\( \mathcal{N} = 1 \) Gauge Theory with Flavor from Fluxes

Yutaka Ookouchi

Department of Physics, Tokyo Institute of Technology, Tokyo 152-8511, Japan

Abstract

Cachazo and Vafa provided a proof of the equivalence of \( \mathcal{N} = 1 \) dynamics obtained by deforming \( \mathcal{N} = 2 \) SYM by addition of certain superpotential terms, with that of type IIB superstring on Calabi-Yau manifold with fluxes. We extend the proof to SYM with massive flavors in fundamental representation of gauge group. For the gauge group we consider \( U(N) \) and \( SO(N)/Sp(N) \) groups. When the adding tree level superpotential is square of adjoint chiral superfield we can derive Affleck-Dine-Seiberg potentials. By turning off the flux, we obtain Seiberg-Witten curves of \( \mathcal{N} = 2 \) theories.

*e-mail address : ookouchi@th.phys.titech.ac.jp
1 Introduction

Recently Dijkgraaf-Vafa proposed that a holomorphic information in $\mathcal{N} = 1$ gauge theories with classical gauge groups derive from matrix model [1]. The relation the Calabi-Yau manifold with flux and matrix model was discussed in [2, 3, 4]. The generalization of D-V duality to the gauge theories with massive flavor was discussed in [5, 6, 7, 8, 9, 11, 10, 12, 13]. Dijkgraaf and Vafa have reached this duality via string theory route using the model with one adjoint matter. So we want to consider the generalization of string duality for the model with massive flavor.

It was discussed that in [15, 16] large $N$ dual of $U(N)$ $\mathcal{N} = 2$ gauge theory deformed by certain tree level superpotential is realized as type IIB string theory on Calabi-Yau threefold with fluxes. The discussion for type IIB superstring was extended in [17, 18, 19, 20, 21]. In the resolved geometry, classical superpotential for adjoint chiral superpotential arises in the gauge theory on $N$ $D5$ branes wrapping in the $S^2$’s and it leads to $\mathcal{N} = 1$ supersymmetric theory. After the conifold transition, $S^2$ shrink and $S^3$ appear in the dual geometry. On this dual geometry, there are 3-form fluxes through $S^3$’s that comes from the $D$ brane charge. As discussed in [22, 23] this flux generates the effective superpotential for glueball superfields. The low energy information is given by extremization of the effective superpotential written as glueball superfields. On the other hand from purely viewpoint of field theory it is given by specializing to the appropriate factorization locus of the Seiberg-Witten curve. In [24] the equivalence of two descriptions were proved for the model with one adjoint chiral superfield.

In this note we generalize Cachazo and Vafa’s discussion to the gauge theory with $N_f$ massive flavors. The point of this generalization is the Riemann surface that has the same genus as no flavor case but different flux. Turning on the various fluxes on the same Riemann surface we can realize various gauge groups and massive flavors. On the other hand turning off the flux in terms of a certain limit, we can obtain $\mathcal{N} = 2$ information from the Calabi-Yau geometry with fluxes.

The organization of this paper is as follows: In section 2, we discuss the geometric engineering for the gauge theory with massive flavors in fundamental representation of gauge group. Then we see the geometric transition for this model and the effect of flux in the dual geometry. In section 3, at first we discuss the effective superpotential from purely field theory viewpoint. Next we discuss that from fluxes. Comparing these two analyses we see the equivalence of two results. In section 4 we explicitly give one-form flux on the Riemann surface. In section 5 we derive the Affleck-Dine-Seiberg potential from Calabi-Yau manifold with flux. In section 6 turning off the flux we reproduce Seiberg-Witten
curves for the $\mathcal{N} = 2$ gauge theories.

Note added: After completion of this note, we have received [14], where the Seiberg-Witten curve derived from matrix model context and pointed out the relation to Calabi-Yau manifold with flux.

## 2 Geometric Transition and Dual Description

### 2.1 Geometric engineering

As discussed in [16], the $U(N)$ gauge theory with massive flavor chiral multiplets in the fundamental representation is obtained in terms of the geometric engineering. At first, we review the simplest case in which the tree level superpotential for the adjoint chiral superfield $\Phi$ is $W_{\text{tree}} = m\Phi^2$. In the type IIB string on the $\mathcal{O}(-1) + \mathcal{O}(-1)$ bundle over $\mathbb{P}^1$, the $U(N)$ gauge theory with $W_{\text{tree}} = m\Phi^2$ is realized by $N$ $D5$ branes wrapped on $\mathbb{P}^1$. For the massive flavor we introduce another $D5$ brane wrapping a holomorphic 2-cycle not intersecting the $\mathbb{P}^1$. The massive flavor comes from strings stretching between this $D5$ brane and the $N$ $D5$ branes wrapped on $\mathbb{P}^1$.

Next we want to discuss the generalization to the gauge theory with the tree level superpotential,

$$W_{\text{tree}} = \sum_{p=1}^{n+1} g_p \text{Tr} \Phi^p.$$  \hfill (2.1)

This generalization was discussed for the case with no flavor in [16]. In this case the geometry becomes the $\mathcal{O}(-2) + \mathcal{O}(0)$ bundle over $\mathbb{P}^1$. Let $z$ denote the coordinate in the north patch of $\mathbb{P}^1$ and $z' = 1/z$ in the south patch. Let $x, x'$ denote the coordinate of $\mathcal{O}(0)$ direction in the north and south patches respectively, and let $u, u'$ denote the coordinates of $\mathcal{O}(-2)$ in the north and south patches respectively. Using these coordinates the geometry is given by

$$z' = \frac{1}{z}, \quad x' = x, \quad u' = uz^2 + W'_{\text{tree}}(x)z.$$  \hfill (2.2)

Note that there are $\mathbb{P}^1$'s at the point $W'_{\text{tree}}(x) \equiv g_{n+1} \prod_{i=1}^{n}(x - a_i) = 0$. $N$ $D5$ branes are distributed these $\mathbb{P}^1$. If $N_i$ branes wrapped on $\mathbb{P}^1$, the gauge symmetry is broken as

$$U(N) \rightarrow \prod_{i=1}^{n} U(N_i) \quad \text{with} \quad \sum_{i=1}^{n} N_i = N.$$  \hfill (2.3)
In this case also for adding the massive flavor we introduce another D5 brane wrapping holomorphic 2-cycles at the points proportional to the mass scale of the massive flavors. Then we get $\mathcal{N} = 1, 4d, U(N)$ gauge theory with $N_f$ massive flavor.

Next we want to generalize the gauge group. As discussed in [19], in order to realize $SO(N)/Sp(N)$ gauge theories, we introduce orientifold projection. Under the orientifolding the coordinates introduced above are transform as

\[(x, u, z) \rightarrow (\bar{u}, \bar{x}, -1/\bar{z}). \quad (2.4)\]

Since this is antipodal map, $\mathbb{P}^1$ at $x = 0$ becomes $\mathbb{R}P^2$. In this geometry world-volume theory on D5-branes is $SO/Sp$ gauge theory with the following tree level superpotential [19],

\[W_{\text{tree}}(\Phi) = \sum_{p=1}^{n+1} \frac{g_{2p}}{2p} \text{Tr} \Phi^{2p} \equiv \sum_{p=1}^{n+1} g_{2p} u_{2p}, \quad (2.5)\]

where $\Phi$ is the chiral superfield in the adjoint representation of $SO(N)/Sp(N)$ gauge group and $u_{2p} \equiv \frac{1}{2p} \text{Tr} \Phi^{2p}$.

We define parameters $a_i$ by

\[W'_{\text{tree}}(x) = \sum_{p=1}^{n} g_{2p} x^{2p-1} = g_{2n+2} x \prod_{i=1}^{n} (x^2 + a_i^2). \quad (2.6)\]

In the classical vacua of this gauge theory, the eigenvalues of $\Phi$ become roots $0, \pm ia_i$'s of $W'(x) = 0$. When $N_0$ D5-branes wrap on $\mathbb{R}P^2$ and $N_i$ D5-branes wrap on the $\mathbb{P}^1$ located at $x = \pm ia_i$, the vacuum of the gauge theory becomes classically $P(x) \equiv \det(x - \Phi) = x^{N_0} \prod_{i=1}^{n} (x^2 + a_i^2)^{N_i}$ and the gauge group breaks as,

\[SO(N) \rightarrow SO(N_0) \times \prod_{i=1}^{n} U(N_i), \quad Sp(N) \rightarrow Sp(N_0) \times \prod_{i=1}^{n} U(N_i), \quad (2.7)\]

where $N = N_0 + \sum_{i=1}^{n} N_i$.

### 2.2 Geometric dual description

#### 2.2.1 No flavor

The geometric dual description of the gauge theory is found via geometric transition [15, 16, 19]. In this transition each of the $S^2_i$ on which $N_i$ D5 branes wrapped have shrunk and have been replaced by the $S^3$. The geometry after the transition is given by

\[W'_{\text{tree}}(x)^2 + f_{n-1}(x) + y^2 + z^2 + v^2 = 0. \quad (2.8)\]
where \( f_{n-1}(x) \) is the degree \( n-1 \) th polynomial defined by \( f_{n-1} \equiv \sum b_i x^i \). In this deformed geometry, the integral basis of the 3-cycles \( A_i, B_i \) satisfy the symplectic pairing. These 3-cycles are constructed as \( \mathbb{P}^1 \) fibration over the line segments between two critical points of \( W_{\text{tree}}'(x)^2 + f_{n-1}(x) x_i^-, x_i^+ \) and \( \infty \) in \( x \)-plane. Therefore we set the three cycle \( A_i \) to be the \( S^2 \) fibration over the line segment between \( a^- \) and \( a^+ \) and three cycle \( B_i \) is constructed as \( S^2 \) fibration over the line segment between \( a^+ \) and \( \Lambda_0 \). Here we introduced the cut-off \( \Lambda_0 \), as these cycles are non-compact.

The periods \( S_i \) and dual periods \( \Pi_i \) for this deformed geometry is given by the integral of holomorphic 3-form \( \Omega \),

\[
S_i = \int_{A_i} \Omega, \quad \Pi_i = \int_{B_i} \Omega = \partial \mathcal{F} / \partial S_i.
\]

The dual periods are expressed in terms of the prepotential \( \mathcal{F} \). Since these 3-cycles are constructed as \( S^2 \) fibration, these periods are written in terms of the integrals over \( x \)-plane as,

\[
S_i = \frac{1}{2 \pi i} \int_{a_i^0}^{a_i^+} \omega, \quad \Pi_i = \frac{1}{2 \pi i} \int_{a_i^0}^{a_i^+} \omega, \quad \omega = dx \left( W'(x)^2 + f_{n-1}(x) \right)^{\frac{1}{2}}.
\]

where \( \omega \) is obtained by integrating \( \Omega \) over the fiber \( \mathbb{P}^1 \). Following the [15, 16] this period \( S_i \) is identified with the glueball superfield of the \( SU(N_i) \) gauge group, \( S_i = \frac{1}{32 \pi^2} \text{Tr} W W^\alpha \).

When the geometric transition occurs, the \( S^2 \)'s on which D5-branes wrap in the resolved geometry, is replaced by RR 3-form fluxes through the special Lagrangian 3-cycles and NSNS 3-form through dual cycle in the deformed geometry. This 3-form fluxes generate the superpotential, and \( \mathcal{N} = 2 \) supersymmetry for the dual theory is broken partially to \( \mathcal{N} = 1 \) supersymmetry [22, 23],

\[
-\frac{1}{2 \pi i} W_{\text{eff}} = \int \Omega \wedge (H_R + \tau H_{NS}),
\]

where \( H_R \) and \( H_{NS} \) are 3-form fluxes and \( \tau \) is the complexified Type IIB string coupling. In the case of dual theory defined through geometric transition, \( H_R \) and \( H_{NS} \) satisfy,

\[
N_i = \int_{A_i} H_R, \quad \alpha = \int_{B_i} H_{NS},
\]

where \( \alpha \) is interpreted as 4d bare gauge coupling constant \( g_0, \alpha \equiv 4 \pi i / g_0^3 \).

Plugging these relations into (2.11), the superpotential for the dual theory is expressed in terms of periods \( S_i \) and dual periods \( \Pi_i \) of the deformed Calabi-Yau manifold such as,

\[
-\frac{1}{2 \pi i} W_{\text{eff}} = \sum_{i=1}^n \tilde{N}_i \Pi_i + \alpha \sum_{i=1}^n S_i.
\]

where as in [1] we used \( \tilde{N}_i \), which has \( N_i \) for \( U(N_i) \), \( N = 2 \) for \( SO/Sp \) respectively.
2.2.2 Adding flavor

Next we want to discuss the case with $N_f$ massive fundamental matters. In [16] effective superpotential for the case $W_{\text{tree}} = \Phi^2$ was given. We extend this effective superpotential for the case with arbitrary tree level superpotential. The effective superpotential is given by the integral of $\omega$,

$$W_{\text{flavor}}^{\text{eff}} = \frac{1}{2} \sum_{a=1}^{N_f} \int_{m_a}^{\Lambda_0} \omega \equiv 2\pi i \sum_{a=1}^{N_f} F_a.$$  \hspace{1cm} (2.14)

This comes from the RR charge of the $D5$ brane wrapping a holomorphic 2-cycle. Under the geometric transition this 2-cycle does not shrink so the geometry after the transition has the same number of the cuts in the $x$-plane as no flavor case. But RR-flux on the $x$-plane differ from the no massive flavor case. Since in the $x$-plane RR flux come through the point of mass scale $m_i$ of massive flavor, we have

$$\oint_{m_a} H_R = -1.$$  \hspace{1cm} (2.15)

where $m_a$ is the mass of the $i$-th flavor chiral superfield.

The effective superpotential depend on the cut off parameter $\Lambda_0$. From the monodromy argument we can see the holomorphic beta function. Under $\Lambda_0 \rightarrow e^{2\pi i} \Lambda_0$, the $\Pi_i$ and $F_a$ change by,

$$\Delta \Pi_i = -2(\sum_{j=1}^{n} S_j), \quad \Delta F_a = -\sum_{j=1}^{n} S_j.$$  \hspace{1cm} (2.16)

The factor two comes from the two copies of $x$-plane connected by branch cuts. But for the $F_a$ there is no this factor because this integral comes from the one “semi 3-cycle”. We thus see that $W_{\text{eff}}$ must depend on the cutoff $\Lambda_0$ as

$$W_{\text{eff}} = \cdots + 2 \sum_{i=1}^{N_i} \sum_{j=1}^{n} S_j \log \Lambda_0 - \sum_{a=1}^{N_f} \sum_{j=1}^{n} S_j \log \Lambda_0 + \alpha \sum_{j=1}^{n} S_j$$

$$= \cdots + ((2N - N_f) \log \Lambda_0 - \alpha) \sum_{j=1}^{n} S_j,$$  \hspace{1cm} (2.17)

where $\cdots$ are the cut-off single valued terms. This log-divergent piece can be renormalized to bare coupling constant $\alpha$. Let us introduce the new parameter $\Lambda$ and assume

$$\alpha = b_0 \log \frac{\Lambda}{\Lambda_0}.$$  \hspace{1cm} (2.18)

We identify $\Lambda$ with dynamically generated scales of $U(N)$ gauge theory and $b_0$ with one-loop holomorphic beta function,

$$b_0 = 2N - N_f.$$  \hspace{1cm} (2.19)
This agrees with the beta function for $\mathcal{N} = 1$ $U(N)$ gauge theory with $N_f$ flavor and one adjoint chiral matter [27].

We can extend this discussion to the $SO/Sp$ gauge theories. We can get beta function for $SO$, $Sp$ gauge theory from the same monodromy argument,

\begin{align}
    b_0 &= 2(N - 2) - N_f \quad \text{for } SO(N) \quad (2.20) \\
    b_0 &= 2(N + 2) - N_f \quad \text{for } Sp(N). \quad (2.21)
\end{align}

3 Effective Superpotential

In this section we give a proof of the equivalence $W_{\text{eff}}$ form the Calabi-Yau geometry with fluxes with the effective superpotential $W_{\text{low}}$ obtained by purely field theory analysis.

3.1 Field theory analysis

We concentrate on the Coulomb branch with classical value of adjoint chiral superfield as

\[ P(x) \equiv \langle \det(x - \Phi) \rangle = \prod_{i=1}^{n}(s - a_i)^{N_i} \quad (3.1) \]

where $a_i$ is the root of $W'_{\text{tree}} = 0$. We discuss field theory analysis for effective superpotential using the Seiberg-Witten geometry with monopole massless constraint. Seiberg-Witten curve for $U(N)$ and $SO(N)/Sp(N)$ gauge theory with $N_f$ flavor were discussed in [26],

\[ y^2 = P(x)^2 - \Lambda b_0 A(x). \quad (3.2) \]

\[ A(x) = \begin{cases} 
\det_{N_f}(x + m), & \text{for } U(N) \\
x^2 \det_{N_f}(x + m), & \text{for } SO(2N) \\
x^4 \det_{N_f}(x + m), & \text{for } SO(2N + 1)
\end{cases} \quad (3.3) \]

The curve for $Sp(2N)$ theory are slightly different from the ones the other gauge groups,

\[ y^2 = \left(x^2 P(x) + 2\Lambda b_0 \text{Pf} m\right)^2 - 4\Lambda^{2b_0} \det_{2N_f}(x + m). \quad (3.4) \]

As in [16] the supersymmetric vacuum necessarily has at least $l = N - n$ mutually local monopoles condensed. We consider a singular point in the moduli space where $l = N - n$ mutually local monopoles massless. This means that $l$ one cycles shrink to zero,

\[ y^2 = P(x)^2 - \Lambda b_0 A(x) = h_i F_{2N - 2l} \equiv \prod_{i=1}^{l}(x + p_i)^2 F_{2N - 2l}. \quad (3.5) \]
Here we concentrate on the $U(N)$ gauge theory. For the $SO/Sp$ case the discussion is almost the same as $U(N)$ case one, thus we describe only results after the $U(N)$ discussion.

The low energy superpotential with this constraint is described as

$$W_{\text{low}} = \sum_{r}^{n+1} g_{r} u_{r} + \sum_{i=1}^{l} \left[ L_{i} \left( P(p_{i}) - \epsilon_{i} \Lambda \frac{\Lambda}{2} \sqrt{A(p_{i})} \right) + Q_{i} \frac{\partial}{\partial p_{i}} \left( P(p_{i}) - \epsilon_{i} \Lambda \frac{\Lambda}{2} \sqrt{A(p_{i})} \right) \right] \quad (3.6)$$

where $L_{i}, Q_{i}$ are Lagrange multipliers and $\epsilon_{i} = \pm 1$. From the equations of motion for $p_{i}$ and $Q_{i}$ we obtain the following equations,

$$Q_{i} = 0, \quad \frac{\partial}{\partial p_{i}} \left( P(p_{i}) - \epsilon_{i} \Lambda \frac{\Lambda}{2} \sqrt{A(p_{i})} \right) = 0. \quad (3.7)$$

The equation of motion for $u_{r}$ is

$$g_{r} + \sum_{i=1}^{l} L_{i} \frac{\partial}{\partial u_{r}} \left( P(p_{i}) - \epsilon_{i} \Lambda \frac{\Lambda}{2} \sqrt{A(p_{i})} \right) = 0. \quad (3.8)$$

$A(p_{i})$ is independent of $u_{r}$ the third term vanish. Using the Newton’s relation we can obtain

$$g_{r} = \sum_{i=1}^{l} \sum_{j=0}^{N} L_{i} p_{i}^{N-j} s_{j-r}. \quad (3.9)$$

With this relation as in [29, 30, 31] we can obtain following relation.

$$W'_{\text{tree}} = \sum_{r=1}^{n+1} g_{r} x^{r-1} = \sum_{r=1}^{n+1} \sum_{i=1}^{l} \sum_{j=0}^{N} x^{r-1} p_{i}^{N-j} s_{j-r} L_{i} - x^{-1} \sum_{i=1}^{l} L_{i} P(p_{i}) + O(x^{-2})$$

$$= \sum_{r=1}^{n+1} \sum_{i=1}^{l} \sum_{j=0}^{N} x^{r-1} p_{i}^{N-j} s_{j-r} L_{i} - x^{-1} \sum_{i=1}^{l} L_{i} \Lambda \frac{\Lambda}{2} \sqrt{A(p_{i})} + O(x^{-2})$$

$$= \sum_{j=0}^{N} \sum_{i=1}^{l} P x^{j-N-1} L_{i} - x^{-1} \sum_{i=1}^{l} L_{i} \Lambda \frac{\Lambda}{2} \sqrt{A(p_{i})} + O(x^{-2})$$

$$= \sum_{i=1}^{l} \frac{P}{x - p_{i}} L_{i} - x^{-1} \sum_{i=1}^{l} L_{i} \Lambda \frac{\Lambda}{2} \sqrt{A(p_{i})} + O(x^{-2}) \quad (3.10)$$

Defining $B_{l-1}$ as in [16],

$$\sum_{i=1}^{l} \frac{L_{i}}{x - p_{i}} = \frac{B_{l-1}}{H_{l}}. \quad (3.11)$$

We can obtain

$$g_{n+1}^{2} F_{2n} = W'_{\text{tree}}^{2} + 2g_{n+1} x^{n-1} \Lambda \frac{\Lambda}{2} \sum_{i=1}^{l} \sqrt{A(p_{i})} L_{i} + O(x^{n-2}) = W'_{\text{tree}}^{2} + f_{n-1} \quad (3.12)$$
Using this relation we can rewrite the constraint (3.5) as
\[ P^2 - \Lambda^b A(x) = \frac{1}{g_{n+1}^2} \left( W'(x)^2 + f_{n-1} \right) H_{2n-2}^2(x). \] (3.13)

Thus deformation parameter \( b_{n-1} \) is described as
\[ b_{n-1} = 2g_{n+1}x^{n-1} \Lambda^b \sum_{i=1}^{l} \sqrt{A(p_i)} L_i. \] (3.14)

On the other hand the derivative of \( W_{\text{low}} \) with respect to \( \Lambda \) is written as
\[ \frac{\partial W_{\text{low}}}{\partial \log \Lambda^b} = -\frac{1}{4g_{n+1}} 2g_{n+1}x^{n-1} \Lambda^b \sum_{i=1}^{l} \sqrt{A(p_i)} L_i = -\frac{1}{4g_{n+1}} b_{n-1}. \] (3.15)

We used (3.14) for the last equality.

Next we consider the classical limit \( \Lambda \to 0 \). Since we are considering the classical vacuum (3.1), classical value of effective superpotential is given by,
\[ W_{\text{low}}(\Lambda \to 0) = \sum_{i=1}^{n} N_i \sum_{k=1}^{n+1} \frac{1}{k} g_k a_i^k. \] (3.16)

For the \( SO/Sp \) case deformation function is even function [19, 20]. In this case we can discuss as in \( U(N) \) case. We describe only the results,
\[ \frac{\partial W_{\text{low}}}{\partial \log \Lambda^b} = -\frac{1}{4g_{2n+2}} b_{2n}. \] (3.17)

Massless monopole constraints for the \( SO/Sp \) case rewrite as follow,
\[ P^2 - \Lambda^b A(x) = \frac{1}{g_{2n+2}^2} \left( W'(x)^2 + f_{2n} \right) H_{2l-2}^2(x), \] (3.18)
\[ \left( x^2 P(x) + 2\Lambda^b Pf_m \right)^2 - 4\Lambda^{2b_0} \det_{2N_f}(x + m) = \frac{1}{g_{2n+2}^2} \left( W'(x)^2 + f_{2n} \right) H_{2l-2}^2(x). \] (3.19)

### 3.2 Geometric dual analysis

Next we consider the derivative of effective superpotential with respect to the \( \Lambda \) and classical values in the limit \( \Lambda \to 0 \). As in [24] we use deformation parameters \( \{ b_{n-1}, \ldots, b_0 \} \) as a change of variable instead of \( \{ S_1, \ldots S_n \} \). The expectation value of \( b_k \) are given by
the minimization of the effective superpotential, $\partial W_{\text{eff}} / \partial b_k = 0$. The $\Lambda$ dependence comes from 2 terms,

$$
\frac{\partial W_{\text{eff}}(\langle b_k \rangle, \log \Lambda)}{\partial \log \Lambda^{2N-N_f}} = \sum_{i=1}^{n} N_i \frac{\partial \Pi_i}{\partial \log \Lambda^{2N-N_f}} + \frac{1}{2} \sum_{a=1}^{N_f} \frac{\partial F_a}{\partial \log \Lambda^{2N-N_f}}
$$

$$
= \sum_{i=1}^{n} S_i = -\frac{1}{4g_{n+1}} b_{n-1}
$$

(3.20)

Since the integral of $\omega$ around $m_i$ is zero, the sum of period $S_i$ becomes the integral of $\omega$ around infinity. This gives $b_{n-1}$ for the residue. Riemann surface that we are considering has two special point located at the two pre-images of infinity. We call these two point as $P$ and $Q$. We can take the classical limit $\Lambda \to 0$,

$$
W_{\text{eff}}(\Lambda \to 0) = \sum_{i=1}^{n} N_i \sum_{k=1}^{n+1} \frac{1}{k} g_k a_i^k - \left( N - \frac{N_f}{2} \right) W_{\text{tree}}(\Lambda_0) - \frac{1}{2} \sum_{a=1}^{N_f} W_{\text{tree}}(m_a)
$$

(3.21)

We could add to definition of $W_{\text{eff}}$ an arbitrary function which is $\Lambda$ independent. We can redefine the effective superpotential for dual geometry as

$$
W_{\text{eff}} = -\sum_{i=1}^{n} N_i \int_{a_i^{-}}^{a_i^{+}} \omega - \alpha \sum_{i=1}^{n} \int_{a_i^{-}}^{a_i^{+}} \omega + \frac{1}{2} \sum_{a=1}^{N_f} \int_{m_a}^{\Lambda_0} \omega + \left( N - \frac{N_f}{2} \right) W_{\text{tree}}(\Lambda_0) + \frac{1}{2} \sum_{a=1}^{N_f} W_{\text{tree}}(m_a)
$$

Thus with this definition we obtain following classical limit.

$$
W_{\text{eff}}(\Lambda \to 0) = \sum_{i=1}^{n} N_i \sum_{k=1}^{n+1} \frac{1}{k} g_k a_i^k
$$

(3.22)

These two results (3.22),(3.23) agree with the results (3.16),(3.15) obtained from field theory.

The result for $SO/Sp$ case is obtained in a similar way,

$$
\frac{\partial W_{\text{eff}}(\langle b_k \rangle, \log \Lambda)}{\partial \log \Lambda_{0}} = \sum_{i=1}^{n} S_i = -\frac{1}{4g_{2n+2}} b_{2n}
$$

(3.23)

where we evaluate the period integral at $P$ and get the residue $b_{2n}$. For the classical limit we have only to replace $N_i$ to $\hat{N}_i$.

4 Flux on Riemann surface

The crucial difference between no flavor case and this case is the flux on the Riemann surface. In section 2.2 we discussed the reduction of period to the integral of Riemann surface. Let us also introduce a one form $h$ for the reduction of $H$,

$$
h = \int_{s^2} H, \quad H = H_{RR} - \tau_{IB} H_{NS}
$$

(4.1)
Using this variable we can write the condition of fluxes.
\[ \oint_{A_i} h = N_i, \quad \oint_{B_i} h = \tau_{YM}, \quad \oint_{m_a} h = -1. \] (4.2)

Let us introduce parameter \( s, t < 1 \) that is the relative number of flavor on the upper and lower sheet of \( x \)-plane. Thus we obtain another condition for \( h \),
\[ \oint_P h = -N + sN_f, \quad \oint_Q h = N + tN_f. \] (4.3)

Thus from these constraints \( h \) should have a pole of order 1 at \( m_a, P \) and \( Q \) with residue \(-1, -N + sN_f \) and \( N + tN_f \) respectively.

The equation of motion for \( b_k, \partial W/\partial b_k = 0 \), give rise to the following equation
\[ N \int_P \eta_k + \sum_{a=1}^{sN_f} \int_P \eta_k + \sum_{b=1}^{tN_f} \int_Q \eta_k = 0, \] (4.4)
where \( \eta_k \) a holomorphic one form defined as \( \eta_k \equiv \partial \omega / \partial b_k \). As in [24] using the Abel’s theorem this implies there is meromorphic function with the following divisor.
\[ \sum_{a=1}^{N_f} m_a - (-N + sN_f)P + (N + tN_f)Q \] (4.5)

For the simplicity we put \( s = 1, t = 0 \) respectively. We can describe this function explicitly. Let us introduce the new function \( z \) defined as
\[ z = P(x) - \frac{1}{g_{n+1}} \sqrt{W^2_{tree} + f_{n-1}H_{N-n}(x)} = P(x) - \sqrt{P(x)^2 - \Lambda^2 \det_{N_f}(x + m)}, \] (4.6)
where we used (3.13) for the last equality. This new function has \( N \) th order pole at \( Q \) and \( N - N_f \) th order zero at \( P \) and 1 th order zero at \( m_a \) respectively. Using this function we can describe the flux \( h \) as
\[ h = \frac{1}{2\pi i} \frac{dz}{z}. \] (4.7)

This function has a pole of order 1 at \( m_a, P \) and \( Q \) with residue \(-1, -N + N_f \) and \( N \), and then give expected relations (4.2), (4.3).

For the \( SO(2N)/Sp(2N) \) case we define function \( z \) as follow,
\[ z = P(x) - x^2 P(x)^2 - \Lambda^2 \det_{2N_f}(x + m), \quad \text{for} \quad SO(2N) \] (4.8)
\[ z = B(x) - \sqrt{B(x)^2 - 2\Lambda^2 \det_{2N_f}(x + m)}, \quad \text{for} \quad Sp(2N) \] (4.9)
where we defined \( B(x) \equiv x^2 P(x) + 2\Lambda \partial P \). These function have \( \hat{N} \) th order pole at \( Q \) and \( \hat{N} - 2N_f \) th order zero at \( P \) and 1 th order zero at \( m_a \) respectively. In this case also one-form flux is given by (4.7) and has a pole of order 1 at \( m_a, P \) and \( Q \) with residue \(-1, -\hat{N} + N_f \) and \( \hat{N} \).
In this subsection we restrict discussion to $W_{\text{tree}} = \frac{1}{2} \Phi^2$ and masses of flavor are all same value. We can reproduce Affleck-Dine-Seiberg superpotential [28] from dual geometry,

\[ f = x^2 + y^2 + z^2 + v^2 - \mu = 0, \quad (5.1) \]

\[ S = \frac{1}{2\pi i} \int_{-\sqrt{\mu}}^{\sqrt{\mu}} dx \sqrt{x^2 - \mu} = \frac{\mu}{4}. \quad (5.2) \]

The effective superpotential for this case is described as

\[ W_{\text{eff}} = S \log \left( \frac{\Lambda^{2N}}{S^N} \right) + NS + W_{\text{flavor}}^{\text{eff}}, \quad (5.3) \]

\[ W_{\text{flavor}}^{\text{eff}} = -\frac{1}{2} \sum_{a=1}^{N_f} \int_{m_a}^{\Lambda_0} dx \sqrt{x^2 - \mu} \]

\[ = N_f \left( -\frac{S}{2} - \frac{m^2}{4} \sqrt{1 - \frac{4S}{m^2}} + S \log \frac{m}{\Lambda_0} + S \log \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4S}{m^2}} \right) \right). \quad (5.4) \]

As discussed in subsection 2.2 since $\log \Lambda_0$ divergent piece can be renormalized to bare coupling, we can replace $\Lambda_0$ to $\Lambda$, which is interpreted as dynamically generated energy scale. Taking into account $W(m_a)$ term discussed in subsection 3.2, this result (5.4) is exactly the same as the result eq (11) in [5] that is obtained by matrix model. This suggest that $W_{\text{flavor}}^{\text{eff}}$ correspond to matrix model free energy with boundary effect. We can integrate out massive glueball superfield from $\partial_s W_{\text{eff}} = 0$. This leads simple equation,

\[ \partial_s W_{\text{eff}} = -\frac{N}{2} \log \frac{S}{\Lambda^2} + N_f \log \left[ \frac{m}{2\Lambda} \left( 1 + \sqrt{1 - \frac{4S}{m^2}} \right) \right] = 0, \quad (5.5) \]

\[ \left( \frac{S}{\Lambda^2} \right)^{2k} - m \left( \frac{S}{\Lambda^2} \right)^k + N_f \frac{S}{\Lambda^2} = 0. \quad (5.6) \]

where $k \equiv \frac{N}{N_f}$. Using this relation we can rewrite the effective superpotential as

\[ W_{\text{eff}} = C \left[ (k - 1)S + \Lambda^{2k} S^{-k+1} - \frac{1}{2m^2} \Lambda^{4k} S^{-2k+2} \right], \quad (5.7) \]

where $C$ is constant factor. In order to recover a matter field, we should use a matching condition $\Lambda^{2k} = m\tilde{\Lambda}^{2k-\frac{2}{3}}$. In terms of this relation (5.7) is rewritten as

\[ W_{\text{eff}} = C(k - 1)S + Cm\tilde{\Lambda}^{2k-\frac{2}{3}} S^{-k+1} - \frac{C}{2} \Lambda^{4k-\frac{4}{3}} S^{-2k+2}. \quad (5.8) \]
The expectation value of \( X = Q \bar{Q} \) field is written as the derivative with respect to mass \( m \),

\[
X = C \Lambda^{2k-\frac{3}{2}} S^{-k+1}. \tag{5.9}
\]

Using this variable we can rewrite effective superpotential and obtain Affleck-Dine-Seiberg potential [28],

\[
W_{\text{eff}} = A(N, N_f) \left( \frac{\Lambda^{3N-N_f}}{\det_{N_f} X} \right)^{\frac{1}{N-N_f}}, \tag{5.10}
\]

where we replaced \( \Lambda \to \Lambda^{\frac{3}{2}} \). Until now we used \( \Lambda \) for simplicity. But actually since \( x \) has dimension 3/2 we have to use \( \Lambda^{\frac{3}{2}} \). For the \( Sp(2N) \) with \( 2N_f \) case we can rewrite more familiar form,

\[
W_{\text{eff}} = A(N, N_f) \left( \frac{\Lambda^{3(N+1)-N_f}}{\text{Pf}_{N_f} X} \right)^{\frac{1}{N+1-N_f}}. \tag{5.11}
\]

Note that this geometric analysis for the gauge theory does not distinguish between \( N_f < N \) and \( N_f > N \), while the gauge theory physics changes drastically. This is the same as the matrix viewpoint discussion in [5, 8, 6]. For \( N_f \leq N-1 \) effective superpotential reproduce the Affleck-Dine-Seiberg superpotential. For \( N_f \geq N+1 \) effective superpotential reproduce the same superpotential.

In [8] this “problem” was discussed. \( N_f \geq N+2 \) the gauge theory is strongly coupled and the correct description is given by its Seiberg dual. The superpotential for this theory is written in terms of the fields, which are dual to the electric meson fields. This explains the unnatural agreement between the CY with flux and the introduction of an ADS-like superpotential for \( N_f \geq N+2 \).

## 6 Seiberg-Witten curve from flux

In this section we reproduce the Seiberg-Witten curve with \( N_f \) massive flavor from geometry with fluxes. The derivation for \( U(N) \) gauge theory with no flavor was discussed in [24]. As in [24] in terms of turning off the fluxes we can consider the \( \mathcal{N} = 2 \) information. For \( U(N) \) gauge theory we consider the tree level superpotential as

\[
W_{\text{tree}} = \sum_{k=1}^{N+1} \frac{g_k}{k} \text{Tr} \Phi^k. \tag{6.1}
\]
We assume the vacuum which breaks $U(N)$ to $U(1)^N$. In this vacuum $N_i = 1$ for $i = 1, \ldots, N$. $\mathcal{N} = 2$ information is the quantities that dose not vanish in the limit $g_{N+1} \to 0$. The massless monopole constraint (3.13) is rewritten as

$$P_N^2 - \Lambda^{b_0} A(x) = \frac{1}{g_{N+1}^2} \left( W'(x)^2 + f_{N-1} \right).$$

(6.2)

Thus the geometry with respect to this vacuum is described as

$$g_{N+1} \left( P_N^2 - \Lambda^{b_0} A(x) \right) + y^2 + z^2 + w^2 = 0.$$  

(6.3)

After the reduction to the $x$-plane and absorbing the $g_{N+1}$ in $y$, we get the Seiberg-Witten curve.

$$y^2 = P_N^2 - \Lambda^{b_0} A(x)$$

(6.4)

As in the case [24] geometry for this case dose not modified quantum mechanically. Although $b_k$ is not zero, the effect only comes from the mass of massive flavor.

For the $SO/Sp$ case we can discuss in a similar way. We also consider the tree level superpotential as

$$W_{\text{tree}} = \sum_{k=1}^{N+1} \frac{g_{2k}}{2k} \text{Tr} \Phi^{2k}$$

(6.5)

The massless monopole constraint equations (3.18), (3.19) are described as

$$P_N^2 - \Lambda^{b_0} A(x) = \frac{1}{g_{2N+2}} \left( W'(x)^2 + f_{2N-2} \right)$$

(6.6)

$$\left( x^2 P(x) + 2 \Lambda^{b_0} Pf \right)^2 - 4 \Lambda^{2b_0} \det_{2N} (x + m) = \frac{1}{g_{2N+2}} \left( W'(x)^2 + f_{2N-2} \right)$$

(6.7)

Then after the reduction to $x$-plane and absorbing the $g_{2N+2}$, we have the Seiberg-Witten curve for $SO/Sp$ gauge theory with massive flavor (3.2), (3.4).

Acknowledgements

The authors are also obliged to Yoshiyuki Watabiki, Hiroyuki Fuji and Takashi Yokono for stimulating discussions.

References


