Ulmann’s geometric phase in presence of isotropic decoherence

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1. INTRODUCTION

Ulmann’s mixed state geometric phase [Ref. Phys. Lett. A 17, 22 (1987)] is analyzed in the case of a qubit affected by isotropic decoherence treated in the Markovian approximation. It is demonstrated that this phase decreases rapidly with increasing decoherence rate and that it is most fragile to weak decoherence for pure initial states. In the unitary and Markovian approximations, Ulmann’s geometric phase [Ref. Phys. Lett. A 98, 263 (1983)] for mixed states has been treated in Refs. [1,2] and [3], respectively. The present work could in principle be realized experimentally. But, until now, no exact expressions for the geometric phase of a qubit have been obtained. The present work could in principle be realized experimentally. But, until now, no exact expressions for the geometric phase of a qubit have been obtained. The present work could in principle be realized experimentally.
by attaching an ancilla $a$. This extension has the property that every unit vector (purification) $|\psi\rangle$ in the extended space may be reduced to a density operator in the space of operators acting on $\mathcal{H}$ as
\[ \rho = \text{Tr}|\psi\rangle\langle\psi|, \]
where $\text{Tr}$ is partial trace over the ancilla. The lift of $\rho$ is defined in terms of Hilbert-Schmidt operators $W : \mathcal{H} \rightarrow \mathcal{H}$ such that $\rho = WW^\dagger$. Such a lift is not unique as $W \mapsto W = WU$ generates the same $\rho$ for any choice of unitary $U$, i.e., we have a gauge freedom in the choice of $U$. Thus, we have an infinite number of ways to perform a purification of $\rho$. Our concern is now to pick out a class of exceptional purifications that defines a natural notion of geometric phase.

First, let us introduce the inner product between any pair $W_1, W_2$ of Hilbert-Schmidt operators as
\[ \langle W_1, W_2 \rangle = \text{Tr}[W_1^\dagger W_2]. \]
In terms of this we may define the quantity
\[ \nu = \langle W_1, \tilde{W}_m \rangle \langle W_m, \tilde{W}_{m-1} \rangle \cdots \langle \tilde{W}_2, W_1 \rangle \equiv \xi \langle \tilde{W}_1, \tilde{W}_m \rangle \]
(corresponding to the ordered set $H = \rho_1, \ldots, \rho_m$ of density operators. Now, exceptional choices are the ones that maximize $\xi$, or, equivalently, that maximize each $\langle \tilde{W}_{j+1}, \tilde{W}_j \rangle$).

By making use of $W = \sqrt{\rho} V$, $V$ being unitary, and the polar decomposition $\sqrt{\rho_{j+1}} \sqrt{\rho_j} = \sqrt{\rho_{j+1} V_j V_j^\dagger}$ we obtain a necessary and sufficient condition for $\xi$ maximal when
\[ \langle \tilde{W}_{j+1}, \tilde{W}_j \rangle = \text{Tr}[\sqrt{\rho_{j+1} \rho_j \sqrt{\rho_{j+1}}}]. \]
It follows that $\nu = \xi \langle \tilde{W}_1, \tilde{W}_m \rangle$ is invariant under the remaining gauge freedom $W_j \rightarrow e^{i\theta_j} U W_j$, $|\theta_j| = 1$ and $U$ a fixed unitarity. Hence, with $W$ being exceptional, $\nu$ depends only upon the ordered set $\Pi$ and we define the corresponding Uhlmann phase as $\phi_2[\Pi] \equiv \arg \nu$. This phase is real-valued, gauge invariant, and, in the $m \rightarrow \infty$ limit, independent of the subdivision of the path. These properties make $\phi_2[\Pi]$ a natural definition of the mixed state geometric phase associated with $\Pi$. Furthermore, by introducing $\phi_{j+1,j} = \arg\langle W_{j+1}, W_j \rangle$ we may write
\[ \phi_2[\Pi] = \phi_{m,m-1} + \cdots + \phi_{3,2} + \phi_{2,1} + \arg\langle \tilde{W}_1, \tilde{W}_m \rangle. \]
This may be simplified by choosing a particular class of gauge that corresponds to parallel lift defined by requiring
\[ W_{j+1}^\dagger \tilde{W}_j > 0, \quad \forall j, \]
which implies $\phi_{j+1,j} = 0$ for each intermediate step between 1 and $m$ in Eq. (6). Note that the main condition, i.e. that $|\xi|$ is maximal, is still true when Eq. (7) holds. For this important class of gauge Uhlmann’s mixed state geometric phase reads
\[ \phi_2[\Pi] = \arg\langle \tilde{W}_1, W_m \rangle. \]
Let us now consider the parallel transport condition in the particular case of a qubit (two-level system). Any qubit state can be written as $\rho = (1/2) (1 + r \cdot \sigma)$, where $r$ is the Bloch vector, the length of which being less than unity for mixed states, and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the standard Pauli matrices. The parallelity lift condition Eq. (7) for any pair $\rho_j$ and $\rho_{j+1}$ of such mixed qubit states [23] is equivalent to
\[ V_{j+1} V_j^{-1} = \sqrt{\rho_{j+1}} \sqrt{\rho_j} \sqrt{\rho_{j+1} V_j V_j^\dagger}. \]
By writing $\sqrt{\rho_j} = a_j + b \cdot \sigma$ and $\sqrt{\rho_{j+1}} = \tilde{b}_j + \tilde{b} \cdot \sigma$, and using features of 2 $\times$ 2 matrices we obtain [13]
\[ V_{j+1} V_j^{-1} = \frac{b_j a_j + b \cdot a + i(b \cdot a) \cdot \sigma}{\sqrt{(b_j a_j + b \cdot a - i(b \cdot a) \cdot \sigma)^2 + |b \cdot a|^2}}. \]
For a continuous path $\Pi : t \in [0, \pi] \rightarrow r(t)$ of qubit states we need to consider the infinitesimal version of Eq. (10). Let $V_j = V$, $V_{j+1} = V + dV$, $\sqrt{\rho_{j+1}} = \sqrt{\rho_j} + d \sqrt{\rho_j}$ and use that $a_j^2 + |a|^2 = \frac{1}{2}$ we obtain to first order in $d\alpha$ the differential equation [13]
\[ d\alpha V_j = 2i \langle \alpha \cdot a \cdot \sigma \rangle V_j. \]
Formally we may write the solution of Eq. (11) as
\[ V = \mathcal{P} \exp \left( 2i \int (\alpha \cdot a \cdot \sigma) V_0 \right), \]
where $\mathcal{P}$ stands for path ordering and $V_0$ is the initial unitarity. Evaluating this path ordered expression and inserting into Eq. (8) yields Uhlmann’s mixed state geometric phase for any qubit state.

In the qubit case, it has been argued [24, 25] that the interior of the Bloch sphere is curved. This result is essentially captured by the mixed state line element [25]
\[ ds^2 = \frac{d\alpha^2}{1 - r^2} + r^2 d\alpha \cdot d\alpha \]
with $r = \alpha$, which shows that as one moves away from the center at $\alpha = 0$, the circumference of the 2-sphere defined by each fixed $\alpha > 0$ grows more slowly than the distance from the center, due to the factor $1/(1 - r^2)$ in front of $d\alpha^2$. Uhlmann’s geometric phase for a qubit is the holonomy that naturally measures this curvature inside the Bloch sphere, just as the standard pure state geometric phase is the holonomy that measures the curvature of the Bloch sphere. In view of this and due to the non-Euclidean behavior of the radius and circumference when moving away from the origin inside the Bloch sphere, there is no reason to expect that the Uhlmann phase should have any direct relation to the solid angle enclosed by the Bloch vector in three dimensional Euclidean space. The result of the calculation in Sec. IV below for a qubit in the depolarization channel may be regarded as an illustration of this intuitive reasoning.
III. DEPOLARIZATION CHANNEL

In the depolarization channel, the environment induces isotropic errors in the qubit state. This may be represented by taking the three errors

(i) \( |\psi\rangle \mapsto \sigma_x |\psi\rangle \), bit flip,

(ii) \( |\psi\rangle \mapsto \sigma_y |\psi\rangle \), phase flip,

(iii) \( |\psi\rangle \mapsto \sigma_z |\psi\rangle \), both bit and phase flip,

to be equally likely, each occurring with probability \( p/3 \). A unitary representation of the channel, using a minimal set of ancilla states \( \{|0\rangle, \ldots, |3\rangle\} \), is given by

\[
U : \rho \otimes |0\rangle \langle 0| \mapsto \left[ \sqrt{1 - \frac{p}{3}} |0\rangle \langle 0| + \frac{p}{3} \left( 1_{2} \otimes |\sigma_{x} + |2_{2} \otimes |\sigma_{y} + |3_{2} \otimes |\sigma_{z} \right) \right]
\]

\[
\times R \rho R^\dagger \left[ \sqrt{1 - \frac{p}{3}} |0\rangle \langle 0| + \frac{p}{3} \left( |\sigma_{x} \otimes 1_{2} + |\sigma_{y} \otimes 2_{2} + |\sigma_{z} \otimes 3_{2} \right) \right].
\]

(14)

where \( R \) is some unitary operation acting only on the qubit. This may be lifted into the pure state evolution by adding another ancilla system \( b \) such that the initial state reads

\[
\rho \otimes |0\rangle \langle 0| = \left( \frac{1 + r}{2} |\psi \rangle \langle \psi | + \frac{1 - r}{2} |\psi \rangle \langle \psi | \right) \otimes |0\rangle \langle 0|
\]

\[
\mapsto |\Psi \rangle = \left( \sqrt{\frac{1 + r}{2}} |\psi \rangle \otimes |0\rangle + \sqrt{\frac{1 - r}{2}} |\psi \rangle \otimes |1\rangle \right) \otimes |0\rangle
\]

(15)

and extending \( U \) to \( U \otimes I_b \).

Similarly, under certain restrictions one may model the map of the system’s density operator as a nonunitary Markovian evolution described by the Lindblad equation [26] (\( \Gamma = 1 \) from now on)

\[
\dot{\rho} = -i[H, \rho] + \sum_{\mu} \left( h_{\mu} \rho L_{\mu}^{\dagger} L_{\mu} - \frac{1}{2} L_{\mu}^{\dagger} L_{\mu} \rho - \frac{1}{2} \rho L_{\mu}^{\dagger} L_{\mu} \right).
\]

(16)

Each term \( L_{\mu}^{\dagger} L_{\mu} \) represents one of the possible errors (quantum jumps), while the sum over \( -\frac{1}{2} h_{\mu}^{\dagger} \rho h_{\mu} - \frac{1}{2} h_{\mu} h_{\mu}^{\dagger} \rho \) is needed to satisfy the normalization condition. In this framework, the depolarization channel may be represented by the Lindblad operators

\[
L_{\mu} = \sqrt{\frac{\Gamma}{3}} \sigma_{\mu}, \quad \mu = 1, 2, 3,
\]

(17)

where \( \Gamma \) is the time-independent decoherence rate. Inserting these \( L_{\mu} \)'s into Eq. (10), we obtain the Lindblad equation for the depolarization channel as

\[
\dot{\rho} = i[H, \rho] - \frac{2}{3} \Gamma \rho \cdot \sigma.
\]

(18)

Assuming the Hamiltonian \( H = \frac{1}{2} \omega \sigma_z \), corresponding to the unitarity \( R = \exp(-\frac{i}{\lambda} \omega \lambda \sigma_z) \), and the initial condition \( \rho(0) = \rho_0(\sin \theta, 0, \cos \theta) \), the solution of Eq. (18) reads

\[
\rho(t) = \rho_0 \exp\left(-\frac{\Gamma}{2} \left(1 - r(t)/r_0\right)\right) \exp\left(-\frac{i}{\lambda} \omega \lambda \sigma_z \right).
\]

(19)

Thus, the Bloch vector precesses uniformly around the \( z \) axis and its length decreases isotropically. The effects of the unitary and Markovian treatments of the depolarizing channel can be formally related at time \( t \) as \( p(t) = \frac{3}{2} \left(1 - r(t)/r_0\right) = \frac{3}{2} \left(1 - e^{-\lambda \lambda \sigma_z /3}\right) \).

IV. UHLHNN’S GEOMETRIC PHASE IN THE DEPOLARIZATION CHANNEL

A. Theoretical analysis

To evaluate Eq. (12) analytically in the depolarization channel, we may choose a path where \( \langle da \times a \rangle \cdot \sigma \) is time-dependent in such a way that it commutes at different times within a set of time intervals. Such a path is

\[
A \rightarrow B : \quad r(t) = r_0 \exp\left(-\frac{\Gamma}{2} \left(1 - r(t)/r_0\right)\right) \exp\left(-\frac{i}{\lambda} \omega \lambda \sigma_z \right),
\]

\[
0 \leq t \leq \pi/(2\omega),
\]

\[
B \rightarrow C : \quad r(t) = r_0 \exp\left(-\frac{\Gamma}{2} \left(1 - r(t)/r_0\right)\right) \exp\left(-\frac{i}{\lambda} \omega \lambda \sigma_z \right),
\]

\[
\pi/(2\omega) \leq t \leq (\varphi + \pi/2)/\omega,
\]

\[
C \rightarrow D : \quad r(t) = r_0 \exp\left(-\frac{\Gamma}{2} \left(1 - r(t)/r_0\right)\right) \exp\left(-\frac{i}{\lambda} \omega \lambda \sigma_z \right),
\]

\[
(\varphi + \pi/2)/\omega \leq t \leq (\varphi + \pi)/\omega
\]

(20)

which is shown in Fig. 1 in the particular case where \( \varphi = \pi/2 \).

The idealized pure state evolution is obtained when \( \Gamma = 0 \) and \( r_0 = 1 \); it is represented by the solid line in Fig. 1 that defines the geometric phase \( -\pi/4 \). In the general case, the third rotation is taken around the direction \( m = (\sin \varphi, -\cos \varphi, 0) \) and the pure state geometric phase becomes \( -\pi/2 \).

To compute \( V \) and thereby Uhlmann’s mixed state geometric phase for any fixed \( \Gamma \), we first note that with \( a(t) = |a(t)| \) it follows that \( da \times a = a^2(t) \text{d}n \times n \) and for the above path we have

\[
A \rightarrow B : \quad \text{d}n \times n = -\omega dt \text{y},
\]

\[
B \rightarrow C : \quad \text{d}n \times n = -\omega dt z,
\]

\[
C \rightarrow D : \quad \text{d}n \times n = -\omega dt m,
\]

(21)
where \( \mathbf{y}, \mathbf{z}, \) and \( \mathbf{m} \) are unit vectors along the three rotation axes. Substituting this into Eq. (12) yields

\[
V_D = \exp \left( -2i \omega \int_{0}^{t_f} \frac{a^2(t)}{\rho^2} \, dt \right) \mathbf{m} \cdot \mathbf{\sigma} \\
\times \exp \left( -2i \omega \int_{0}^{t_f} \frac{a^2(t)}{\rho^2} \, dt \sigma_z \right) \\
\times \exp \left( -2i \omega \int_{0}^{t_f} \frac{a^2(t)}{\rho^2} \, dt \sigma_y \right) V_A \\
\equiv e^{-i\mathbf{m} \cdot \mathbf{\sigma} t \sigma_z} e^{-i\mathbf{m} \cdot \mathbf{\sigma} \omega t} V_A, \tag{22}
\]

where we have used that \( V_0 = V_A \). Now, from \( \sqrt{\rho^2} = \rho \) and \( \sqrt{\rho} = a_0 + \mathbf{m} \cdot \mathbf{\sigma} \), we obtain \( a^2(t) = \left( 1 - \sqrt{1 - r^2(t)} \right)/4 \) so that the angles \( \mu, \kappa, \) and \( \chi \) have the generic form

\[
2\omega \int_{t_k}^{t_f + \theta \omega} a^2(t) \, dt = \frac{\Delta \varphi}{2} \\
-\frac{\omega}{2} \int_{t_k}^{t_f + \theta \omega} \sqrt{1 - r_0^2 e^{-4\Delta \varphi}} \, dt \tag{23}
\]

with \( t_k = 0, \pi/(2\omega), [\varphi + \pi]/\omega \) and \( \Delta \varphi = \pi/2, \varphi, \pi/2 \), respectively. The integral can be evaluated by using

\[
\frac{8 \Gamma}{3} \int \sqrt{1 - r_0^2 e^{-4\Delta \varphi}} \, dt = -2 \sqrt{1 - r_0^2 e^{-4\Delta \varphi}} \\
+ \ln \left( \frac{1 + \sqrt{1 - r_0^2 e^{-4\Delta \varphi}}}{1 - \sqrt{1 - r_0^2 e^{-4\Delta \varphi}}} \right) = 2 \left[ 4a^2(t) - 1 + \ln \frac{a(t)}{a(0)} \right] \tag{24}
\]

with the concomitant integration limits for \( \mu, \kappa, \) and \( \chi \). Let us further introduce

\[
\sqrt{\rho A} = a_0(0) + a(0) \sigma_z \equiv \alpha + \beta \sigma_z, \\
\sqrt{\rho D} = a(t) \left( \frac{\varphi + \pi}{\omega} \right) + a \left( \frac{\varphi + \pi}{\omega} \right) \sigma_z \\
\equiv \frac{\varphi}{\omega} + \eta \sigma_z. \tag{25}
\]

In terms of Eqs. (22) and (25) the geometric phase for the path \( A \rightarrow B \rightarrow C \rightarrow D \) becomes

\[
\phi_g = \arg \text{Tr} \left[ \sqrt{\rho A} \sqrt{\rho D} V_A V_A^\dagger \right] = -\arctan \left[ \frac{\alpha \eta + \beta \eta}{\alpha \nu + \beta \eta} \right] \\
\times \left( \frac{\sin \kappa + \sin[\varphi - \kappa] \tan \chi \tan \mu}{\cos \kappa + \cos[\varphi - \kappa] \tan \chi \tan \mu} \right). \tag{26}
\]

To further analyze this general analytic result, let us consider the following important special cases:

1. Pure and unitary case: Here, \( r(t) = 1, \forall t > 0 \), which implies that \( \alpha = \beta = \nu = \eta = \frac{\varphi}{\omega}, \mu = \chi = \pi/4, \) and \( \kappa = \varphi/2 \) yielding \( \phi_g = -\varphi/2 \). This is the expected pure state geometric phase for a qubit enclosing the solid angle \( \varphi \) on the Bloch sphere.

2. Mixed and unitary case: Here, \( \Gamma = 0 \) so that \( r(t) = r_0 \neq 1, \forall t > 0 \), which implies that \( \rho D = \rho A \), \( \mu = \chi = \pi(1 - \sqrt{1 - r_0^2})/4 \equiv \zeta, \) and \( \kappa = \varphi(1 - \sqrt{1 - r_0^2})/2 \). Inserting this into Eq. (26) we obtain

\[
\phi_g = -\arctan \left[ \frac{r_0 \sin[\varphi - \kappa] \tan^2 \zeta}{\cos \kappa + \cos[\varphi - \kappa] \tan^2 \zeta} \right]. \tag{27}
\]

This case has also been analyzed in the context of Mach-Zehnder interferometry in [15], where it was shown that the interference pattern for \( r_0 > 0 \) [27] is shifted by the mixed state geometric phase

\[
\gamma_g = -\arctan \left[ r_0 \tan \frac{\varphi}{2} \right]. \tag{28}
\]

where the solid angle enclosed by the Bloch vector is \( \varphi \). The mixed state geometric phases in Eqs. (27) and (28) match only if \( \tan \zeta = 1 \) or \( \kappa = \varphi/2 = n\pi, \) an integer. The former case corresponds to pure states and holds for any \( \varphi \). As \( 0 < \kappa < \varphi/2 \) for \( r_0 > 0 \), the latter case has solution only for \( n = 0 \) so that \( \kappa = \varphi/2 \), which again corresponds to pure states. Thus, for the present evolution, \( \phi_g \) and \( \gamma_g \) only match for pure states. In fact, one may show that

\[
\frac{\sin \kappa + \sin[\varphi - \kappa] \tan^2 \chi}{\cos \kappa + \cos[\varphi - \kappa] \tan^2 \chi} \leq \frac{\sin \varphi}{\cos \varphi} \leq \frac{\sin \varphi}{\cos \varphi} \tag{29}
\]

\[
\Rightarrow |\phi_g| \leq |\gamma_g|
\]
with equality in the limit of pure states. This suggests that the mixed state geometric phase proposed in [15] is more robust to reduction of the length of the Bloch vector than that proposed by Uhlmann.

3. Pure initial state: This case is characterized by \( \alpha = \beta = \frac{1}{2} \), which yields

\[
\phi_0 = -\arctan \left( \frac{\sin \chi \sin[\varphi - \chi] \tan \chi \tan \mu}{\cos \chi + \cos[\varphi - \chi] \tan \chi \tan \mu} \right).
\] (30)

Thus, from Eq. (29) it follows that \( |\phi_0| \leq \frac{\varphi}{2} \) for pure initial states. This reduction of the geometric phase value as a function of the decoherence efficiency parameter \( \Gamma/\omega \) is illustrated in Fig. 2 in the case where \( \varphi = \pi/2 \). We see that the geometric phase decreases rapidly with increasing decoherence rate. In particular, we note that when the decoherence rate is as large as the precession time-scale (i.e. \( \Gamma/\omega \approx 1 \)), the geometric phase has practically vanished.

\[\text{Fig. 2: Uhlmann's mixed state geometric phase} \] (vertical axis) \( \text{as a function of the decoherence efficiency parameter} \) \( \Gamma/\omega \) (horizontal axis), \( \Gamma \) and \( \omega \) being the decoherence rate and gate frequency, respectively. The precession angle around the \( z \) axis is restricted to \( \pi/2 \).

4. Small decoherence rate \( \Gamma/\omega \ll 1 \): First, in the case where \( r_0 = 1 \) we may expand the right-hand side of Eq. (24) to lowest order in the small quantity \( \sqrt{1 - e^{-\Gamma \tau}} \) yielding

\[
2\omega \int_{t_0}^{t_0 + \Delta \varphi/\omega} a^2(t) \, dt = \frac{\Delta \varphi}{2} + \mathcal{O}\left( \sqrt{\frac{\Gamma}{\omega}} \right).
\] (31)

On the other hand, when \( 0 < r_0 \neq 1 \) and \( r_0^2 (8 \Gamma / 3) \tau_{\text{max}} \ll 1 - r_0^2 \), one may take \( \sqrt{1 - r_0^2 e^{-\Gamma \tau}} \approx \sqrt{1 - r_0^2} + \mathcal{O}(\Gamma \tau) \) so that

\[
2\omega \int_{t_0}^{t_0 + \Delta \varphi/\omega} a^2(t) \, dt = \frac{\Delta \varphi}{2} - \frac{\Delta \varphi^2}{4} + \mathcal{O}(\Gamma \tau).
\] (32)

Comparison of Eqs. (31) and (32) shows that Uhlmann's mixed state geometric phase in the case of weak isotropic decoherence is most fragile for pure or nearly pure initial states.

B. Proposed experiment

To test the above predictions experimentally requires control over the dynamics of the ancilla [22]. This is very difficult in the Markovian case, where the ancilla typically has a complicated structure with many degrees of freedom so as to quickly forget the information acquired from the qubit. However, one could imagine a few-qubit implementation adapted to the unitary representation of the depolarization channel, in case of which Uhlmann's geometric phase could be realized interferometrically.

The general idea behind such a realization is based upon purification lift of the path \( t \mapsto \rho(t) \) by adding ancilla systems in such a way that partial trace over these ancillas is \( \rho(t) \). Such a measurement scheme may be constructed as follows:

(i) Represent the ancilla states \( |\rho_A, \mu_3 = 0, \ldots, 3 \rangle \) with a pair of qubits so that \( |0_a \rangle \otimes |0_b \rangle \otimes |0_{a2} \rangle \otimes |1_{a2} \rangle = |0_a \rangle \otimes |1_a \rangle \), \( |2_a \rangle = |1_a \rangle \otimes |0_a \rangle \), and \( |5_a \rangle = |1_a \rangle \otimes |1_a \rangle \) and write the initial state Eq. (15) as

\[
|\Psi_A \rangle = \sqrt{\frac{1 + r_A}{2}} |0 \rangle \otimes |0_b \rangle + \sqrt{\frac{1 - r_A}{2}} |1 \rangle \otimes |1_b \rangle \otimes |0_a \rangle \otimes |0_a \rangle.
\] (33)

where \( r_A = r(0) \), and \( |\psi \rangle = |0 \rangle \) according to Fig. 1.

(ii) Take \( \Psi_A \) through the depolarization channel around the path of Fig. 1 yielding the final state in Schmidt form

\[
|\Psi_B \rangle = \sqrt{\frac{1 + r_D}{2}} |0 \rangle \otimes |D \rangle + \sqrt{\frac{1 - r_D}{2}} |1 \rangle \otimes |D^+ \rangle.
\] (34)

with \( r_D = r(|\varphi + \pi|/\omega) \) and \( \langle D | D^+ \rangle = 0 \), where \( |D \rangle, |D^+ \rangle \in \mathcal{H}_b \otimes \mathcal{H}_a \) are normalized and read explicitly
\[ |D\rangle = \sqrt{\frac{1 + r_A}{1 + r_D}} \left( \sqrt{1 - p_D} |0_a\rangle \otimes |0_a\rangle \otimes |0_a\rangle + \frac{p_D}{3} (|0_a\rangle \otimes |1_a\rangle \otimes |1_a\rangle \right) \\
+ \sqrt{\frac{1 - r_A}{1 + r_D}} \left( \sqrt{1 - p_D} |1_a\rangle \otimes |0_a\rangle \otimes |1_a\rangle - i \frac{p_D}{3} (|0_a\rangle \otimes |1_a\rangle \otimes |0_a\rangle \right) \\
|D^\perp\rangle = \sqrt{\frac{1 + r_A}{1 - r_D}} \left( \sqrt{1 - p_D} |0_a\rangle \otimes |0_a\rangle \otimes |0_a\rangle + i \frac{p_D}{3} (|0_a\rangle \otimes |1_a\rangle \otimes |0_a\rangle \right) \\
+ \sqrt{\frac{1 - r_A}{1 - r_D}} \left( \sqrt{1 - p_D} |1_a\rangle \otimes |0_a\rangle \otimes |0_a\rangle - \frac{p_D}{3} (|0_a\rangle \otimes |1_a\rangle \otimes |0_a\rangle \right), \] (35)

where \( p_D = \frac{3}{2}(1 - r_D/r_A). \)

(iii) Perform a unitary transformation under which [28]

\[ |D\rangle \rightarrow |0_a\rangle \otimes |0_a\rangle \otimes |0_a\rangle \]
\[ |D^\perp\rangle \rightarrow |1_a\rangle \otimes |0_a\rangle \otimes |0_a\rangle \] (36)

so as to obtain the state

\[ |\tilde{\Psi}_D\rangle = \left( \sqrt{\frac{1 + r_D}{2}} |0\rangle \otimes |0_b\rangle + \sqrt{\frac{1 - r_D}{2}} |1\rangle \otimes |1_b\rangle \right) \otimes |0_a\rangle \otimes |0_a\rangle. \] (37)

(iv) Expose \( |\tilde{\Psi}_D\rangle \) to \( V_D V_A^\dagger \otimes I_b \otimes I_a \) and let the resulting state interfere with \( e^{i\delta} |\Psi_A\rangle \), \( \delta \) being a variable \( U(1) \) shift. The intensity \( I \) reads

\[ I \propto \left| e^{i\delta} |\Psi_A\rangle + V_D V_A^\dagger \otimes I_b \otimes I_a |\tilde{\Psi}_D\rangle \right|^2 \\
= 2 + 2 \Im \left( \langle \Psi_A | V_D V_A^\dagger \otimes I_b \otimes I_a |\tilde{\Psi}_D\rangle e^{-i\delta} \right), \] (38)

where

\[ \langle \Psi_A | V_D V_A^\dagger \otimes I_b \otimes I_a |\tilde{\Psi}_D\rangle \]

Thus Uhlmann’s geometric phase for the path in Fig. 1 is realized as the Pancharatnam relative phase [29] that shifts the interference oscillations obtained by applying a variable \( U(1) \) shift to one of the interfering states. Such an experiment could in principle be implemented using, e.g., ion traps or NMR techniques, by letting three qubits act as the ancilla systems \( a \) and \( b \) for a fourth qubit in the reduced mixed state \( p \).

V. CONCLUSIONS

We have computed Uhlmann’s mixed state geometric phase for a qubit affected by the depolarization channel and have described how this phase could in principle be realized experimentally. A rapid decrease of this phase with increasing decoherence rate has been demonstrated. For weak decoherence we have found that Uhlmann’s geometric phase is most fragile for pure or nearly pure states. In the unitary case, we have also demonstrated that the mixed state geometric phase proposed by Uhlmann seems to be more sensitive to reduction of the length of the Bloch vector than that proposed in Ref. [15]. In this context, it would be interesting to compare the results of the present analysis with the extension of the mixed state geometric phase in [15] to the decoherence case [30]. We hope that this work may lead to further studies of the Uhlmann phase for various quantum channels as well as to experiments of mixed state geometric phases in the presence of decoherence.

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[23] As the eigenvalues of $\rho$ and $\rho_{\pm 1}$ for $|\rho| < 1$ are nonvanishing, the existence of their inverse is ensured. The pure state limit $(|\rho| \to 1)$ of this approach is well-defined due to continuity.
[27] $r_0$ has to be nonvanishing as the mixed state geometric phase $\gamma_0$ in [15] is indeterminate for degenerate density operators.
[28] As is apparent from Eq. (35), this transformation depends upon $r_A$ and $r_B$, which therefore have to be known.