The Quantum Supersymmetric Vector Multiplet 
and Some Problems in Non-Abelian Supergauge Theory

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Abstract

We consider the supersymmetric vector multiplet in a purely quantum framework. We obtain some discrepancies with respect to the literature in the expression of the superpropagator and we prove that the model is consistent only for positive mass. The gauge structure is constructed purely deductive and leads to the necessity of introducing scalar ghost superfields, in analogy to the usual gauge theories. The construction of a consistent supersymmetric gauge theory based on the vector model depends crucially one the definition of gauge invariance. We find some significant difficulties to impose a supersymmetric gauge invariance condition for the usual expressions from the literature.

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1 Introduction

The supersymmetric gauge theories are constructed using the so-called vector supersymmetric multiplet \([7]\) (see also \([33]\), \([32]\), \([31]\), \([12]\), \([1]\), \([8]\), \([16]\), \([27]\), \([20]\), \([21]\), \([24]\), etc.) The justification for this choice comes from the analysis of the unitary irreducible representations of the \(N = 1\) supersymmetric extension of the Poincaré group; there are two irreducible massive representations

\[
\begin{align*}
\Omega_{1/2} &\sim [m, 0] \oplus [m, 1/2] \oplus [m, 1/2] \oplus [m, 1], \\
\Omega_1 &\sim [m, 1/2] \oplus [m, 1] \oplus [m, 1] \oplus [m, 3/2],
\end{align*}
\]

(1.1)

containing a spin 1 system (see for instance \([23]\); here \([m, s]\) is the irreducible representation of mass \(m\) and spin \(s\) of the Poincaré group.) The standard vector multiplet is constructed such that the one-particle subspace of the Fock space carries the “simplest” representation \(\Omega_{1/2}\). We will prove in this paper that it is very hard to build a corresponding fully consistent quantum theory with all the usual properties. The other possibility is to construct a supersymmetric multiplet for which the associated Fock space has \(\Omega_1\) as the one-particle subspace of the Fock space. The construction in this case is natural and straightforward; the content of this multiplet is a (complex) spin 1 and a spin 3/2 fields (more precisely a Rarita-Schwinger field without the transversality conditions.) We have proved in \([13]\) that the second multiplet can be the basis for a supersymmetric extension of quantum gauge theory because its gauge structure involving ghosts, anti-ghosts and unphysical scalar (Goldstone) fields is very similar to the ordinary gauge theory.

In this paper we consider in detail the \(\Omega_{1/2}\) vector model (so from no on, when we say “the vector model” we mean the the \(\Omega_{1/2}\) vector model.) We intend to give a rigorous treatment of all aspects of this model using the Epstein-Glaser framework. This seems to be a rather difficult task. In this paper we start this program analysing the layout of the model, that is the construction of the quantum multiplet, its gauge structure and the expression of the interaction Lagrangian (or, in the language of perturbation theory, the first order chronological product). The analysis will be performed entirely in the quantum framework \([15]\), \([19]\), \([18]\), \([5]\), \([11]\), \([13]\) avoiding the usual approach based on quantizing a classical supersymmetric theory. In this way we avoid the complications associated to the proper mathematical definition of a super-manifold \([9]\), \([6]\) and we do not need a quantization procedure.

The main results are the following. First we show that the vector model is consistent only for positive mass. Next, we determine the gauge structure of the vector model: it coincides essentially with the expression from the literature but, because the mass of the multiplet is positive we need to introduce some scalar ghost superfields. We are also able to determine the general expressions for the Feynman super-propagators in a purely deductive way; some discrepancy with the standard literature appears. Finally, we investigate the expression for the interaction Lagrangian consistent with the conventional approaches based on path integral quantization.

It is important to outline the mathematical framework used in this paper from the very beginning. The description of higher spin fields will be done in the indefinite metric approach (Gupta-Bleuler). That is, we construct a Hilbert space \(\mathcal{H}\) with a non-degenerate sesqui-linear
form $<\cdot,\cdot>$ and a gauge charge operator $Q$ verifying $Q^2 = 0$; the form $<\cdot,\cdot>$ becomes positively defined when restricted to a factor Hilbert space $\mathcal{H}_{\text{phys}} \equiv \text{Ker}(Q)/\text{Im}(Q)$ which will be the physical space of our problem. The interaction Lagrangian $t(x)$ will be some Wick polynomial acting in the total Hilbert space $\mathcal{H}$ and verifying the conditions

$$[Q,t(x)] = i\partial_\mu T^\mu(x)$$

(1.2)

for some Wick polynomials $t^\mu(x)$; this condition guarantees that the interaction Lagrangian $t(x)$ factorises to the physical Hilbert space $\text{Ker}(Q)/\text{Im}(Q)$ in the adiabatic limit, i.e. after integration over $x$; the condition (1.2) is equivalent to the usual condition of (free) current conservation. The condition (1.2) has far reaching physical consequences: under some reasonable additional assumptions one can prove that the usual expression of the interaction Lagrangian for a Yang-Mills model is unique, up to trivial terms. It is desirable to generalize this scheme to supersymmetric theories.

In the supersymmetric framework one postulates that the basic supersymmetric multiplets should be organized in superfields i.e. fields dependent on space-time variables and some auxiliary Grassmann parameters. It is showed in [13] that one can consistently replace fields by superfields: one has a canonical map $w \mapsto sw \equiv W$ mapping a ordinary Wick monomial $w(x)$ into its supersymmetric extension $W(x,\theta,\bar{\theta})$; in particular this map associates to every field of the model a superfield. Moreover, one postulates that the interaction Lagrangian $t$ should be of the form

$$t(x) \equiv \int d\theta^2 d\bar{\theta}^2 T(x,\theta,\bar{\theta})$$

(1.3)

for some supersymmetric Wick polynomial $T$. This hypothesis makes possible the generalization of the Epstein-Glaser approach to the supersymmetric case as it is showed in [13].

Concerning the gauge invariance of the model there are two possible attitudes. One is to impose only (1.2); this “minimal” possibility is certainly consistent from the physical point of view but in this case one loses the unicity results concerning the interaction Lagrangian. One can hope to keep this unicity result if one finds out a supersymmetric generalization of (1.2). A natural candidate would be the relation:

$$[Q,T(x,\theta,\bar{\theta})] = i\partial_\mu T^\mu(x,\theta,\bar{\theta}) + \ldots$$

(1.4)

where by $\ldots$ we mean total divergence expressions in the Grassmann variables. It is clear that (1.4) implies (1.2) but not conversely. We call (1.4) the condition of supersymmetric gauge invariance. In [13] we have showed that the stronger condition (1.4) can be imposed and indeed the unicity argument concerning the interaction Lagrangian holds. However for the $\Omega_{1/2}$ vector model the situation is not so good. If one uses only the “minimal” gauge invariance condition (1.2) then one loses the unicity of the interaction Lagrangian. If one tries to impose the supersymmetric version (1.4) one finds out that the usual expressions for the interaction Lagrangian suggested by the literature do not fulfil it. Of course it is in principle possible to find alternative expression for the interaction Lagrangian such that (1.4) is true, but this possibility seems to be rather improbable. So our results concerning the construction of the interaction Lagrangian for the $\Omega_{1/2}$ vector model must be considered as a criticism of
the traditional approaches based on the path integral formalism: a rigorous approach produces some differences and negative results.

The structure of the paper is the following. In Section 2 we give a brief but general discussion about supersymmetric multiplets and the associated superfields. In Section 3 we give a detailed description of the vector multiplet in a purely quantum framework. We find out differences with respect to the literature in the expression of the super-propagator. In Section 4 we first remained the basic facts about the chiral multiplet (they will be used as scalar ghost superfields) and then we construct in analogy the ghost and the anti-ghost superfields. In particular we prove that so-called Wess-Zumino gauge is not a supersymmetric decomposition of the vector superfield into a chiral, anti-chiral and a “physical” part. In Section 5 we use these superfields for the construction of the gauge structure. In Section 6 we study the interaction Lagrangian which can be inferred from the expression appearing in the formal path integral quantization method (combined with the Faddeev-Popov trick). We have to add to it supplementary terms containing the scalar ghosts superfields and argue that it cannot fulfil the supersymmetric gauge invariance condition (1.4). In Section 7 we investigate the possibility of using the linear vector multiplet in a supersymmetric gauge theory.

One can conclude that the new vector multiplet proposed for the first time in [13] and based only on chiral superfields is a more natural object and it remains as a serious candidate for a possible supersymmetric extension of the standard model.

2 Quantum Supersymmetric Theory

We remind here the definition of a supersymmetric theory in a pure quantum context. We will not consider extended supersymmetries here. We follow closely [13].

The conventions are the following: (a) we use summation over dummy indices; (b) we raise and lower Minkowski indices with the Minkowski pseudo-metric $g_{\mu\nu} = g^{\mu\nu}$ with diagonal $1, -1, -1, -1$; (c) we raise and lower Weyl indices with the anti-symmetric $SL(2,\mathbb{C})$-invariant tensor $\epsilon_{ab} = -\epsilon^{ab}$; $\epsilon_{12} = 1$; (d) we denote by $\sigma^\mu$ the usual Pauli matrices with elements denoted by $\sigma^\mu_{ab}$ and the convention $\sigma^0 = 1$; (e) we introduce the notations:

$$
\theta \lambda \equiv \theta^a \lambda_a, \quad \bar{\theta} \bar{\lambda} \equiv \bar{\theta}_a \bar{\lambda}^a, \\
\theta^2 \equiv \theta \theta, \quad \bar{\theta}^2 \equiv \bar{\theta} \bar{\theta} \\
\theta \sigma^\mu \bar{\lambda} \equiv \theta^a \sigma^\mu_{ab} \bar{\lambda}^b.
$$

(2.1)

Suppose that we have a quantum theory of free fields; this means that we have the following construction:

- $\mathcal{H}$ is a Hilbert space of Fock type (associated to some one-particle Hilbert space describing some choice of elementary particles) with the scalar product $(\cdot, \cdot)$;
- $\Omega \in \mathcal{H}$ is a special vector called the vacuum;
- $U_{a,A}$ is a unitary irreducible representation of $inSL(2,\mathbb{C})$ the universal covering group of the proper orthochronous Poincaré group such that $a \in \mathbb{R}^4$ is translation in the Minkowski space and $A \in SL(2,\mathbb{C})$;
• \( b_j, \ j = 1, \ldots, N_B \) (resp. \( f_A, \ A = 1, \ldots, N_F \)) are the quantum free fields of integer (resp. half-integer) spin. We assume that the fields are linearly independent up to equations of motion;

• The equations of motion do not connect distinct fields.

In practice, one considers only particles of spin \( s \leq 2 \). For the standard vector model we will consider only \( s \leq 1 \). In \cite{13} we have considered the a more unusual case namely \( 1 \leq s \leq 3/2 \). The fact that we work only with free fields is very natural from the point of view of \( S \)-matrix perturbation theory in the sense of Bogoliubov \cite{3}. Even if one considers a more general case, namely a Wightman theory like in \cite{15}, one still have a Fock space structure generated by the asymptotic fields and some natural assumption show that the supersymmetric structure of the interacting theory is preserved for the free fields.

As we have said in the Introduction, if one considers higher-spin fields (more precisely \( s \geq 1 \)), as we do here and we have done in \cite{13}, it is necessary to extend somewhat this framework: one considers in \( \mathcal{H} \) besides the usual positive definite scalar product a non-degenerate sesquilinear form \( \langle \cdot, \cdot \rangle \) which becomes positively defined when restricted to a factor Hilbert space \( \text{Ker}(Q)/\text{Im}(Q) \) where \( Q \) is some gauge charge. We denote with \( A^\dagger \) the adjoint of the operator \( A \) with respect to \( \langle \cdot, \cdot \rangle \).

It is convenient to make the description of quantum fields more explicit. According to the usual treatment of the standard vector model, we have to consider that we have the following fields.

• A set of Bosonic scalar fields \( b^{(j)} \) of mass \( m_j \), \( j = 1, \ldots, s \) respectively which can be taken Hermitian without losing generality. This means that we have the following relations:

\[
(b^{(j)})^\dagger = b^{(j)}, \quad j = 1, \ldots, s
\]  

\[
(\partial^2 + m_j^2)b^{(j)} = 0, \quad j = 1, \ldots, s
\]  

\[
[b^{(j)}(x), b^{(k)}(y)] = -i \delta_{jk} D_{m_j}(x - y),
\]  

where \( D_m \) is the Pauli-Jordan causal distribution of mass \( m \).

• A Bosonic real vector field \( b_\mu \) of mass \( m \) verifying

\[
(b_\mu)^\dagger = b_\mu
\]

\[
(\partial^2 + m^2)b_\mu = 0
\]

\[
[b_\mu(x), b_\nu(y)] = i g_{\mu\nu} D_m(x - y).
\]  

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• A set of Fermi fields $f^{(A)}_a$ of spin $1/2$ and of mass $M_A$, $A = 1, \ldots, f$ which can be taken without losing generality to be Majorana fields; here $a = 1, 2$ is a Weyl index so, the transformation property of the fields with respect to the group $SL(2, \mathbb{C})$ is given by the representation $(1/2, 0)$. We define

$$\bar{f}^{(A)}_a \equiv (f^{(A)}_a)^\dagger \quad A = 1, \ldots, f, \quad a = 1, 2$$

(2.8)

i.e. the bared indices correspond to the representation $(0, 1/2)$ of the group $SL(2, \mathbb{C})$. We also suppose that these fields obey Dirac equation:

$$i \sigma^\mu_{ab} \partial_\mu f^{(A)b}_a = M_A f^{(A)}_a, \quad -i \sigma^\mu_{ab} \partial_\mu f^{(A)a}_b = M_A \bar{f}^{(A)}_b, \quad A = 1, \ldots, f$$

(2.9)

and the usual causal anticommutation relation:

$$\{f^{(A)}_a(x), f^{(B)}_b(y)\} = \delta_{AB} \epsilon_{ab} M_A D_{M_A} (x - y),$$

$$\{f^{(A)}_a(x), \bar{f}^{(B)}_b(y)\} = \delta_{AB} \sigma^\mu_{ab} \partial_\mu D_{M_A} (x - y).$$

(2.10)

All these fields have bona fide representations in Fock spaces; they can be found in standard literature. It is appropriate to clarify now the connection between Majorana and Dirac fields.

**Proposition 2.1** (i) Suppose that the Weyl spinor $f_a$ verifies only Klein-Gordon equation:

$$(\partial^2 + m^2)f_a = 0$$

(2.11)

of positive mass $m$ and the causal anticommutation relations:

$$\{f_a(x), f_b(y)\} = 0,$$

$$\{f_a(x), \bar{f}_b(y)\} = \frac{1}{2m^2} \sigma^\mu_{ab} \partial_\mu D_m (x - y).$$

(2.12)

Then the bi-spinor

$$\psi \equiv \begin{pmatrix} f_a \\ \bar{g}^b \\ \end{pmatrix}$$

(2.13)

where

$$\bar{g}_b \equiv -\frac{i}{m} \sigma^\mu_{ab} \partial_\mu f^a$$

(2.14)

verifies Dirac equation.

$$(i \gamma \cdot \partial + m)\psi = 0;$$

(2.15)

here $\gamma^\mu$ are the Dirac matrices (in the chiral representation). Conversely, if $\psi$ is a Dirac bi-spinor, its upper component $f_a$ is restricted only by the Klein-Gordon equation and its lower component is determined as above.
(ii) Let us consider the Weyl spinor $f_a$ verifying Klein-Gordon equation and let us define the following Majorana spinors:

$$\xi_a^{(1)} = i m f_a + \sigma^\mu_{ab} \partial_\mu \bar{f}_b$$

$$\xi_a^{(2)} = m f_a + i \sigma^\mu_{ab} \partial_\mu \bar{f}_b. \quad (2.16)$$

Then the spinors $\xi_a^{(j)}, \quad j = 1, 2$ verify the Dirac equation:

$$i \sigma^\mu_{ab} \bar{\xi}_a^{(j)} b = m \xi_a^{(j)} = i \sigma^\mu_{ab} \bar{\xi}_a^{(j)\alpha} = m \bar{\xi}_a^{(j)} \quad (2.17)$$

and the usual causal anti-commutation relations:

$$\{ \xi_a^{(j)}(x), \xi_b^{(k)}(y) \} = i \delta_{jk} \epsilon_{ab} m D_m(x - y)$$

$$\{ \xi_a^{(j)}(x), \bar{\xi}_b^{(k)}(y) \} = \delta_{jk} \sigma^\mu_{ab} \partial_\mu D_m(x - y) \quad (2.18)$$

and conversely, if two spinors $\xi_a^{(j)}, \quad j = 1, 2$ verify the preceding relations, then the spinor

$$f_a \equiv \frac{1}{2m} (\xi_a^{(2)} - i \xi_a^{(1)}) \quad (2.19)$$

verifies Klein-Gordon equation and the causal anticommutation relation (2.12). The correspondence $f_a \leftrightarrow (\xi_a^{(j)})_{j=1}^2$ is one-one.

The proof is elementary and shows that every (four-component) Dirac bi-spinor can be described by a Weyl spinor verifying only Klein-Gordon equation; such a spinor can in turn be described by two Majorana spinors. It follows that we do not lose generality if we consider all Fermi fields of spin 1/2 to be Majorana spinors.

Now we define the notion of supersymmetry invariance of the system of Bosonic and Fermionic fields considered above. Suppose that in the Hilbert space $\mathcal{H}$ we also have the operators $Q_a, \quad a = 1, 2$ such that:

(i) the following relations are verified:

$$Q_a \Omega = 0, \quad \bar{Q}_a \Omega = 0 \quad (2.20)$$

and

$$\{Q_a, Q_b\} = 0, \quad \{Q_a, \bar{Q}_b\} = 2 \sigma^\mu_{ab} P_\mu, \quad [Q_a, P_\mu] = 0, \quad U^{-1}_A Q_a U_A = A^b_a Q_b. \quad (2.21)$$

Here $P_\mu$ are the infinitesimal generators of the translation group given by

$$[P_\mu, b] = -i \partial_\mu b, \quad [P_\mu, f] = -i \partial_\mu f. \quad (2.22)$$

and

$$\bar{Q}_b \equiv (Q_b)\dagger. \quad (2.23)$$

(ii) The following commutation relations are true:

$$i [Q_a, b] = p(\partial) f, \quad \{Q_a, f\} = q(\partial) b \quad (2.24)$$
where \( b = (b^{(j)}, b^\mu) \) (resp. \( f = (f_a^{(A)}, \bar{f}_a^{(A)}) \)) is the collection of all integer (resp. half-integer) spin fields and \( p, q \) are matrix-valued polynomials in the partial derivatives \( \partial_a \) (with constant coefficients). These relations express the tensor properties of the fields with respect to (infinitesimal) supersymmetry transformations.

If these conditions are true we say that \( Q_a \) are **super-charges** and \( b, f \) are forming a **supersymmetric multiplet**. The notion of **irreducibility** can be defined for any supersymmetric multiplet if we consider the quantum fields as a modulus over the ring of partial differential operators. As emphasised in [13], the matrix-valued operators \( p \) and \( q \) are subject to various constraints. Let us describe them in this context.

- From the compatibility of (2.24) with Lorentz transformations it follows that these polynomials are Lorentz covariant.

- Next, we start from the fact that the Hilbert space of the model is generated by vectors of the type

\[
\Psi = \prod b(x_p) \prod f(x_q) \Omega \in \mathcal{H}.
\]  

The action of the supercharges \( Q_a \), \( \bar{Q}_{\bar{a}} \) is determined by (2.24): one commutes the supercharge operators to the right till they hit the vacuum and then one applies (2.20). However, the supercharges are not independent: they are constrained by the relations from (2.21) and we should check that we do not get a contradiction. The consistency relations are given by the (graded) Jacobi identities combined with (2.21) and the relation (2.22):

As a result we must have:

\[
\begin{align*}
\{Q_a, [Q_b, b]\} &= -(a \leftrightarrow b) \\
[Q_a, \{Q_b, f\}] &= -(a \leftrightarrow b), \\
\{Q_a, [\bar{Q}_{\bar{b}}, b]\} + \{\bar{Q}_{\bar{b}}, [Q_a, b]\} &= -2i \sigma_{ab}^\mu \partial_\mu b, \\
[Q_a, \{\bar{Q}_{\bar{b}}, f\}] + [\bar{Q}_{\bar{b}}, \{Q_a, f\}] &= -2i \sigma_{ab}^\mu \partial_\mu f.
\end{align*}
\]  

- The equation of motion (2.3), (2.6) and (2.9) are supersymmetric invariant, i.e. if we take the commutator of the supercharges \( Q_a \) and \( \bar{Q}_{\bar{a}} \) with the equations (2.3) and (2.6) we obtain zero modulo (2.9); also if we take the anticommutator of the equations of motion (2.9) with the supercharges \( Q_a, \bar{Q}_{\bar{a}} \) we get zero modulo (2.3), (2.6).

- The (anti)commutation relations have the implication that one and the same vector from the Hilbert space \( \mathcal{H} \) can be expressed in the form (2.25) in two distinct ways. This means that the supercharges are well defined via (2.21) iff some new consistency relations are valid following again from graded Jacobi identities; the non-trivial ones are of the form:

\[
[b(x), \{f(y), Q_a\}] = -\{f(y), [Q_a, b(x)]\}
\]  

(2.27)
• If a gauge supercharge $Q$ is present in the model, then it is usually determined by relations of the type (2.24) involving ghost fields also, so it means that we must impose consistency relations of the same type as above. Moreover, it is desirable to have

\[ \{Q, Q_a\} = 0, \quad \{Q, \bar{Q}_a\} = 0; \quad \text{(2.28)} \]

this implies that the supersymmetric charges $Q_a$ and $\bar{Q}_a$ factorizes to the physical Hilbert space $\mathcal{H}_{\text{phys}} = \text{Ker}(Q)/\text{Im}(Q)$. This implies new consistency relations of the type (2.26) with one of the supercharges replaced by the gauge charge:

\[ \{Q_a, [Q, b]\} = -\{Q, [Q_a, b]\} \quad [Q_a, \{Q, f\}] = -\{Q, \{Q_a, f\}\}. \quad \text{(2.29)} \]

• A relation of the type (2.27) must be also valid for the gauge charge:

\[ [b(x), \{f(y), Q\}] = -\{f(y), [Q, b(x)]\}. \quad \text{(2.30)} \]

• To have $Q^2 = 0$ we must also impose

\[ \{Q, [Q, b]\} = 0 \quad [Q, \{Q, f\}] = 0. \quad \text{(2.31)} \]

**Remark 2.2** All these conditions are of pure quantum nature i.e. they can be understood only for a pure quantum model. Some of them do not have a classical analogue so we can interpret the obstacles in constructing supersymmetric quantum models (associated to some classical supersymmetric theories) as some quantum anomalies.

It seems to be an essential point to describe supersymmetric theories in *superspace* [28], [29]. We do this in the following way. We consider the space $\mathcal{H}_G \equiv G \otimes \mathcal{H}$ where $G$ is a Grassmann algebra generated by Weyl anticommuting spinors $\theta_a$ and their complex conjugates $\bar{\theta}_a = (\theta_a)^*$ and perform a Klein transform such that the Grassmann parameters $\theta_a$ are anti-commuting with all Fermionic fields, the supercharges and the gauge charge. The field operators acting in $\mathcal{H}_G$ are called superfields. Of special interest are the superfields constructed as in [4], [5] according to the formulæ:

\[
B(x, \theta, \bar{\theta}) \equiv W_{\theta, \bar{\theta}} b(x) W^{-1}_{\theta, \bar{\theta}}, \\
F(x, \theta, \bar{\theta}) \equiv W_{\theta, \bar{\theta}} f(x) W^{-1}_{\theta, \bar{\theta}},
\]

where

\[ W_{\theta, \bar{\theta}} \equiv \exp \left( i\theta^a Q_a - i\bar{\theta}^a \bar{Q}_a \right) \quad \text{(2.33)} \]

and we interpret the exponential as a (finite) Taylor series.

It is a remarkable fact that only such type of superfields are really necessary, so in the following, when referring to superfields we mean expressions given by (2.32). We will call them super-Bose and respectively super-Fermi fields. For convenience we will denote frequently the ensemble of Minkowski and Grassmann variables by $X = (x, \theta, \bar{\theta})$. 

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More generally one starts from Wick monomials defined in $\mathcal{H}$ and by multiplication with Grassmann variables we obtain super-Wick monomials in the extended Fock space $\mathcal{H}_G$. It appeared from the analysis of [13] that it is worthwhile to define a canonical map associating to any Wick monomial $w$ in $\mathcal{H}$ a super-Wick monomials acting in $\mathcal{H}_G$ according to the formula:

$$(sw)(x, \theta, \bar{\theta}) \equiv W_{\theta, \bar{\theta}} w(x) W_{\bar{\theta}, \theta}^{-1};$$

(2.34)

(here $s$ stands for “sandwich formula” or for “supersymmetric extension”.) From now on by supersymmetric Wick monomials we mean only expressions of the type $sw$.

Now we have some elementary results from [13] which will be repeatedly used in the computations; for simplicity we denote by $[,]$ the graded commutator.

**Lemma 2.3** (i) Let us define the operators:

$$D_a \equiv \frac{\partial}{\partial \theta^a} + i \sigma^\mu \hat{\theta}^\mu \partial_\mu, \quad \bar{D}_a \equiv -\frac{\partial}{\partial \bar{\theta}^a} - i \sigma^\mu \theta^b \bar{\theta}^b \partial_\mu. \quad (2.35)$$

Then if $W \equiv s(w)$ the following formulæ are true:

$$i[Q_a, W(x, \theta, \bar{\theta})] = D_a W(x, \theta, \bar{\theta}), \quad i[\bar{Q}_a, W(x, \theta, \bar{\theta})] = \bar{D}_a W(x, \theta, \bar{\theta}) \quad (2.36)$$

(ii) Let us define the operators

$$D_a \equiv \frac{\partial}{\partial \theta^a} - i \sigma^\mu \hat{\theta}^\mu \partial_\mu, \quad \bar{D}_a \equiv -\frac{\partial}{\partial \bar{\theta}^a} + i \sigma^\mu \theta^b \bar{\theta}^b \partial_\mu \quad (2.37)$$

acting on any superfield (or super–Wick polynomials). Then for any Wick monomial $w(x)$ the following relations are true:

$$D_a sw = i s([Q_a, w]), \quad \bar{D}_a sw = i s([\bar{Q}_a, w]). \quad (2.38)$$

(iii) The operators $D_a$ and $\bar{D}_a$ verify the following formulæ:

$$(D_a T)^\dagger = \pm \bar{D}_a T^\dagger,$$

$$\{D_a, D_b\} = 0, \quad \{\bar{D}_a, \bar{D}_b\} = 0, \quad \{D_a, \bar{D}_b\} = -2 i \sigma^\mu_{ab} \partial_\mu$$

(2.39)

where in the first formula the sign $+(-)$ corresponds to a super-Bose (-Fermi) field. The operators $D$ verify relations of the same type.

Let us comment on the physical interpretation of the formulæ (2.36 ). If the Wick polynomial $W(x, \theta, \bar{\theta})$ verifies (2.36 ) then let us define

$$w(x) \equiv \int d\theta^2 d\bar{\theta}^2 W(x, \theta, \bar{\theta});$$

(2.40)

it follows from (2.36 ) that we have

$$[Q_a, w(x)] = i \partial_\mu w^\mu_a(x) \quad [\bar{Q}_a, w(x)] = i \partial_\mu \bar{w}^\mu_a(x)^\dagger \quad (2.41)$$
where
\[ w'^{a} \equiv \sigma^{a} \bar{b} \int d\theta^{2} d\bar{\theta}^{2} \bar{\theta}^{\bar{\phi}} W(x, \theta, \bar{\theta}); \] (2.42)

the equations (2.41) are exactly the supersymmetry postulate used in [11]. It is not clear if the converse is true i.e. suppose we have (2.41) for some Wick polynomial \( w \); then is it possible to find out a supersymmetric Wick polynomial \( W \) such that we have (2.40) and (2.36)?

Concerning the meaning of (2.38) let us consider for an arbitrary supersymmetric Wick monomial \( W(x, \theta, \bar{\theta}) \) the operation of restriction to the “initial value” (in the Grassmann variables):
\[ (r W)(x) \equiv W(x, 0, 0). \] (2.43)

Then the formulæ (2.38) imply
\[ D^{a} W = i s([Q^{a}, r W]), \quad \bar{D}^{\bar{a}} W = i s([\bar{Q}^{\bar{a}}, r W]). \] (2.44)

These equations can be regarded as a system of partial differential equations (in the Grassmann variables) and this system determines uniquely the supersymmetric Wick monomials \( W \) if one knows the “initial values” \( w = r W \). (If there are two solutions, then their difference verifies the associated homogeneous equation which tells that there is no dependence on the Grassmann variables; but the “initial values” for the difference is zero.)

For another point of view concerning supersymmetric Hilbert spaces we refer to the recent paper [22].

We close this Section mentioning that for the construction of the ghost and anti-ghost multiplets (which are needed in the construction of supersymmetric gauge theories) one must consider that the integer (resp. half-integer) spin fields have Fermi-Dirac (resp. Bose-Einstein) statistics and to invert everywhere the rôle of commutators and anti-commutators. The scheme presented above is reproduced with minimal changes.

### 3 The Vector Multiplet

By definition, the vector multiplet has the content described in Section 2: the Bosonic fields are some (real) scalars \( b^{(j)}, \quad j = 1, \ldots, s \) and a real vector field \( b_{\mu} \); the Fermionic fields are some Majorana fields of spin one-half \( f^{(A)}, \quad A = 1, \ldots, f \). Let us consider
\[ C \equiv \sum_{j} \gamma_{j} b^{(j)} \] (3.1)

for some real constants \( \gamma_{j} \), not all of them zero (i.e. in vector notations \( \vec{\gamma} \neq 0 \) because otherwise we would have \( C = 0 \) also). In particular \( C \) can be one of the scalar fields \( b^{(j)} \). Now we define the following superfield: \( V \equiv s(C) \) i.e.
\[ V(x, \theta, \bar{\theta}) \equiv W_{\theta, \bar{\theta}} C(x) W_{\theta, \bar{\theta}}^{-1}. \] (3.2)

It is clear that one has the reality condition
\[ V^{\dagger} = V; \] (3.3)
this is not a restriction: is $V$ is not Hermitian we consider its Hermitian and anti-Hermitian parts separately.

Moreover the generic expression of $V$ must be

$$V(x, \theta, \bar{\theta}) = C(x) + \theta \chi(x) + \bar{\theta} \bar{\chi}(x) + \theta^2 \phi(x) + \bar{\theta}^2 \phi^\dagger(x)$$

$$+ (\theta \sigma^\mu \bar{\theta}) v_\mu(x) + \theta^2 \bar{\theta} \lambda(x) + \bar{\theta}^2 \theta \lambda(x) + \theta^2 \bar{\theta}^2 d(x)$$  \hspace{1cm} (3.4)

where, from Lorentz covariance arguments, we must have:

$$\chi = \sum_A \alpha_A f^{(A)} \quad \lambda = \sum_A \beta_A f^{(A)}$$

$$d = \sum_j \delta_j b^{(j)} + \delta_0 \partial^\mu b_\mu$$

$$\phi = \sum_j \rho_j b^{(j)} + \rho_0 \partial^\mu b_\mu$$

$$v_\mu = \tau_0 b_\mu + \sum_j \tau_j \partial_\mu b^{(j)}$$  \hspace{1cm} (3.5)

for some (complex) numbers $\vec{\alpha}, \vec{\beta}, \vec{\rho}, \vec{\tau}, \delta_0, \rho_0, \tau_0$. From the reality condition we must have:

$$d^\dagger = d, \quad v_{\mu}^\dagger = v_\mu$$  \hspace{1cm} (3.6)

so $\vec{\delta}, \vec{\tau}, \delta_0, \tau_0$ must be real.

Now we determine the action of the supercharges on the components of the multiplet.

**Proposition 3.1** In the preceding conditions, the following relations are true:

$$i [Q_a, C] = \chi_a$$

$$\{Q_a, \chi_b\} = 2i \epsilon_{ab} \phi$$

$$\{Q_a, \bar{\chi}_b\} = -i \sigma^{\mu}_{ab} (v_\mu + i \partial_\mu C)$$

$$[Q_a, \phi] = 0$$

$$i [Q_a, \phi^\dagger] = \lambda_a - \frac{i}{2} \sigma^{\mu}_{ab} \partial_\mu \bar{\chi}_b$$

$$i [Q_a, v_\mu] = \sigma^{\mu}_{ab} \bar{\lambda}_b - \frac{i}{2} \partial_\mu \lambda_a - \sigma^{\mu\rho}_{ab} \partial_\rho \lambda_b$$

$$\{Q_a, \lambda_b\} = i \epsilon_{ab} \left( 2d + \frac{i}{2} \partial^\rho v_\rho \right) - i \sigma^{\mu\rho}_{ab} \partial_\mu v_\rho$$

$$\{Q_a, \bar{\lambda}_b\} = \sigma^{\mu}_{ab} \partial_\mu \phi$$

$$[Q_a, d] = -\frac{1}{2} \sigma^{\mu}_{ab} \partial_\mu \bar{\lambda}_b$$  \hspace{1cm} (3.7)

These relations are compatible with the Jacoby identities (2.26).
If $C$ verifies the Klein-Gordon equation (for mass $m$)
\[
(\partial^2 + m^2)C = 0 \quad (3.8)
\]
then the superfield $V$ verifies the Klein-Gordon equation
\[
(\partial^2 + m^2)V = 0 \quad (3.9)
\]
so all the components of the multiplet are verifying Klein-Gordon equation of mass $m$. These equations are compatible with the supersymmetry action i.e. they are left invariant by the supercharges $Q_a$ and $\bar{Q}_{\bar{a}}$.

The multiplet $(C, \phi, v_\mu, d, \lambda_a, \chi_a)$ is irreducible; in particular it follows that the indices $j$ and $A$ take four values, $C$ and $d$ are real scalar fields of mass $m$, $b_\mu$ is a real vector field of mass $m$, $\phi$ is a complex scalar field of mass $m$ and $\chi_a$ and $\lambda_a$ are Dirac fields of mass $m$ (both of them being equivalent to a pair of Majorana fields in the sense of proposition 2.1).

**Proof:** We use the relation (2.36)
\[
i[Q_a, V(x, \theta, \bar{\theta})] = D_a V(x, \theta, \bar{\theta}) \quad i[\bar{Q}_{\bar{a}}, V(x, \theta, \bar{\theta})] = \bar{D}_{\bar{a}} V(x, \theta, \bar{\theta}) \quad (3.10)
\]
and if we introduce in both hand sides the expression (3.4) of the vector superfield we obtain by straightforward computations the action of the supercharges (3.7). The verification of the relations (2.26) is long but also straightforward. One can avoid this long computations if one derives immediately from the preceding relation that:
\[
\{Q_a, [Q_b, V]\} = -(a \leftrightarrow b)
\]
\[
\{Q_a, [\bar{Q}_{\bar{b}}, V]\} = \{\bar{Q}_{\bar{b}}, [Q_a, V]\} = -2i \sigma_{ab}^\mu \partial_\mu V; \quad (3.11)
\]
if we introduce here the expression (3.4) of $V$ and consider the various coefficients of the Grassmann variables, then one obtains (2.26).

The assertion concerning the Klein-Gordon equation is immediate. The irreducibility of the multiplet follows by *reductio ad absurdum*. One admits that a relation of the type
\[
\alpha C + \beta \phi + \bar{\beta} \phi^\dagger + \gamma \partial^\mu v_\mu + \delta d = 0 \quad (3.12)
\]
is true (this being the most general linear dependence between the Bosonic fields compatible with Lorentz covariance; higher derivatives are excluded if one uses Klein-Gordon and Dirac equations). Then one commutes twice with the supercharges and discovers some contradictions. If one has a relations between the Fermi fields, the the anticommutator with the supercharges gives a relation of the preceding type.

**Remark 3.2** The relations (3.7) are, essentially, those from the literature - see for instance [12] formula (3.6.5) - in our notations. If one makes the change of fields [26]
\[
\lambda'_a \equiv \lambda_a + \frac{i}{2} \sigma_{ab}^\mu \partial_\mu \chi^b \\
d' \equiv d - \frac{m^2}{4} C
\]

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then the (3.7) acquires a somewhat simple form:

\[ i [Q_a, C] = \chi_a \]
\[ \{Q_a, \chi_b\} = 2i \epsilon_{ab} \phi \]
\[ \{Q_a, \bar{\chi}_b\} = -i \sigma_{ab}^\mu (v^\mu + i \partial_\mu C) \]
\[ [Q_a, \phi] = 0 \]
\[ i [Q_a, \phi^\dagger] = \chi'_a - i \sigma_{ab}^\mu \partial_\mu \bar{\chi}^b \]
\[ i [Q_a, v^\mu] = \sigma_{ab}^\mu \bar{\chi}^b - i \partial^\mu \chi_a \]
\[ \{Q_a, \chi'_b\} = 2i \epsilon_{ab} d^l - 2i \sigma_{ab}^\mu \partial_\mu v_\rho \]
\[ \{Q_a, \bar{\chi}'_b\} = 0 \]
\[ [Q_a, d^l] = -\frac{1}{2} \sigma_{ab}^\mu \partial_\mu \bar{\chi}^b. \] (3.14)

It is clear that the one-particle Hilbert space of the vector multiplet is bigger than the representation \( \Omega_{1/2} \) described by formula (1.1). To obtain the “physical” Fock space associated to \( \Omega_{1/2} \) we have to follow the idea outlined in the Introduction, namely to extend the Hilbert space of the vector field with some superghost and anti-superghost fields and find a gauge charge operator \( Q \) such that the “physical” Fock space is given by the formula \( \text{Ker}(Q)/\text{Im}(Q) \).

A hint about this construction is given by

**Proposition 3.3** The vector superfield \( V \) can be written as follows

\[ V = \sum_{j=0}^{2} P_j V \] (3.15)

where the expressions \( P_j, \ j = 0, 1, 2 \) are given by

\[ P_0 \equiv -\frac{1}{8m^2} \mathcal{D}^a \mathcal{D}^2 \mathcal{D}_a, \quad P_1 \equiv \frac{1}{16m^2} \mathcal{D}^2 \mathcal{D}^2, \quad P_2 \equiv \frac{1}{16m^2} \mathcal{D}^2 \mathcal{D}^2. \] (3.16)

The expressions \( P_j, \ j = 0, \ldots, 2 \) are projectors on the mass shell i.e. they verify

\[ P_j P_k = 0, \quad \forall j \neq k, \]
\[ P_j^2 V = P_j V, \quad \forall j. \] (3.17)

The components \( V_j \equiv P_j V, \ j = 1, 2 \) of \( V \) verify

\[ \mathcal{D}_a V_1 = 0, \quad \bar{\mathcal{D}}_a V_2 = 0. \] (3.18)

**Proof:** The proof follows from some elementary identities verified by \( \mathcal{D}_a \) and \( \bar{\mathcal{D}}_a \):

\[ \mathcal{D}^2 \mathcal{D}^2 + \bar{\mathcal{D}}^2 \mathcal{D}^2 - 2 \mathcal{D}^a \mathcal{D}^2 \mathcal{D}_a = -16\partial^2, \]
\[ (\mathcal{D}^2 \mathcal{D}^2)^2 = -16\partial^2 \mathcal{D}^2 \mathcal{D}^2, \quad (\bar{\mathcal{D}}^2 \mathcal{D}^2)^2 = -16\partial^2 \bar{\mathcal{D}}^2 \mathcal{D}^2 \]
\[ \mathcal{D}_a \mathcal{D}_b \mathcal{D}_c = 0, \quad \bar{\mathcal{D}}_a \bar{\mathcal{D}}_b \bar{\mathcal{D}}_c = 0. \] (3.19)
The proof of these identities is elementary (see for instance [26]).
The relations (3.18) are called chiralities (resp. anti-chiralities) conditions.
We now have directly from (2.38) the following

**Proposition 3.4** Let us define

\[
V_\mu \equiv s(v_\mu) \quad D \equiv s(d) \quad D' \equiv s(d') \quad \Phi \equiv s(\phi)
\]

\[
X_a \equiv s(x_a), \quad \Lambda_a \equiv s(\lambda_a) \quad \Lambda'_a \equiv s(\lambda'_a).
\]

Then the following relations are true:

\[
\begin{align*}
D_a V &= X_a \\
\bar{D}_a X_b &= -2\epsilon_{ab}\Phi \\
D_a \bar{X}_b &= \sigma_{ab}^\mu (V_\mu + i \partial_\mu V) \\
\bar{D}_a \Phi &= 0 \\
D_a \Phi^\dagger &= \Lambda'_a - i \sigma_{ab}^\mu \partial_\mu \bar{X}^b \\
D_a V^\mu &= \sigma_{ab}^\mu \bar{X}^b - i \partial^\mu X_a \\
D_a \Lambda'_b &= -2\epsilon_{ab} D' + 2\sigma_{\mu \rho}^ab \partial_\mu V_\rho \\
\bar{D}_a \Lambda'_b &= 0 \\
D_a D' &= -i \frac{2}{\sigma_{ab}^\mu \partial_\mu \bar{X}^b}.
\end{align*}
\]

Using these relations one can express all associated superfields from (3.20) as some polynomial in the operators \(D_a\) and \(\bar{D}_a\) applied to \(V\); in particular the following algebraic relations:

\[
\Phi = -\frac{1}{4} D^2 V \quad \partial_\mu V^\mu = \frac{i}{16} [\bar{D}^2, D^2] V \\
D = \frac{1}{32} (D^2\bar{D}^2 + \bar{D}^2 D^2) V - \frac{m^2}{4} V
\]

(3.22)
can be obtained. One can determine by direct computation the expression of the superfield \(V_0\):

\[
V_0 = -\frac{2}{m^2} D'
\]

\[
D' = d' - \frac{i}{2} \theta \sigma^\mu \partial_\mu \bar{\lambda}' + \frac{i}{2} \partial_\mu \lambda' \sigma^\mu \bar{\theta} - \frac{1}{2} (\theta \sigma^\mu \bar{\theta}) (m^2 g_{\mu \rho} + \partial_\mu \partial_\rho) v^\rho \\
- \frac{m^2}{4} (\theta^2 \bar{\theta} \lambda' + \bar{\theta}^2 \theta \lambda') - \frac{m^2}{4} \theta^2 \bar{\theta}^2 d'
\]

(3.23)

so the superfield \(V_0\) contains only one Majorana spinor \(\lambda'\), one scalar field \(d'\) and a real vector field

\[
v'_\mu \equiv \left( g_{\mu \rho} + \frac{1}{m^2} \partial_\mu \partial_\rho \right) v^\rho
\]

(3.24)

verifying the transversality condition

\[
\partial^\mu v'_\mu = 0.
\]

(3.25)

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If one applies the superfield $V_0(x)$ to the vacuum one can see that $\Omega_{1/2}$ is generated. So, to obtain the “physical” Hilbert space associated to $\Omega_{1/2}$ one has to eliminate the chiral and the anti-chiral parts of $V$. As above, one can determine by direct computation that the superfields $V_1$ and $V_2$ do not contain particles of spin 1; indeed the field $v_\mu$ appears in the two superfields only through the combination $\partial^\mu v_\mu$.

We now determine the supercommutator of the vector field. We have the following result.

**Theorem 3.5** The vector multiplet exists only for $m > 0$. In this case, the generic form of the causal (anti)commutators of the fields are:

\[
[C(x), C(y)] = -i \, D_m(x - y) \\
[C(x), d(y)] = -i \, \alpha \, D_m(x - y) \\
[C(x), \phi(y)] = -i \, \beta \, D_m(x - y) \\
[\phi(x), \phi^+(y)] = -i \left( \alpha + \frac{m^2}{4} \right) \, D_m(x - y) \\
[\phi(x), d(y)] = \frac{m^2 \beta}{4} \, D_m(x - y) \\
[\phi(x), v_\mu(y)] = i \, \beta \, \partial_\mu D_m(x - y) \\
[d(x), d(y)] = -\frac{im^4}{16} \, D_m(x - y) \\
[v_\mu(x), v_\rho(y)] = i \, \partial_\mu \partial_\rho \, D_m(x - y) + i \left( \frac{m^2}{2} - 2\alpha \right) \, g_{\mu\rho} \, D_m(x - y)
\]

\[
\{\chi_a(x), \chi_b(y)\} = 2\beta \, \epsilon_{ab} \, D_m(x - y), \\
\{\chi_a(x), \bar{\chi}_b(y)\} = \sigma^\mu_{ab} \, \partial_\mu D_m(x - y) \\
\{\lambda_a(x), \lambda_b(y)\} = -\frac{m^2 \beta}{2} \, \epsilon_{ab} \, D_m(x - y), \\
\{\lambda_a(x), \bar{\lambda}_b(y)\} = \frac{m^2}{4} \, \sigma^\mu_{ab} \, \partial_\mu D_m(x - y) \\
\{\chi_a(x), \lambda_b(y)\} = -2i\alpha \, \epsilon_{ab} \, D_m(x - y), \\
\{\chi_a(x), \bar{\lambda}_b(y)\} = i\beta \, \sigma^\mu_{ab} \, \partial_\mu D_m(x - y)
\]

(3.26)

and all other (anti)commutators are zero. Here $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$ are two free parameters constrained only by the inequalities

\[
|\alpha| \leq \frac{m^2}{4} \quad |\beta| \leq \frac{m}{2} \quad |Im(\beta)| \leq \frac{m}{4} + \frac{\alpha}{4}.
\]

(3.27)

**Proof:** Starting directly from (2.4) + (2.7) + (2.10) and (3.1) + 3.5) we immediately get:

\[
[C(x), C(y)] = -i \, |\gamma|^2 \, D_m(x - y) \\
[C(x), d(y)] = -i \, \bar{\gamma} \cdot \bar{\delta} \, D_m(x - y) \\
[C(x), \phi(y)] = -i \, \bar{\gamma} \cdot \bar{\rho} \, D_m(x - y)
\]
Let us consider the causal commutator vector superfield. We first have:

\[ [C(x), v_\mu(y)] = -i \bar{\gamma} \cdot \bar{\tau} \partial_\mu D_m(x - y) \]
\[ [\phi(x), \phi(y)] = -i \left( |\rho|^2 - m^2 \rho_0^2 \right) D_m(x - y) \]
\[ [\phi(x), \phi^\dagger(y)] = -i \left( \bar{\rho} \cdot \bar{\rho} - m^2 |\rho_0|^2 \right) D_m(x - y) \]
\[ [\phi(x), d(y)] = -i \left( \bar{\rho} \cdot \bar{\tau} - m^2 \rho_0 \tau_0 \right) D_m(x - y) \]
\[ [\phi(x), v_\mu(y)] = i \left( \rho \cdot \tau - m^2 \rho_0 \tau_0 \right) D_m(x - y) \]
\[ [d(x), d(y)] = -i \left( |\delta|^2 - m^2 \delta_0^2 \right) D_m(x - y) \]
\[ [d(x), v_\mu(y)] = i \left( \delta \cdot \tau - \delta_0 \tau_0 \right) D_m(x - y) \]

Now we consider the first equation. Because \( \bar{\gamma} \neq 0 \) we can rescale \( C \) (and implicitly \( V \)) and make \( |\bar{\gamma}| = 1 \); in this way we arrange that we have the first equation of (3.26). Next we define the real numbers \( \alpha, \gamma \) and the complex number \( \beta \) according to

\[ \alpha \equiv \bar{\gamma} \cdot \delta \quad \beta \equiv \bar{\gamma} \cdot \rho \quad \gamma \equiv \bar{\gamma} \cdot \bar{\tau}. \] (3.29)

If we consider all non-trivial Jacobi identities (2.27) we obtain after some computation that \( \gamma = 0 \) and the rest of the relations of (3.26).

If \( m = 0 \) we get from (3.26) \( \{ \lambda_a(x), \lambda_b(y) \} = 0 \) so using (3.28) we get \( \bar{\beta} \cdot \bar{\beta}^* = 0 \); this implies \( \bar{\beta} = 0 \) so \( \lambda_a = 0 \), absurd. So we must have \( m > 0 \). In this case the inequalities from the statement follow from the Cauchy-Schwartz inequalities: one splits all complex vectors in the real and complex part and considers all pair of vectors so obtained.

We now give an alternative way of computing the causal (anti)commutation relations having the advantage of being more abstract and giving directly the causal super-commutator of the vector superfield. We first have:

**Proposition 3.6** Let us consider the causal commutator

\[ [V(X_1), V(X_2)] = -i D(X_1; X_2) \ 1. \] (3.30)

Then the expression \( D(X_1; X_2) \) is a distribution in the variables \( x_j \) and a polynomial in the Grassmann variables \( \theta_j, \quad j = 1, 2 \) and verifies the following properties:

(a) it is Poincaré covariant; in particular it depends only on the difference \( x_1 - x_2 \);
(b) it has causal support;
(c) verifies Klein-Gordon equation:

\[(\partial^2 + m^2)D(X_1; X_2) = 0; \] (3.31)

(d) verifies the Hermiticity condition:

\[\overline{D(X_1; X_2)} = -D(X_2; X_1); \] (3.32)

(e) verifies the antisymmetry condition:

\[D(X_2; X_1) = -D(X_1; X_2). \] (3.33)

(f) verifies the consistency condition:

\[(D_a^1 + D_a^2)D(X_1; X_2) = 0; \] (3.34)

**Proof:** All properties except (f) are immediate. If we start from the Jacobi identity

\[
[Q_a, [V(X_1), V(X_2)]] + [V(X_1), [V(X_2), Q_a]] + [V(X_2), [Q_a, V(X_1)]] = 0 \] (3.35)

and we use (3.10) to obtain

\[(D_a^1 + D_a^2) [V(X_1), V(X_2)] = 0 \] (3.36)

and (f) follows.

**Proposition 3.7** The generic solution of the conditions (a)-(f) of the preceding proposition is

\[D(X_1; X_2) = \exp[i (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \partial_\mu] E(\theta_1 - \theta_2; \bar{\theta}_1 - \bar{\theta}_2; x_1 - x_2) \] (3.37)

where the expression \(E\) is a distribution in the variable \(x\) and a polynomial in the Grassmann variables \(\zeta \equiv \theta_1 - \theta_2\) and verifies the following properties:

(a) it is Lorentz covariant;
(b) it has causal support;
(c) verifies Klein-Gordon equation:

\[(\partial^2 + m^2)E = 0; \] (3.38)

(d) verifies the Hermiticity condition:

\[\overline{E(\zeta; \bar{\zeta}; x)} = -E(-\zeta; -\bar{\zeta}; -x); \] (3.39)

(e) verifies the antisymmetry condition:

\[E(\zeta; \bar{\zeta}; x) = -E(-\zeta; -\bar{\zeta}; -x); \] (3.40)
Proof: One rewrites the consistency condition (f) in the variables
\[ \theta \equiv \frac{1}{2}(\theta_1 + \theta_2), \quad \zeta = \theta_1 - \theta_2 \] (3.41)
and obtains
\[ \left( \frac{\partial}{\partial \theta^a} + i \sigma^\mu_{ab} \tilde{\zeta}^b \partial_\mu \right) D = 0. \] (3.42)

This equation can be “integrated” to
\[ D(\theta, \bar{\theta}; \zeta, \bar{\zeta}; x) = \exp[i (\zeta \sigma^\mu \bar{\theta} - \bar{\theta} \sigma^\mu \zeta) \partial_\mu] E(\zeta, \bar{\zeta}; x); \] (3.43)

In particular the expressions \( D_j \) do not contain terms with odd number of Grassmann variables. In consequence any Bose field are commuting with any Fermi field.

Proposition 3.8 The solutions of the problem from proposition 3.6 generate a real vector space of dimension 4; a basis in this space can be taken to be:
\[ D_1(X_1; X_2) = \exp[i (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \partial_\mu] D_m(x_1 - x_2) \]
\[ D_2(X_1; X_2) = (\theta_1 - \theta_2)^2 (\bar{\theta}_1 - \bar{\theta}_2)^2 D_m(x_1 - x_2) \]
\[ D_3(X_1; X_2) = \exp[i (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \partial_\mu] [(\theta_1 - \theta_2)^2 + (\bar{\theta}_1 - \bar{\theta}_2)^2] D_m(x_1 - x_2) \]
\[ D_4(X_1; X_2) = i \exp[i (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \partial_\mu] [(\theta_1 - \theta_2)^2 - (\bar{\theta}_1 - \bar{\theta}_2)^2] D_m(x_1 - x_2) \] (3.44)

Proof: The generic form of \( E \) is, from Lorentz covariance considerations:
\[ E = A_1 + \zeta^2 A_2 - \bar{\zeta}^2 A_3 + \zeta \sigma^\mu \tilde{\zeta} \partial_\mu A_4 + \zeta^2 \bar{\zeta}^2 A_5 \] (3.45)

where \( A_j, \ j = 1, \ldots, 5 \) are numerical distributions. One can impose now the restrictions (a) to (e) from the preceding proposition and gets quite easily the result from the statement.

It follows that the general solution of the problem (a) to (f) from the proposition 3.6) is of the form:
\[ D(X_1; X_2) = \sum_{j=1}^{4} c_j D_j(X_1; X_2). \] (3.46)

After some computations one can match this expression with (3.26); we have:
\[ c_1 = 1, \quad \alpha = c_2, \quad \beta = c_3 - i c_4 \] (3.47)
so the super-order of singularity is:
\[ \omega(D(X_1; X_2)) = -2. \] (3.48)

We also give some interesting relations verified by the expressions \( D_j(X_1; X_2) \). We have:
Proposition 3.9  The following relations are true:

\[ D^2 D_1 = -\frac{m^2}{2} (D_3 + i D_4) \]
\[ D^2 D_2 = -2 (D_3 + i D_4) \]
\[ D^2 D_3 = -4 D_1 - m^2 D_2 - 4i D_5 \]
\[ D^2 D_4 = -4i D_1 - i m^2 D_2 + 4 D_5 \]
\[ D^2 D_5 = -i m^2 (D_3 + i D_4) \]

(3.49)

where we have defined

\[ D_5(X_1; X_2) = \exp[i \left( \theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1 \right) \partial_\mu] \left( \theta_1 - \theta_2 \right) \sigma^\rho (\bar{\theta}_1 - \bar{\theta}_2) \partial_\rho D_m(x_1 - x_2) \]

(3.50)

and the operator \( D^2 \) pertains to the variable \( X_1 \).

Proof: If we consider only the conditions (a) to (d) from the proposition 3.6 we obtain a complex vector space of dimension 5 with a basis given by \( D_j \), \( j = 1, \ldots, 5 \). Now it is clear that the expressions \( D^2 D_j \) are also verifying the conditions (a) to (d), so we must have relations of the following type:

\[ D^2 D_j = \sum_{k=1}^{5} c_{jk} D_k \]

(3.51)

for some complex coefficients \( c_{jk} \). To determine these coefficients we make \( \theta_2 \to 0, \bar{\theta}_2 \to 0 \) and some simple computations.

In perturbation theory we need the expression of the Feynman super-propagator. This can be obtained from the expression of the super-causal distribution \( D \) by distribution splitting [25], [10]; in this simple case this amounts to make the replacement

\[ D_m(x) \quad \rightarrow \quad D_m^F(x) \]

(3.52)

where at the right-hand side we have the usual expression of the Feynman propagator. This means that we have

\[ D^F(X_1; X_2) = \sum_{j=1}^{4} c_j D_j^F(X_1; X_2). \]

(3.53)

where the expressions \( D_j^F(X_1; X_2) \) are obtained from (3.44) with the substitution (3.52).

Let us emphasise now an important departure from the standard literature. The expression of the super-propagator appearing in the standard literature is \( D_2^F(X_1; X_2) \) - see for instance [7] formula (5.23). But one can immediately see that the choice \( c_1 = 0 \) is in conflict with the basic theorem about the structure of the super-causal distribution \( D \). Indeed, if the causal super-commutator is \( D_2 \) this means in particular that we have \([C(x), C(y)] = 0\) which implies \( C = 0, \ V = 0 \). So, it seems that the formal manipulations based on the formal path integral integration are not completely safe: one can obtain in this way formal theories which do not have a \textit{bona fide} representation in a Hilbert space. These theories are in obvious conflict with good old fashion quantum mechanics! Another departure from the standard literature is the proof that the model exists only for \( m > 0 \).
4 Chiral Multiplets

4.1 The Chiral Scalar Multiplet

The decomposition given by Proposition 3.3 shows that the ghost and the antighost superfields should be constrained by two restrictions: they should not generate spin 1 particles and they should obey the chirality condition. These two conditions determine what it is called in the literature the scalar chiral superfield. (We will also need the ghost version of such a superfield.)

By definition a **scalar chiral superfield** is a superfield $H(x, \theta, \bar{\theta})$ verifying the following conditions:

(a) it corresponds to a multiplet of fields of spin $s \leq 1/2$;
(b) it is a scalar with respect to Poincaré transformations;
(c) it verifies the chirality condition:

$$D_a H = 0; \quad (4.1)$$

the anti-chirality condition is obviously

$$\bar{D}_a H = 0. \quad (4.2)$$

If $H$ is a chiral superfield, then $H^\dagger$ is an anti-chiral superfield and vice-versa so we can study only one of them, say chiral superfields. It is easy to determine the generic form of a chiral superfield; from Lorentz covariance we get

$$H(x, \theta, \bar{\theta}) = h(x) + 2 \bar{\theta} \bar{\psi}(x) + i (\theta \sigma^\mu \bar{\theta}) \partial_\mu h(x) + \bar{\theta}^2 f(x) - i \bar{\theta}^2 \theta \sigma^\mu \partial_\mu \bar{\psi}(x) + \frac{m^2}{4} \theta^2 \bar{\theta}^2 h(x). \quad (4.3)$$

where $h, f$ and $\psi$ have expressions of the type (3.5):

$$h = \sum_j \rho_j b^{(j)}, \quad f = \sum_j \delta_j b^{(j)}, \quad \psi = \sum_A \alpha_A f^{(A)} \quad (4.4)$$

for some real scalar fields $b^{(j)}$ and some Majorana fields $f^{(A)}$. These are free fields i.e. Klein-Gordon, respectively Dirac equations are verified. We proceed as for the vector multiplet.

**Proposition 4.1** Let us suppose that $H = s(h)$. Then the action of the supercharges on the chiral multiplet, consistent with the conditions (2.26) is:

$$i [Q_a, h] = 0, \quad i [Q_a, h^\dagger] = 2\psi_a$$

$$[Q_a, f] = -2 \sigma^\mu_{ab} \partial_\mu \bar{\psi}^b \quad [Q_a, f^\dagger] = 0$$

$$\{Q_a, \psi_b\} = i \epsilon_{ab} f^\dagger \quad \{Q_a, \bar{\psi}_b\} = \sigma^\mu_{ab} \partial_\mu h. \quad (4.5)$$

**As a result, if the field $h$ is of mass $m$ all the fields of the multiplet must be of mass $m$.**

**Proof:** As for the vector superfield we have from (2.36)

$$i [Q_a, H(x, \theta, \bar{\theta})] = D_a H(x, \theta, \bar{\theta}) \quad i [\bar{Q}_a, H(x, \theta, \bar{\theta})] = \bar{D}_a H(x, \theta, \bar{\theta}) \quad (4.6)$$
and if we introduce the expression (4.3) we get the action of the supercharges. The consistency conditions (2.26) are verified by direct computation. The last assertion is elementary.

If we do not impose additional constraints, then we must have in the general scheme from Section 2 (namely the relations (2.2) - (2.10)) $s = 4$ and $f = 2$ so the chiral multiplet is reducible: we have twice the representation

$$\Omega_0 \sim [m, 0] \oplus [m, 0] \oplus [m, 1/2]$$

(4.7)
of the super-Poincaré group. Also from (2.38) we have

**Proposition 4.2** Let us define

$$F \equiv s(f), \quad \Psi_a \equiv s(\psi_a).$$

(4.8)

Then the following relations are true:

$$\mathcal{D}_a H = 0, \quad \mathcal{D}_a H^\dagger = 2 \Psi_a$$

$$\mathcal{D}_a F = -2i \sigma^\mu_{ab} \partial_\mu \bar{\Psi}^b \quad \mathcal{D}_a F^\dagger = 0$$

$$\mathcal{D}_a \Psi_b = -\epsilon_{ab}F, \quad \mathcal{D}_a \bar{\Psi}^b = i \sigma^\mu_{ab} \partial_\mu H.$$ (4.9)

As a consequence we have

$$\bar{\mathcal{D}}^2 H = -4 F \quad \mathcal{D}^2 F = -4m^2 H \quad \mathcal{D}^2 \Psi_a = 0.$$ (4.10)

In particular we have the “equation of motion”

$$\mathcal{D}^2 \bar{\mathcal{D}}^2 H = 16m^2 H.$$ (4.11)

It was proved in [17] (see also [13]) that any supersymmetric multiplet of fields with spin $s \leq 1/2$ is a sum of Wess-Zumino multiplets [34]. By definition, a Wess-Zumino multiplet is composed from a complex scalar field $h$ and a spin $1/2$ Majorana field $f_a$ of the same mass $m$. In particular $f_a$ verifies Dirac equation. The relations (2.24) are in this case by definition:

$$[Q_a, h] = 0 \quad i [Q_a, h^\dagger] = 2 f_a$$

$$\{Q_a, f_b\} = -i \epsilon_{ab} h, \quad \{Q_a, \bar{f}_b\} = \sigma^\mu_{ab} \partial_\mu h.$$ (4.12)

This multiplet is irreducible. The causal (anti)commutators are:

$$[h(x), h(y)^\dagger] = -2i \alpha D_m(x - y),$$

$$\{f_a(x), f_b(y)\} = i \alpha \epsilon_{ab} m D_m(x - y),$$

$$\{f_a(x), \bar{f}_b(y)\} = \alpha \sigma^\mu_{ab} \partial_\mu D_m(x - y)$$ (4.13)

and the other (anti)commutators are zero; here $\alpha \in \mathbb{R}^+$ is a arbitrary parameter. One can prove them if one starts from the first relation and uses the consistency conditions (2.27).

We now show that the chiral multiplet is a sum of two Wess-Zumino multiplets.
Proposition 4.3  In the conditions of proposition 4.1 let us define

\[ \psi_a^{(+)} \equiv i \sigma^{\mu}_{ab} \partial_{\mu} \bar{\psi}^b + m \psi_a \quad h^{(+)} \equiv mh - f^\dagger \]
\[ \psi_a^{(-)} \equiv \sigma^{\mu}_{ab} \partial_{\mu} \bar{\psi}^b + i m \psi_a \quad h^{(-)} \equiv i \left( mh + f^\dagger \right). \]  \hspace{1cm} (4.14)

The the couples \((h_a^{(\pm)}, \psi^{(\pm)})\) are two Wess-Zumino multiplets.

Proof: By direct computation one proves that for both couples the relations (4.12) are verified. Then one notices that

\[ h = \frac{1}{2m} \left[ h^{(+)} + i h^{(-)} \right] \quad f = \frac{1}{2} \left[ h^{(-)} - h^{(+)} \right] \quad \psi_a = \frac{1}{2m} \left[ \psi_a^{(+)} - i \psi_a^{(-)} \right] \]  \hspace{1cm} (4.15)

so the correspondence \((\psi_a, h) \leftrightarrow (\psi_a^{(\pm)}, h^{(\pm)})\) is one-one.

One usually obtains the Wess-Zumino multiplet from the chiral multiplet by imposing

\[ \psi_a^{(-)} = 0 \quad h^{(-)} = 0 \]  \hspace{1cm} (4.16)

which are equivalent to the equation of motion namely:

\[ \bar{D}^2 H = 4mH^\dagger. \]  \hspace{1cm} (4.17)

The determination of the causal (anti)commutation relation for the chiral multiplet follows the usual pattern.

Proposition 4.4  The generic causal (anti)commutators for the chiral multiplet are

\[ [h(x), h^\dagger(y)] = -i \alpha' D_m(x - y) \]
\[ [h(x), f(y)] = \beta' D_m(x - y) \]
\[ [f(x), f^\dagger(y)] = -i m^2 \alpha' D_m(x - y) \]
\[ \{ \psi_a(x), \psi_b(y) \} = -\frac{\beta'}{2} \epsilon_{ab} D_m(x - y), \]
\[ \{ \psi_a(x), \bar{\psi}_b(y) \} = \frac{\alpha'}{2} \sum_{ab} \sigma^{\mu}_{ab} \partial_{\mu} D_m(x - y) \]  \hspace{1cm} (4.18)

and the other (anti)commutators are zero. Here \(\alpha' \in \mathbb{R}^+, \beta' \in \mathbb{C}\) are arbitrary parameters.

Proof: We start with the first two relations which can be justified from (4.4). If we use the consistency conditions (2.27) we obtain the other (anti)commutators. ■

Remark 4.5  If require that the two Wess-Zumino multiplets associated to the chiral multiplet in the sense of the proposition 4.3 are completely decoupled i.e. they (anti)commute one with the other, then we must impose \(\text{Re}(\beta) = 0\). However, the general consistency conditions from Section 2 are valid for arbitrary \(\beta\) so we will not restrict this parameter.
There is an alternative expression of the chiral superfield (4.3)

\[ H(x, \theta, \bar{\theta}) = \exp(\theta \sigma^\mu \bar{\theta} \partial_\mu) \left[ h(x) + 2 \bar{\theta} \psi(x) + \bar{\theta}^2 f(x) \right] \] (4.19)

which can be used to determine the commutation relations of the superfields:

\[ [H(X_1), H(X_2)] = \beta^I D_+(X_1; X_2) \]
\[ [H(X_1), H^\dagger(X_2)] = -i \alpha^I D_-(X_1; X_2) \] (4.20)

where

\[ D_+(X_1; X_2) = (\bar{\theta}_1 - \bar{\theta}_2)^2 \exp[i (\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1) \partial_\mu] \ D_m(x_1 - x_2) \]
\[ D_-(X_1; X_2) = \exp[i (\theta_1 \sigma^\mu \bar{\theta}_1 + \theta_2 \sigma^\mu \bar{\theta}_2 - 2 \bar{\theta}_2 \sigma^\mu \bar{\theta}_1) \partial_\mu] \ D_m(x_1 - x_2) \] (4.21)

or, if we use the causal super-distribution \( D_j, j = 1, \ldots, 5 \) introduced in the preceding Section:

\[ D_+ = \frac{1}{2} (D_3 - i D_4) \quad D_- = D_1 - \frac{m^2}{2} D_2 + i D_5. \] (4.22)

The super-order of singularities are better than the general formula from [13], namely:

\[ \omega(D_-) = -2, \quad \omega(D_+) = -3. \] (4.23)

We end this subsection with a critical analysis of the so-called Wess-Zumino gauge. It is asserted in the literature (see for instance [16]) that one can write a vector superfield in the form

\[ V = V_{WZ} + H + H^\dagger \] (4.24)

where \( H \) is a chiral superfield and \( V_{WZ} \) has the generic form

\[ V_{WZ}(x, \theta, \bar{\theta}) = (\theta \sigma^\mu \bar{\theta}) \ w_\mu(x) + \theta^2 \ \bar{\theta} \omega(x) + \bar{\theta}^2 \ \theta \omega(x) + \theta^2 \bar{\theta}^2 \ v(x); \] (4.25)

here \( v \) is a real scalar fields, \( w_\mu \) is a real vector field and \( \omega_\alpha \) is a Dirac spinor. If one introduces in (4.24) the explicit expressions (3.4) and (4.3) then one immediately get the relations

\[ C = h + h^\dagger \quad \chi_\alpha = 2 \psi_\alpha \quad \phi = f^\dagger. \] (4.26)

We show that the decomposition (4.24) is not supersymmetric invariant. For this we simply take the (anti)commutators of the supercharges \( Q_\alpha \) and \( \bar{Q}_\dot{\alpha} \) with the three preceding relations. In particular we must have

\[ [\bar{Q}_\dot{\alpha}, \phi - f^\dagger] = 0. \] (4.27)

But if we use (3.7) and (4.5) we immediately get

\[ \chi'_\alpha = 0 \] (4.28)

i.e. a contradiction. The conclusion is that the Wess-Zumino gauge is not a legitimate supersymmetric decomposition: if one supposes that \( V \) is a superfields then \( H \) cannot be a superfield and vice-versa.
4.2 Chiral Ghost and Antighost Multiplets

To define in a consistent super-symmetric way ghost and anti-ghost fields one only has to invert the statistics assignment: we assume that the ghost multiplet is build from some scalar fields \( u^{(j)} \) which are Hermitian and respect Fermi-Dirac statistics; their Majorana partners \( f^{(A)}_a \) are Bosons. The anti-ghost multiplet has a similar structure but we change the Hermiticity properties in agreement to the usual conventions [25]: the scalar fields \( \tilde{u}^{(j)} \) are anti-Hermitian and respect Fermi-Dirac statistics; their anti-Majorana partners \( \tilde{f}^{(A)}_a \) are Bosons:

\[
(\tilde{u}^{(j)})^\dagger = -\tilde{u}^{(j)} \quad (f^{(A)}_a)^\dagger = \tilde{f}^{(A)}_a.
\]  

These are free fields i.e. Klein-Gordon, respectively Dirac equations are verified. Now the changes in the preceding arguments are minimal. It is easy to determine the generic form of a chiral ghost and anti-ghost superfields; we get

\[
U(x, \theta, \bar{\theta}) = u(x) + 2i \, \bar{\theta} \zeta(x) + i \, (\theta \sigma^\mu \bar{\theta}) \, \partial_\mu u + \bar{\theta}^2 \, g(x) + \bar{\theta}^2 \, \theta \sigma^\mu \partial_\mu \zeta(x) + \frac{m^2}{4} \, \bar{\theta}^2 \, \theta^2 \, u(x) \quad (4.30)
\]

and respectively

\[
\tilde{U}(x, \theta, \bar{\theta}) = \tilde{u}(x) - 2i \, \bar{\theta} \tilde{\zeta}(x) + i \, (\theta \sigma^\mu \bar{\theta}) \, \partial_\mu \tilde{u} + \bar{\theta}^2 \, \tilde{g}(x) - \bar{\theta}^2 \, \theta \sigma^\mu \partial_\mu \tilde{\zeta}(x) + \frac{m^2}{4} \, \bar{\theta}^2 \, \theta^2 \, \tilde{u}(x) \quad (4.31)
\]

where \( u, g \) (resp. \( \tilde{u}, \tilde{g} \)) are linear combinations of \( u^{(j)} \) (resp. of \( \tilde{u}^{(j)} \)) and \( \zeta \) (resp. \( \tilde{\zeta} \)) are linear combinations of \( f^{(A)} \) (resp. \( \tilde{f}^{(A)} \)).

Instead of the formulæ of the proposition 4.1 we get:

\[
\{ Q_a, u \} = 0 \quad \{ Q_a, u^\dagger \} = 2\zeta_a
\]

\[
\{ Q_a, g \} = -2i \, \sigma^\mu_{ab} \, \partial_\mu \tilde{\zeta}^\dagger \quad \{ Q_a, g^\dagger \} = 0
\]

\[
[ Q_a, \zeta_b ] = \epsilon_{abg} g^\dagger \quad i[ Q_a, \tilde{\zeta}_b ] = \sigma^\mu_{ab} \, \partial_\mu u
\]  

and respectively:

\[
\{ Q_a, \tilde{u} \} = 0 \quad \{ Q_a, \tilde{u}^\dagger \} = 2\tilde{\zeta}_a
\]

\[
\{ Q_a, \tilde{g} \} = 2i \, \sigma^\mu_{ab} \, \partial_\mu \zeta^\dagger \quad \{ Q_a, \tilde{g}^\dagger \} = 0
\]

\[
[ Q_a, \zeta_b ] = \epsilon_{abg} \tilde{g}^\dagger \quad i[ Q_a, \tilde{\zeta}_b ] = -\sigma^\mu_{ab} \, \partial_\mu \tilde{u}.
\]  

In particular all fields of the same multiplet must have the same mass.

If we define

\[
G \equiv s(g), \quad Z_a \equiv s(\zeta_a)
\]  

then we have:

\[
\mathcal{D}_a U = 0 \quad \mathcal{D}_a U^\dagger = 2i \, Z_a
\]

\[
\mathcal{D}_a G = 2 \, \sigma^\mu_{ab} \, \partial_\mu Z^\dagger_b \quad \mathcal{D}_a G^\dagger = 0
\]

\[
\mathcal{D}_a Z_b = i \, \epsilon_{ab} G^\dagger \quad \mathcal{D}_a \tilde{Z}_b = \sigma^\mu_{ab} \, \partial_\mu U.
\]  

(4.35)
and as a consequence we have

\[ \mathcal{D}^2 U = -4 G \quad \mathcal{D}^2 G = -4m^2 \quad \mathcal{D}^2 Z_a = 0. \] (4.36)

In particular we have the “equation of motion”

\[ \mathcal{D}^2 \mathcal{D}^2 U = 16m^2 U. \] (4.37)

Analogously if we define

\[ \tilde{G} \equiv s(\tilde{g}), \quad \tilde{Z}_a \equiv s(\tilde{\zeta}_a) \] (4.38)

then we have:

\[ \mathcal{D}_a \tilde{U} = 0 \quad \mathcal{D}_a \tilde{U}^\dagger = 2i \tilde{Z}_a \]
\[ \mathcal{D}_a \tilde{G} = -2 \sigma_{ab} \partial_\mu \tilde{Z}_b \quad \mathcal{D}_a \tilde{G}^\dagger = 0 \]
\[ \mathcal{D}_a \tilde{Z}_b = i \epsilon_{ab} \tilde{G}^\dagger \quad \mathcal{D}_a \tilde{Z}_b^\dagger = -\sigma_{ab} \partial_\mu \tilde{U}. \] (4.39)

and as a consequence we have

\[ \mathcal{D}^2 \tilde{U} = -4 \tilde{G} \quad \mathcal{D}^2 \tilde{G} = -4m^2 \tilde{U} \quad \mathcal{D}^2 \tilde{Z}_a = 0. \] (4.40)

In particular we have the “equation of motion”

\[ \mathcal{D}^2 \mathcal{D}^2 \tilde{U} = 16m^2 \tilde{U}. \] (4.41)

The ghost multiplet is a sum of elementary ghost multiplets built from a complex scalar field \( u \) with Fermi statistics and a Majorana spinor \( \psi \) with Bose statistics of the same mass \( m \) such that we have instead of (4.12) the following relations [13]:

\[ \{Q_a, u\} = 0 \quad \{Q_a, u^\dagger\} = 2\zeta_a \]
\[ [Q_a, \zeta_b] = -m \epsilon_{ab} u \quad i [Q_a, \tilde{\zeta}_b] = \sigma_{ab} \partial_\mu u \] (4.42)

and for the anti-ghost multiplet of mass \( m \) we have instead:

\[ \{Q_a, \tilde{u}\} = 0 \quad \{Q_a, \tilde{u}^\dagger\} = 2\tilde{\zeta}_a \]
\[ [Q_a, \tilde{\zeta}_b] = m \epsilon_{ab} \tilde{u} \quad i [Q_a, \zeta_b] = -\sigma_{ab} \partial_\mu \tilde{u} \] (4.43)

The decomposition of the chiral ghost and antighost multiplets into irreducible ones goes as above and the formulae are:

\[ \zeta_a^{(+)} \equiv i \sigma_{ab}^\mu \partial_\mu \tilde{Z}_b + m \zeta_a \quad u^{(+)} \equiv m u - g^\dagger \]
\[ \zeta_a^{(-)} \equiv \sigma_{ab}^\mu \partial_\mu \tilde{Z}_b + i m \zeta_a \quad u^{(-)} \equiv -i (m u + g^\dagger) \] (4.44)

and respectively:

\[ \tilde{\zeta}_a^{(+)} \equiv i \sigma_{ab}^\mu \partial_\mu \tilde{\psi} + m \tilde{\zeta}_a \quad \tilde{u}^{(+)} \equiv -(m \tilde{u} + \tilde{g}^\dagger) \]
\[ \tilde{\zeta}_a^{(-)} \equiv \sigma_{ab}^\mu \partial_\mu \tilde{\psi} + i m \tilde{\zeta}_a \quad \tilde{u}^{(-)} \equiv i (m \tilde{u} - \tilde{g}^\dagger). \] (4.45)
The reduction of the chiral ghost and antighost multiplets can be done imposing supersymmetric equations of motion:

\[
\bar{\mathcal{D}}^2 U = 4m U^\dagger \quad \mathcal{D}^2 \tilde{U} = 4m \tilde{U}^\dagger. \tag{4.46}
\]

Up to now, the ghost and the antighost multiplets can be considered of distinct masses. However to consider the commutation relations we remember that for usual gauge theories [25] one has to consider that the ghost and the anti-ghost fields are of the same mass and verify commutation relations of the following type:

\[
\{ u^{(j)}(x), \bar{\tilde{u}}^{(k)}(y) \} = -i \delta_{jk} \ D_m(x - y)
\]

\[
[f_a^{(A)}(x), \tilde{g}_b^{(B)}(y)] = -i \delta_{AB} \epsilon_{ab} \ D_m(x - y)
\]

\[
[f_a^{(A)}(x), \tilde{f}_b^{(B)}(y)] = -\delta_{AB} \sigma_{ab}^\mu \partial_\mu D_m(x - y). \tag{4.47}
\]

Then we have from (2.30) the generic causal (anti)commutator relations for the chiral multiplets:

\[
\{ u(x), \tilde{u}^\dagger(y) \} = i\alpha'' \ D_m(x - y)
\]

\[
\{ u(x), \tilde{u}(y) \} = \beta'' \ D_m(x - y)
\]

\[
\{ g(x), \tilde{g}^\dagger(y) \} = i \ m^2 \alpha'' \ D_m(x - y)
\]

\[
\{ g(x), \tilde{u}(y) \} = \beta'' \ D_m(x - y)
\]

\[
[\zeta_a(x), \tilde{\zeta}_b(y)] = -\frac{\beta''}{2} \epsilon_{ab} D_m(x - y)
\]

\[
[\zeta_a(x), \bar{\tilde{\zeta}}_b(y)] = -\frac{\alpha''}{2} \sigma_{ab}^\mu \partial_\mu D_m(x - y) \tag{4.48}
\]

and the other (anti)commutators are zero. Here $\alpha'', \beta'' \in \mathbb{C}$ are arbitrary parameters.

Finally we note the relations:

\[
U(x, \theta, \tilde{\theta}) = \exp(\theta \sigma^\mu \tilde{\theta} \partial_\mu) \ [u(x) + 2i \ \tilde{\theta} \tilde{\zeta}(x) + \tilde{\theta}^2 \ g(x)]
\]

\[
\tilde{U}(x, \theta, \tilde{\theta}) = \exp(\theta \sigma^\mu \tilde{\theta} \partial_\mu) \ [\tilde{u}(x) - 2i \ \tilde{\theta} \tilde{\zeta}(x) + \tilde{\theta}^2 \ \tilde{g}(x)] \tag{4.49}
\]

which can be used to determine the commutation relations of the superfields:

\[
\{ U(X_1), \tilde{U}(X_2) \} = \beta'' \ D_+(X_1; X_2)
\]

\[
\{ U(X_1), \tilde{U}^\dagger(X_2) \} = i \alpha'' \ D_-(X_1; X_2) \tag{4.50}
\]

where $D_\pm$ are given by (4.21).

5 The Gauge Charge and the Gauge Supermultiplet

In ordinary quantum gauge theory, one gauges away the unphysical degrees of freedom of a vector field $v_\mu$ using ghost fields. Suppose that the vector field is of positive mass $m$; then one enlarges the Hilbert space with three ghost fields $u, \tilde{u}, \phi$ such that:
• All three are scalar fields;
• All them have the same mass $m$ as the vector field.
• The Hermiticity properties are:

$$
\phi^\dagger = \phi, \quad u^\dagger = u, \quad \tilde{u}^\dagger = -\tilde{u}
$$

(5.1)

• The first two ones $u, \tilde{u}$ are Fermionic and $\phi$ is Bosonic.
• The commutation relations are:

$$
[\phi(x), \phi(y)] = -i \, D_m(x - y), \quad \{u(x), \tilde{u}(y)\} = -i \, D_m(x - y)
$$

(5.2)

and the rest of the (anti)commutators are zero.

Then one introduces the gauge charge $Q$ according to:

$$
Q\Omega = 0, \quad Q^\dagger = Q,
$$

$$
[Q, v_\mu] = i \partial_\mu u, \quad [Q, \phi] = i \, m \, u
$$

$$
\{Q, u\} = 0, \quad \{Q, \tilde{u}\} = -i \, (\partial^\mu v_\mu + m \, \phi).
$$

(5.3)

It can be proved that this gauge charge is well defined by these relations i.e. it is compatible with the (anti)commutation relations. Moreover one has $Q^2 = 0$ so the factor space $\text{Ker}(Q)/\text{Im}(Q)$ makes sense; it can be proved that this is the physical space of an ensemble of identical particles of spin 1. For details see [25], [10].

In [13] we have generalised this structure for a new vector multiplet corresponding to the representation $\Omega_1$. We try to do the same thing here for the standard vector multiplet analysed in detail in Section 3. First, it is natural to expect that the Hilbert space of the model should be enlarged as above, containing beside the vector multiplet $V$ a pair of ghost and antighost multiplets $U, \tilde{U}$ and a scalar ghost multiplet $H$. The definition of the gauge charge $Q$ have to verify the consistency relations from Section 2. These relations can be written in a compact way using superfields; the non-trivial ones are

- from (2.30):

$$
[V(X_1), \{U(X_2), Q\}] = - \{U(X_2), [Q, V(X_1)]\}
$$

(5.4)

$$
[V(X_1), \{\tilde{U}(X_2), Q\}] = - \{\tilde{U}(X_2), [Q, V(X_1)]\}
$$

(5.5)

$$
[H(X_1), \{U(X_2), Q\}] = - \{U(X_2), [Q, H(X_1)]\}
$$

(5.6)

$$
[H(X_1), \{\tilde{U}(X_2), Q\}] = - \{\tilde{U}(X_2), [Q, H(X_1)]\};
$$

(5.7)

- from (2.29):

$$
\{Q_a, [Q, V]\} = - \{Q, [Q_a, V]\}
$$

$$
\{Q_a, [Q, H]\} = - \{Q, [Q_a, H]\}
$$

$$
[Q_a, \{Q, U\}] = - [Q, \{Q_a, U\}]
$$

$$
\left[Q_a, \{Q, \tilde{U}\}\right] = - \left[Q, \{Q_a, \tilde{U}\}\right];
$$

(5.8)
\[
\{ Q, [Q,V] \} = 0, \quad \{ Q, [Q,H] \} = 0,
\]
\[
[Q, \{ Q,U \}] = 0, \quad [Q, \{ Q,\bar{U} \}] = 0.
\] (5.9)

(One has to add, of course the relations where some of the superfields are replaced by their hermitian conjugate). Let us try to define the gauge charge \( Q \) postulating
\[
Q\Omega = 0, \quad Q^\dagger = Q
\] (5.10)
and
\[
[Q, V] = U - U^\dagger \quad \{ Q, U \} = 0.
\] (5.11)

The action of the gauge charge on \( V \) is natural if we take into account the discussion following relation (3.23); it is also consistent with the self-adjointness postulated above. Moreover it is in accordance with the usual formulæ [33]. In our context it is important that the relation (5.4) and the relevant relations (5.8) and (5.9) are identically verified.

It will be useful to translate (5.11) in terms of the component fields of the multiplet. It is easy to obtain
\[
[Q, v_\mu] = i\partial_\mu (u + u^\dagger),
\]
\[
[Q, C] = u - u^\dagger, \quad [Q, \phi] = -g^\dagger, \quad [Q, d] = \frac{m^2}{4} (u - u^\dagger),
\]
\[
\{ Q, \chi_a \} = 2i \zeta_a \quad \{ Q, \lambda_a \} = \sigma^\mu_{ab} \partial_\mu \bar{\zeta}^b
\]
\[
\{ Q, u \} = 0 \quad \{ Q, g \} = 0 \quad [Q, \zeta_a] = 0.
\] (5.12)

If we consider only the fields \( v_\mu \) and \( u \) then we are back in the framework of (5.3) with the substitution \( u \to u + u^\dagger \).

We now need the action of the gauge charge on the antighost superfield. It is easier to express the consistency relation (5.5) in component fields. We have:
\[
[C(x_1), \{ \bar{u}(x_2), Q \}] = -\{ \bar{u}(x_2), [Q, C(x_1)] \}
\]
\[
[d(x_1), \{ \bar{u}(x_2), Q \}] = -\{ \bar{u}(x_2), [Q, d(x_1)] \}
\]
\[
[\phi(x_1), \{ \bar{u}(x_2), Q \}] = -\{ \bar{u}(x_2), [Q, \phi(x_1)] \}
\]
\[
[\phi^\dagger(x_1), \{ \bar{u}(x_2), Q \}] = -\{ \bar{u}(x_2), [Q, \phi^\dagger(x_1)] \}
\]
\[
[v_\mu(x_1), \{ \bar{u}(x_2), Q \}] = -\{ \bar{u}(x_2), [Q, v_\mu(x_1)] \}.
\] (5.13)

Because \( \bar{u} \) is a Fermi field, the expression \( \{ Q, \bar{u} \} \) must be a Bose field. From Poincaré covariance arguments we have the ansatz
\[
\{ Q, \bar{u} \} = \delta_1 C + \delta_2 d + \delta_3 \phi + \delta_4 \phi^\dagger + \delta_5 \partial^\mu v_\mu - i h
\] (5.14)

where \( \delta_j, j = 1, \ldots, 5 \) are some complex number and \( h \) is a complex scalar field, such that \( H = s(h) \). We will prove later that one cannot take \( h = 0 \). If we substitute (5.14) in (5.13)
and use (3.26) and (4.48) we get the following system of equations for the numbers $\delta_j$:

\[
\begin{align*}
\delta_1 + \alpha \delta_2 + i \beta \delta_3 - i \bar{\beta} \delta_4 &= \alpha'' \\
\alpha \delta_1 + \frac{m^4}{16} \delta_2 + i \frac{m^2}{4} \beta \delta_3 - i \frac{m^2}{4} \bar{\beta} \delta_4 &= \frac{m^2}{2} \alpha'' \\
\beta \delta_1 + \frac{m^2}{4} \beta \delta_2 - i \left( \frac{m^2}{4} + \alpha \right) \delta_4 + i m^2 \beta \delta_5 &= 0 \\
\bar{\beta} \delta_1 + \frac{m^2}{4} \bar{\beta} \delta_2 + i \left( \frac{m^2}{4} + \alpha \right) \delta_3 - i m^2 \beta \delta_5 &= \beta'' \\
\beta \delta_3 + \bar{\beta} \delta_4 - 2 \left( \frac{m^2}{4} + \alpha \right) \delta_5 &= -i \bar{\alpha}''.
\end{align*}
\]

(5.15)

Now we apply the operator $s$ (the supersymmetric extension) to the relation (5.14) and obtain

\[
\{ Q, \tilde{U} \} = \delta_1 V + \delta_2 D + \delta_3 \Phi + \delta_4 \Phi^\dagger + \delta_5 \partial^\mu V_\mu - i H
\]

(5.16)

or, if we use (3.22)

\[
\{ Q, \tilde{U} \} = \lambda_1 V + \lambda_2 D^2 V + \lambda_3 \bar{D}^2 V + \lambda_4 D^2 \bar{D}^2 V + \lambda_5 D^2 \bar{D}^2 V - i H
\]

(5.17)

where

\[
\begin{align*}
\lambda_1 &= \delta_1 - \frac{m^2}{4} \delta_2 \\
\lambda_2 &= -\frac{1}{4} \delta_3 \\
\lambda_3 &= -\frac{1}{4} \delta_4 \\
\lambda_4 &= \frac{1}{32} (\delta_2 - 2i \delta_5) \\
\lambda_5 &= \frac{1}{32} (\delta_2 + 2i \delta_5).
\end{align*}
\]

(5.18)

One can obtain the expressions of these parameters in a different way using directly the relations (5.5) and the relations derived at proposition 3.9.

Let us apply the operator $D_a$ to the relation (5.17) and take into account that $\tilde{U}$ is a chiral superfield. We get

\[
\lambda_1 D_a V + \lambda_3 D_a \bar{D}^2 V + \lambda_5 D_a D^2 \bar{D}^2 V - i D_a H = 0
\]

(5.19)

and it follows that we should have

\[
\lambda_j = 0, \quad j = 1, 3, 5
\]

(5.20)

and $H$ must be a chiral superfield. If we redefine $\lambda \equiv \lambda_4, \lambda' \equiv \lambda_2$ then we obtain

\[
\{ Q, \tilde{U} \} = \lambda D^2 \bar{D}^2 V + \lambda' D^2 V - i H.
\]

(5.21)

Moreover, the system (5.15) reduces to two equations which can be taken as the definition of the parameters $\alpha'$ and $\beta'$ appearing in the causal anti-commutation relations of the ghost and antighost superfields (4.48) or (4.50):

\[
\begin{align*}
\alpha'' &= 4(4\alpha + m^2)\lambda + 4i \beta \bar{\lambda}' \\
\beta'' &= 16m^2 \bar{\beta} \lambda - i(4\alpha + m^2) \lambda'.
\end{align*}
\]

(5.22)
The presence of the superfield $H$ makes possible the fulfilment of all conditions listed above. Indeed, let us determine the action of the gauge charge on the chiral superfield $H$. This follows from the last relation (5.9):

$$[Q, H] = -16i \, m^2 \, \lambda U + i \, \lambda' \bar{D}^2 U^\dagger;$$

(5.23)

in particular, this shows that we cannot have $H = 0$.

Next, one can compute the causal commutation relation for the superfield $H$. Using (5.7) and the relation with $H \to H^\dagger$ one can obtain relations of the type (4.20) with

$$\alpha' = 4[(4 \alpha + m^2)(4m^2|\lambda|^2 + |\lambda'|^2) + 32m^2\lambda \text{Im}(\beta\lambda')]$$

$$\beta' = 16[16m^4\beta \lambda^2 - 4\beta (\lambda')^2 - i\lambda\lambda'(4\alpha + m^2)]$$

(5.24)

so we have the supplementary condition $\alpha' > 0$. However, one can get rid of the parameter $\lambda$ if one performs the rescalings

$$\tilde{U} \to -16\lambda\tilde{U} \quad H \to -16m \, H;$$

(5.25)

In this way the preceding relations are

$$\{Q, U\} = -\frac{1}{16} \, \bar{D}^2 \bar{D}^2 V + \lambda' \, \bar{D}^2 V - i \, m \, H$$

(5.26)

and

$$[Q, H] = i \, m \, U - \frac{i}{m} \lambda' \bar{D}^2 U^\dagger$$

(5.27)

and the parameters $\alpha', \beta'$ are rescaled by a factor $\frac{1}{m^2}$. The parameters $\alpha, \beta, \lambda'$ remain arbitrary and all other relations of consistency are valid so the gauge structure of the (quantum) vector field is completely determined.

If we want to have complete analogy to the usual ghost structure associated to a massive vector field (see for instance [25], [10]) we get new conditions on the parameter $\lambda'$. Let us note from (5.12) that the ghost field relevant to $v_{\mu}$ is $u + u^\dagger$; if we split $u$ into the hermitian and the anti-Hermitian part $u = u_1 + i \, u_2$, such that $u_j^\dagger = u_j$; then the relevant (Hermitian) ghost field is $2 \, u_1$. Suppose that we decompose $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$ where now $\tilde{u}_j^\dagger = -\tilde{u}_j$; then the anti-ghost field associated to $v_{\mu}$ should be $2 \, \tilde{u}_1$ or, up to a sign $\tilde{u} - \tilde{u}^\dagger$. To have complete analogy to the usual action of the gauge charge on the antighost field we compute the expression $\{Q, \tilde{u} - \tilde{u}^\dagger\}$ and we get directly from (5.14):

$$\{Q, \tilde{u} - \tilde{u}^\dagger\} = (\delta_1 - \bar{\delta}_1) \, C + (\delta_2 - \bar{\delta}_2) \, d + (\delta_3 - \bar{\delta}_3) \, \phi + (\delta_4 - \bar{\delta}_4) \, \phi^\dagger + (\delta_5 - \bar{\delta}_5) \, \partial^\mu v_{\mu}$$

$$-i \, (h + h^\dagger).$$

(5.28)

This should be compared with the last relation (5.3) which shows that we must take

$$\delta_1 = \bar{\delta}_1 \quad \delta_2 = \bar{\delta}_2 \quad \delta_3 = \bar{\delta}_4 \quad \delta_5 = -\bar{\delta}_5$$

(5.29)

and the scalar ghost field relevant to $v_{\mu}$ should be $h + h^\dagger$.  

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The preceding conditions give
\[ \lambda' = 0 \]  
(5.30)
so the gauge structure of the vector field should be
\[ v_{\mu}, u + u^\dagger, \bar{u} - \bar{u}^\dagger, h + h^\dagger. \]  
(5.31)

The gauge transformations of \( \bar{U} \) and \( H \) became very simple in this case:
\[ \{Q, \bar{U}\} = -\frac{1}{16} D^2 \bar{D} V - i m H \]
\[ [Q, H] = i m U. \]  
(5.32)
In this particular case (5.24) and (5.22) become
\[ \alpha' = -(4\alpha + m^2) \]
\[ \beta' = -m^2 \bar{\beta} \]  
(5.33)
and respectively
\[ \alpha'' = -\left(\alpha + \frac{m^2}{4}\right) \]
\[ \beta'' = m^2 \bar{\beta}; \]  
(5.34)
in particular the condition \( \alpha' > 0 \) gives \( 4\alpha + m^2 < 0. \)

The expression \( \{Q, \bar{U}\} \) should be compared with the expression (6.2.23) from [12]. The difference is due to the fact that our definition of chirality corresponds to the definition of anti-chirality in the standard literature and we have \( m > 0. \)

### 6 The Problem of Gauge Invariant Couplings

To be able to construct a supersymmetric extension of a gauge model, let us remind the reader some important difference between the classical and quantum treatment of gauge theories. In the classical framework, we start form a Lie algebra \( \mathfrak{g} \) with basis \( e_j, j = 1, \ldots, r \) and with the Lie bracket \([\cdot, \cdot]\); the structure constants in this basis will be denoted by \( f_{jkl} \). The basic variables of a gauge model are some classical fields \( v_{\mu} : \mathbb{R}^4 \to \mathfrak{g} \) called the gauge potentials. We denote the set of all gauge potential by \( \mathcal{M} \); its elements are also called mathematical configuration.

On this set there is an action of the gauge group, more precisely the associated gauge algebra: \( \text{Gau}(\mathfrak{g}) \) which is by definition the set of smooth maps \( \xi : \mathbb{R}^2 \to \mathfrak{g} \) with the pointwise Lie bracket. The action is non-linear:
\[ (\xi \cdot v)_{\mu}(x) = [\xi(x), v_{\mu}(x)] + \partial_{\mu} \xi(x). \]  
(6.1)

By definition the physical configurations are described by the factor set \( \mathcal{M}_{\text{phys}} \equiv \mathcal{M}/\text{Gau}(\mathfrak{g}) \). In this context the operation of choosing a gauge is perfectly meaningful: it means to choose a section of the fibre bundle \( \mathcal{M} \to \mathcal{M}_{\text{phys}} \); every point of the section will represent a physical configuration.
If one tries to find out a (classical) Lagrangian $L$ which is invariant with respect to this transformation, so the classical trajectories will factorise to the set of physical configuration $\mathcal{M}_{\text{phys}}$, then one essentially obtains the expression

$$L_{YM} = <f^\mu_\nu, f^\mu_\nu>$$

(6.2)

where $<\cdot, \cdot>$ is the Killing-Cartan form and

$$f^\mu_\nu \equiv \partial_\mu v_\nu - \partial_\nu v_\mu + [v_\mu, v_\nu]$$

(6.3)

is the well-known field strength. (If one considers Lagrangians which are invariant with respect to gauge transformations up to a total divergence one also gets Chern-Simons terms). The proper mathematical framework for this scheme is the fibre bundle theory. The inclusion of the ghost fields is done considering the cotangent bundle. This means that we have to enlarge the configuration space $\mathcal{M}$ adding some Grassmann valued variables $u, \bar{u}$: $\mathbb{R}^4 \rightarrow g \otimes G$ where $G$ is some Grassmann algebra. In this way the gauge transformation given above is extended to the classical BRST transformation and the invariance of the Lagrangian with respect to the BRST transformation is achieved by adding the Faddeev-Popov term:

$$L_{FP} = <v_\mu, [u, \partial^\mu \bar{u}]>$$

(6.4)

and a gauge fixing term

$$L_{gf} = \frac{1}{2\xi} <\partial^\mu v_\mu, \partial^\mu v_\mu>.$$  

(6.5)

Now, in quantum mechanics the meaning of a non-linear transformation is less clear. However, a “miracle” happens [25], [10]! Let we consider that: (a) the fields $v_\mu, u, \bar{u}$ are quantum free fields with the usual assignment of spin and statistics; (b) the total Lagrangian has terms of order $j = 2, 3, 4$

$$L = L_{YM} + L_{FP} = \sum_{j=2}^{4} L^{(j)};$$

(6.6)

we promote the tri-linear terms from the total Lagrangian to the status of interaction Lagrangian, in the sense of perturbation theory by adding Wick ordering:

$$T(x) =: L^{(3)} :$$

(6.7)

and (c) we consider only the linear part of the BRST transformation as a quantum operator. In this way the formulæ (5.3) appear. Then one can show that formula (1.2) from the Introduction is true for some Wick polynomial $T^\mu$ and moreover, the condition of gauge invariance in the second order generates the terms of order fourth $L^{(4)}$ of the Lagrangian $L$. So, the condition of quantum gauge invariance generates in a natural way the expression $L$ (up to the kinematical part which is quadratic piece $L^{(2)}$ of $L$; this piece of $L$ is encoded in the structure of the Fock space).

It is natural to try the same idea in a supersymmetric context. For this we start from the classical supersymmetric Lagrangian. It is argued (see [12] formulæ ((6.2.12) and (6.2.20) that
the corresponding terms should have the following form. Suppose that \( V, U, \tilde{U} \) has values in \( \mathfrak{g} \) i.e.
we have in fact \( r \) vector superfields \( V_j \) grouped in the Lie-valued expression \( V(X) \equiv e_j V_j(X) \)
(we sum over the dummy indices) and similarly for \( U \) and \( \tilde{U} \). Then the classical interaction
Lagrangian is taken to be the sum of

\[
L_{YM} = - \frac{1}{2} \left( e^{-V} D^a e^V \right) \bar{D}^2 \left( e^{-V} D^a e^V \right) + H.c.
\]

\[
L_{FP} = (\tilde{U} + \tilde{U}^\dagger) L_{V/2} \left( (U + U^\dagger) + \coth L_{V/2} (U - U^\dagger) \right)
\] (6.8)

and a gauge fixing term; here \( L \) is the Lie derivative. In analogy to the pure Yang-Mills case
we compute the tri-linear terms and obtain, up to a super-divergence i.e. an expression of the

type

\[
D_a T^a + D_a \bar{T}^\alpha
\] (6.9)

the following interaction Lagrangian:

\[
T = \sum_{j=1}^{2} T^{(j)}
\] (6.10)

with

\[
T^{(1)} \equiv f_{jkl}^{(1)} \left[ : V_j (D^a V_k) (\bar{D}^2 D_a V_k) : - H.c. \right]
\]

\[
T^{(2)} \equiv f_{jkl}^{(2)} : V_j (U_k + U_k^\dagger)(\tilde{U}_l + \tilde{U}_l^\dagger) :
\] (6.11)

and where \( f_{jkl}^{(j)}, j = 1, 2 \) are some constants proportional to the structure constants \( f_{jkl} \). Let us
note that this Lagrangian is non-renormalizable: it has the supersymmetric canonical dimension
5. In principle one can hope that the gauge invariance condition will eliminate the arbitrariness
in every order of the perturbation theory such that the series of the exponential from
the classical expression \( L_{YM} \) is reconstructed perturbatively (as one get the fourth degree term
of the usual Yang-Mills Lagrangian in the second order of the perturbation theory). If one
computes the corresponding expressions \( t \) and \( t^\mu \) (see the Introduction) by integrating out
the Grassmann variables one gets, up to finite renormalizations, the usual expressions from the
literature [25], [10]:

\[
\int d\theta^2 d\bar{\theta}^2 T^{(1)} = 4i \ f_{jkl}^{(1)} : v_j^\mu v_k^\nu f_{\nu \mu} : + \cdots
\]

\[
\int d\theta^2 d\bar{\theta}^2 T^{(2)} = \frac{i}{2} \ f_{jkl}^{(2)} : v_j^\mu (u_k + u_k^\dagger) (\tilde{u}_l - \tilde{u}_l^\dagger) : + \cdots
\] (6.12)

where by \( \cdots \) we mean terms containing the superpartners from the corresponding multiplets.
It seems encouraging that the last term is in agreement with the gauge structure (5.31). In the
usual case [25], [10] the gauge invariance is restored by adding new couplings with some scalar
ghost fields; in our case these couplings must include the Wick monomials

\[
: (h_j + h_j^\dagger) (u_k + u_k^\dagger) (\tilde{u}_l - \tilde{u}_l^\dagger) : \\
: (h_j + h_j^\dagger) \partial_\mu (h_k + h_k^\dagger) v_\mu^l : \\
: (h_j + h_j^\dagger) v_{k\mu} v_\mu^l :
\] (6.13)
Guided by this argument we study the gauge invariance of a Lagrangian of the form

\[ T = \sum_{j=1}^{6} T^{(j)} \]  

(6.14)

where the new terms are

\[ T^{(3)} = f^{(3)}_{jkl} : (H_j + H_j^\dagger) (U_k - U_k^\dagger) (\tilde{U}_l + \tilde{U}_l^\dagger) : \]
\[ T^{(4)} = f^{(4)}_{jkl} : (H_j + H_j^\dagger) (H_k - H_k^\dagger) V_l : \]
\[ T^{(5)} = f^{(5)}_{jkl} : (H_j + H_j^\dagger) V_k (\mathcal{D}^2 \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^2 \mathcal{D}^2) V_l : \]
\[ T^{(6)} = f^{(6)}_{jkl} : (H_j + H_j^\dagger) V_k V_l : \]  

(6.15)

and we impose the supersymmetric gauge invariance condition (1.4) from the Introduction.

The naturalness of the new terms follows from the explicit expressions for \( dQ_T^{(i)} \): we have

\[ dQ_T^{(1)} = - f^{(1)}_{jkl} : (U_j + U_j^\dagger) V_k (\mathcal{D}^2 \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^2 \mathcal{D}^2) V_l : + 16 f^{(1)}_{jkl} m_l^2 : (U_j + U_j^\dagger) (H_k + H_k^\dagger) V_l : \]
\[ + \mathcal{D}_a T^a + \bar{\mathcal{D}}_{\bar{a}} \tilde{T}^{\bar{a}} \]  

(6.16)

where

\[ T_a \equiv - f^{(1)}_{jkl} : (U_j + U_j^\dagger) V_k (\mathcal{D}^2 \mathcal{D}_a V_l) : - 2 f^{(1)}_{jkl} : U_j (\mathcal{D}_b V_k) \mathcal{D}_a \tilde{\mathcal{D}}^b V_l : \]  

(6.17)

and the complete antisymmetry of the constants \( f^{(1)}_{jkl} \) was used. Also

\[ dQ_T^{(2)} = f^{(2)}_{jkl} : (U_j - U_j^\dagger) V_k (U_k + U_k^\dagger) (\tilde{U}_l + \tilde{U}_l^\dagger) : \]
\[ + \frac{1}{16} f^{(2)}_{jkl} : V_j (U_k + U_k^\dagger) (\mathcal{D}^2 \tilde{\mathcal{D}}^2 + \tilde{\mathcal{D}}^2 \mathcal{D}^2) V_l : \]
\[ + i f^{(2)}_{jkl} m_l : V_j (U_k + U_k^\dagger) (H_l - H_l^\dagger) : \]  

(6.18)

Then one can see that to compensate the various terms one is forced to introduce the new terms \( T^{(j)}, j = 3, \ldots, 6 \). The new terms seem to be a logical choice because if we integrate out the Grassmann variables we obtain, essentially, the usual couplings of the scalar ghosts \([25], [10]\) listed above. If one requires that the expression \( T = \sum_{j=1}^{6} T^{(j)} \) does verify the supersymmetric gauge invariance condition (1.4) then one obtains the solution

\[ f^{(1)}_{jkl} = 0 \quad f^{(6)}_{jkl} = 0 \]

(6.19)

If we compute the corresponding Lagrangian \( t(x) \) we find out a strange solution of the gauge invariance condition: there is no pure Yang-Mills coupling but one has monomials with canonical dimension 6 (they are produced by \( T^{(5)} \)).

The negative result which we have obtained can be traced to the gauge structure. Indeed the cancelation of the coefficient of the Wick monomial : \((U_j - U_j^\dagger) (U_k + U_k^\dagger) (\tilde{U}_l + \tilde{U}_l^\dagger) :\)
in the supersymmetric gauge invariance condition (1.4) implies the third relation from (6.19).
However if we impose only (1.2) a weaker condition follows. Indeed we have
\[ \int d\theta^2 d\bar{\theta}^2 : (U_j - U_j^\dagger) (U_k + U_k^\dagger) : (\bar{U}_l + \bar{U}_l^\dagger) = \frac{1}{4} (m_j^2 + m_l^2 - m_k^2) : (u_j + u_j^\dagger) (u_k + u_k^\dagger) (\bar{u}_l - \bar{u}_l^\dagger) : \]
But the condition of cancelation of the coefficient of this Wick monomial gives the weaker
condition
\[ (m_j^2 + m_l^2 - m_k^2) (f_{jkl}^{(2)} - i m_k f_{jkl}^{(3)}) = (j \leftrightarrow k) \]
(6.21)
because of the antisymmetry in \( j \) and \( k \) obtained after integrating out the Grassmann variables.
The same argument works for the annulation of the coefficient of the Wick monomial : \( (H_j + H_j^\dagger) (U_k - U_k^\dagger) (H_l - H_l^\dagger) \) : which gives the last relation (6.19). However, because
\[ \int d\theta^2 d\bar{\theta}^2 : (H_j + H_j^\dagger) (U_k - U_k^\dagger) (H_l - H_l^\dagger) = \frac{1}{4} (m_k^2 + m_l^2 - m_j^2) : (h_j + h_j^\dagger) (h_k + h_k^\dagger) (h_l + h_l^\dagger) : \]
the condition (1.2) gives only
\[ (m_k^2 + m_l^2 - m_j^2) (f_{jkl}^{(4)} + i m_k f_{jkl}^{(3)}) = -(j \leftrightarrow l) \]
(6.23)
because of the symmetry property in \( j \) and \( l \).

We have tried in vain to circumvent this no-go result taking for granted the expressions \( T^{(j)} , j = 1, 2 \) which are suggested by the existing literature. To obtain weaker conditions from the gauge invariance condition (1.4) it seems that one is forced to change the expression \( T^{(2)} \); a possible choice would be
\[ T^{(2)} \equiv f_{jkl}^{(2)} : V_j (U_k - U_k^\dagger) (\bar{U}_l - \bar{U}_l^\dagger) : \]
(6.24)
because after integrating out the Grassmann variables we again obtain the usual expression; moreover in the expression \( d_Q T^{(2)} \) we have now the trilinear ghost term \( f_{jkl}^{(2)} : (U_j - U_j^\dagger) (U_k - U_k^\dagger) (\bar{U}_l - \bar{U}_l^\dagger) : \) with some antisymmetry property in \( j \) and \( k \). However, then one is forced to change the expression \( T^{(1)} \) too. A possible choice would be
\[ T^{(1)} \equiv f_{jkl}^{(1)} : V_j V_k \partial_\mu V_l^\mu : \]
(6.25)
Adding coupling with the scalar ghost superfields and imposing the supersymmetric gauge invariance condition (1.4) one obtains again after integration of the Grassmann variables a strange solution with anomalous couplings.

We find these arguments rather convincing for a negative result. We conjecture that one cannot find out a solution \( T \) verifying the supersymmetric gauge invariance condition (1.4) and such that after integration of the Grassmann variables we obtain the usual Yang-Mills interaction between the gauge Bosons and the ghost fields.

One can save the model with \( f_{jkl}^{(1)} \neq 0 \) if one imposes only (1.2) but in this case one can prove that one can add to \( T \) many other supersymmetric Wick monomials so the arbitrariness of the
The interaction Lagrangian is rather large. Moreover one does not have a fully supersymmetric gauge invariance property.

We mention in the end that one can use the Ω_{1/2} vector multiplet to construct a supersymmetric extension of quantum electrodynamics. This can be done as follows. We take two Wess-Zumino multiplets \((\phi^{(j)}, f^{(j)}_a), j = 1, 2\) verifying the relations from Subsection 4.1, in particular the relations (4.5). Then we define the left and right fields

\[
\phi_L \equiv \phi^{(1)} + i\phi^{(2)} \quad \phi_R \equiv \phi^{(1)} - i\phi^{(2)}
\]

\[
f_{La} \equiv f^{(1)}_a - if^{(2)}_a \quad f_{Ra} \equiv f^{(1)}_a + if^{(2)}_a,
\]

the expressions \(f_{L,R}\) are the left and right components of the electron field. Next, we define two chiral superfields \(\Phi_L \equiv s(\phi_L), \Phi_R \equiv s(\phi_R)\) using the sandwich formula. If we consider now the interaction Lagrangian

\[
T = \left( \Phi_L^\dagger \Phi_L - \Phi_R^\dagger \Phi_R \right) V
\]

between the vector superfield \(V\) and these chiral superfields, then one can prove rather easy two facts: (i) this Lagrangian is gauge invariant in the sense (1.2); (ii) after integrating out the Grassmann variables one obtains the usual expression for the QED interaction Lagrangian

\[
\int d\theta^2 d\bar{\theta}^2 T = v_\mu \left( f_L \sigma^\mu \bar{f}_L - f_R \sigma^\mu \bar{f}_R \right) + \cdots
\]

7 The Linear Vector Model

We try to circumvent the negative result from the preceding Section by choosing a gauge. This operation has a perfectly well meaning in the classical field theory context: it means to choose a section of the fibre bundle \(M \rightarrow M_{\text{phys}}\); every point of the section will represent a physical configuration. In the quantum context, the relations (5.12) are considered as a proof that by choosing conveniently the expressions \(u, g, \psi_a\) one can make equal to zero the fields \(C, \phi, \chi\) and the longitudinal part of \(v_\mu\).

In the quantum context, we proceed as follows. Guided by Proposition 3.3 we impose the following restriction on the vector field:

\[
\mathcal{D}^2 \mathcal{D}^2 V = 0;
\]

this implies that the chiral and antichiral components \(V_1, V_2\) of \(V\) are zero so we have \(V = V_0 = -\frac{2}{m^2} D'\). We call \(V\) in this case the linear vector model. One can express the preceding condition in component fields; it is easy to get: that the condition (7.1) is equivalent to:

\[
\partial^\mu v_\mu = 0 \quad d = -\frac{m^2}{2} C \quad \phi = 0 \quad \chi_a = i \frac{m^2}{\sigma_{ab} \partial_\mu \bar{\lambda}^b}.
\]

From these constraints it follows that we have \(d' = 2d, \quad \chi'_a = 2\chi_a\). This reduction of the multiplet is consistent. Indeed, the fields are now the scalar field \(d\), the transversal vector field
\( v_\mu \) and the Dirac field \( \lambda_a \) and if we act with this field on the vacuum we get the representation \( \Omega_{1/2} \). One can easily obtain the following action of the supercharges from (3.7) if we use (7.2):

\[
[Q_a, d'] = -\frac{1}{2} \sigma_\mu^a \partial_\mu \bar{\lambda}^{\dot{b}} \quad \Leftrightarrow \quad [Q_a, d] = -\frac{1}{2} \sigma_\mu^a \partial_\mu \bar{\lambda}^b
\]

\[
i [Q_a, v^\mu] = \sigma_\rho^a \left( \delta_\mu^\rho + \frac{1}{m^2} \partial^\mu \partial_\rho \right) \bar{\lambda}^{\dot{b}} = 2 \sigma_\rho^a \left( \delta_\mu^\rho + \frac{1}{m^2} \partial^\mu \partial_\rho \right) \bar{\lambda}^b
\]

\[
\{Q_a, \lambda^b_1\} = 2i \epsilon_{ab} d' - 2i \sigma_\mu^a \partial_\mu v_\rho \quad \Leftrightarrow \quad \{Q_a, \lambda^b_1\} = 2i \epsilon_{ab} d - i \sigma_\mu^a \partial_\mu v_\rho
\]

\[
\{Q_a, \lambda^b_1\} = 0 \quad \Leftrightarrow \quad \{Q_a, \bar{\lambda}^\dot{b}_1\} = 0;
\]

(7.3)

let us note that the second relation is consistent with the transversality condition \( \partial^\mu v_\mu = 0 \). One can check directly that the consistency relations (2.26) are true. The causal (anti)commutation relation can be obtained from (3.26) if we take into account the restrictions (7.2); we have a solution iff

\[
\alpha = -\frac{m^2}{4} \quad \beta = 0;
\]

(7.4)

the explicit form is:

\[
[d(x), d(y)] = -\frac{im^4}{16} D_m(x - y)
\]

\[
[v_\mu(x), v_\nu(y)] = i \left( \partial_\mu \partial_\nu + m^2 g_{\mu \nu} \right) D_m(x - y)
\]

\[
\{\lambda_a(x), \lambda_b(y)\} = 0
\]

\[
\{\lambda_a(x), \bar{\lambda}^{\dot{b}}_1(y)\} = \frac{m^2}{4} \sigma_\mu^{a\dot{b}} \partial_\mu D_m(x - y);
\]

(7.5)

let us remark that the second relations is compatible with the restriction \( \partial^\mu v_\mu = 0 \).

One usually rejects this transversality condition because then the perturbative theory of the vector field \( v_\mu \) is non-renormalizable: the causal distribution of \( v_\mu \) has order of singularity 0 instead of -2. In the supersymmetric context the situation is different. Indeed, the causal (anti)commutation given above do not change the super-order of singularity of the causal distribution; we still have \( \omega(D(X_1; X_2)) = -2 \) so one can try to build a perturbation theory for the transversal vector model. One does not need in this case the ghost fields so one can build the perturbation theory starting from the Lagrangian \( T = T^{(1)} \) from (6.11). However, this Lagrangian in non-renormalizable even in the supersymmetric context and there is no symmetry requirement which could restrict the arbitrariness in higher orders of perturbation theory. Moreover, by integrating out the Grassmann variables one gets an expression for the interaction Lagrangian which is different from the standard expression from the literature: no ghost fields are needed.
conclusions

We have succeeded to construct in a rigorous way the quantum supersymmetric vector multiplet. Some differences from the literature already appear: the mass of the multiplet should be necessarily strictly positive and the Feynman propagator is different. The gauge structure associated to this multiplet is, essentially the same as in the standard literature; scalar superghosts have to be included because the mass is strictly positive. However, the expression of the interaction Lagrangian suggested by the literature does not verify a supersymmetric gauge invariance condition (1.4) as it is suggested by the BRST invariance of the classical action. We are conjecturing that a no-go result can be obtained if one studies systematically all possible Wick monomials (as it is done for the ordinary gauge models in [25], [10]). The ways out of this negative result are: (a) to change the gauge structure from Section 5; (b) to abandon (1.4) and replace it by the weaker condition (1.2); (c) to relax the conditions (2.21). The second possibility means to accept a model which is not gauge invariant in a supersymmetric sense (and this is the origin of the appearance of many free parameters). The first possibility cannot be ruled out but we did not succeed to find a natural replacement of the gauge structure contained in the formulae (5.10), (5.11) and (5.32). The last possibility is suggested by the analysis of the model as a classical field theory: one requires that (2.21) are valid only “up to gauge transformations”. We did not succeed to give a rigorous formulation of this idea in our pure quantum context.

We emphasize again that the (new) $\Omega_{3/2}$ vector multiplet introduced in [13] is gauge invariant in a supersymmetric sense and reproduces the usual Yang-Mills Lagrangian after the integration of the Grassmann variables.
References


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