate valuable discussions with Stephen Bartlett and Dominic Berry. This project has been supported by an Australian Research Council Large Grant and by a Macquarie University Research Grant.

[17] Susskind L and Glogower J 1964 Physics 1 49
[19] Bartlett S D 2002 private communication