In this paper we argue that the dark energy is composed of a new form of matter that is not part of the standard model of particle physics. We propose that this dark energy is a consequence of a modified version of Einstein's general relativity, which allows for the existence of a new scalar field. We show that this scalar field can account for the observed acceleration of the universe and provides an explanation for the cosmological constant problem.

**1. Introduction**

The observational evidence for dark energy has been accumulating at an accelerating rate. The supernova surveys ofaf and the cosmic microwave background anisotropies have provided strong evidence for a cosmological constant, or dark energy, that is presently accelerating the expansion of the universe. The Friedmann equation, which describes the expansion of the universe, can be written as:

\[ H^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \]

where \( H \) is the Hubble parameter, \( \rho \) is the energy density, \( G \) is the gravitational constant, and \( k \) is the curvature of the universe. The first term on the right-hand side represents the effect of the dark energy, while the second term represents the effect of gravity.

The cosmological constant problem arises because the observed value of the energy density of dark energy is much smaller than what would be expected from a simple application of general relativity. The problem is that the energy density of dark energy is about 100 times smaller than the energy density of visible matter.

In this paper we propose a new form of matter that can account for the dark energy. This new form of matter is a scalar field that is not part of the standard model of particle physics. We show that this scalar field can account for the observed acceleration of the universe and provide an explanation for the cosmological constant problem.

**References**

II. INITIAL CONDITIONS FOR THE MODE EVOLUTION

Refs. [1, 2, 3] consider a scalar field, \( \phi \), with a non-linear dispersion relation that is linear in the sub-Planckian regime and approaches a decreasing exponential at trans-Planckian wave-numbers (i.e. for \( k_{\text{phys}} \gtrsim k_c, k_c \sim M_P \) being a fundamental characteristic scale). This scalar field is assumed to describe the density (scalar) perturbations and/or the primordial gravitational waves. The “tail” modes are thus interpreted as a bath of gravitons of super-Planckian wavelengths and sub-Hubble frequencies. This scalar field is treated as a test-field (its back-reaction on the background is neglected) and is quantized on the curved cosmological background. Assuming that the “tail” modes of this field are initially in a well chosen vacuum state as \( \eta \to -\infty \) (\( \eta \) denoting conformal time), the occupation number at late times \( (\eta \to +\infty) \) of quanta extracted out of the vacuum by the dynamical background has been calculated in Ref. [1]. This occupation number can then be used to calculate the energy density stored today in the “tail”. This is the thread of the calculation performed in Ref. [1], which we now follow in some detail. This discussion will take us to the two main arguments that we bring forward against this model (given in this Section and the following).

The equation of motion of a scalar field \( \phi \equiv \mu/a \) in a Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time with scale factor \( a(\eta) \) reads:

\[
\mu_k'' + \left[ \omega^2 - \left( 1 - 6\xi \frac{a''}{a} \right) \right] \mu_k = 0, \tag{1}
\]

where \( \xi \) is a coupling parameter to gravity and a prime denotes differentiation with respect to conformal time. \( k = a k_{\text{phys}} \) is the co-moving wave-number and \( \omega = a \omega_{\text{phys}} \) is the co-moving frequency. \( \xi = 0 \) for tensor perturbations degrees of freedom and \( \xi = 1/6 \) for a conformally coupled field. The scale factor is taken to be a power law in conformal time, \( a \equiv |\eta/\eta_c|^\beta \), and the following dispersion relation, parameterized by two parameters \( \epsilon_1 \) and \( \epsilon_3 \) with \( \epsilon_3 = 4 - 2\epsilon_1 \) in order to insure that the dispersion relation is linear for small wave-numbers, is introduced:

\[
\omega_{\text{phys}}^2 = \frac{\epsilon_1}{1 + e^\beta} + \frac{\epsilon_3 e^{\beta}}{(1 + e^\beta)^2}, \tag{2}
\]

with \( k \approx (k_{\text{phys}}/k_c)^{1/\beta} \equiv A[\eta] \) with \( A \equiv (k/k_c)^{1/\beta} \). A problem with this dispersion relation is that it depends on the power-law index \( \beta \) of the scale factor. If taken literally, this means that the dispersion relation, or the physical frequency of a given mode, jumps discontinuously as the scale factor power-law index \( \beta \) changes between various cosmological eras (e.g. inflation / radiation domination / matter domination). Moreover, one easily sees that the above dispersion relation has a pathological behavior in the radiation (\( \beta = -1 \)) or matter (\( \beta = -2 \)) dominated eras. In fact, for \( \beta < 0 \), it implies \( \omega_{\text{phys}} / k_{\text{phys}} \to 0 \) as \( k_{\text{phys}} \to 0 \), whereas one should instead reach the linear dispersion relation in that regime with \( \omega_{\text{phys}} / k_{\text{phys}} \to 1 \). Since Ref. [1] focused on the case of de Sitter space-time with \( \beta = 1 \), we set \( \beta = 1 \) in the above dispersion relation, i.e., the above \( x \) should be understood as \( x \equiv k_{\text{phys}} / k_c \). This reformulated dispersion relation thus coincides with that used in Ref. [1] for de Sitter space. However, in the matter dominated era, for instance, we have \( x \equiv k_{\text{phys}} / k_c \propto \eta^{-2} \) and the general class of solutions to the field equation obtained in Ref. [1] does not hold anymore. The linear dependence of \( x \) on \( \eta \) is lost for background metrics other than de Sitter, but the linear dependence of \( \omega_{\text{phys}} \) on \( k_{\text{phys}} \) is preserved in the small \( k_{\text{phys}} \) limit for all metrics, which is obviously an imperative. The field equation can finally be rewritten as:

\[
\mu_k'' + \left\{ k^2 \left[ \frac{\epsilon_1}{1 + e^\beta} + \frac{\epsilon_3 e^{\beta}}{(1 + e^\beta)^2} \right] - (1 - 6\xi) \frac{\beta (\beta + 1)}{\eta^2} \right\} \mu_k = 0, \tag{3}
\]

with, again, \( x(\eta) \equiv k_{\text{phys}} / k_c \equiv k / [a(\eta) k_c] \). The solution to this equation depends on the value of \( \xi \) and \( \beta \). In Ref. [1], the contribution of the \( [a''/a] \) term is assumed to be negligible at early times. However the above equation shows that this is not the case; denoting by \( \Omega_k^2 \) the term in curly brackets in Eq. (3), one has \( \Omega_k^2 \approx k^2 (\epsilon_1 + \epsilon_3) e^{-k \int [1/(k a)]} - \beta (\beta + 1) (1 - 6\xi)/\eta^2 \) as \( \eta \to -\infty \) and the term \( a''/a \propto \eta^{-2} \) is always dominant in that limit if \( \xi \neq 1/6 \) (\( \beta \geq 1 \).
\( \eta \to -\infty \). Therefore, in the limit \( \xi \neq 1/6 \) and \( \eta \to -\infty \) the two independent solutions to the field equation are power-laws in \( \eta \):

\[
\mu_k \propto \left( \frac{\eta}{\eta_k} \right)^{\alpha \pm}, \quad \alpha \pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \beta(1 + \beta)(1 - 6\xi)}.
\]

This is an important point as it implies that the mode function does not behave as a plane wave in the limit \( \eta \to -\infty \) when \( \xi \neq 1/6 \). The solution to the field equation in this limit is reminiscent of the mode freezing in inflationary theories for fields with linear dispersion relations and \( \xi = 0 \) when the mode exits the horizon.

It is also argued that the term in \( d^4/a \) can be absorbed at late times in a redefinition of the dispersion relation. However this cannot be correct since, by construction, the dispersion relation \( \omega_{\text{phys}}(k_{\text{phys}}) \) depends only on \( k_{\text{phys}} \), i.e. time only enters via \( k_{\text{phys}} \). Therefore, one can absorb a term \( d^4/a \propto \eta^{-2} \) into \( \omega_{\text{phys}}(a^2) \) only if \( \beta = 1 \) (de Sitter space-time), as inspection of Eq. (3) reveals. In effect, the curly bracket of Eq. (3) can then be rewritten as \( a^{-2} \) times a function of \( \eta \propto k_{\text{phys}} \).

But, in that particular case, the redefined modified dispersion relation does not have an exponential shape anymore, since \( \omega_{\text{phys}} \) approaches a constant \((\sim H)\) as \( \omega_{\text{phys}} \to +\infty \). However, this should not give the impression that the corresponding solution is a plane wave since, evidently, the co-moving frequency \( \omega \) which enters Eq. (1) still behaves as \( \propto \eta^{-2} \). Moreover in the case of a matter or radiation dominated cosmology, one cannot absorb the scale factor term in the dispersion relation.

Nevertheless one can also assume \( \xi = 1/6 \). In that case, it is possible to find an exact solution to the equation of motion in de Sitter space-time. Indeed, for a conformally coupled field, the term \( d^4/a \) disappears from the field equation and the equation becomes simpler. Notice however that the scalar field cannot correspond to tensor perturbations degrees of freedom since these are minimally coupled to the metric. Let us consider \( \xi = 1/6 \) for the moment. A solution to the field equation, given in Ref. [1] reads:

\[
\mu_k^{(im)}(\eta) = C^{(im)}(1 + e^{-x})^4 z F_1 \left( b + d + \frac{1}{2} - b + d + \frac{1}{2} 2d + 1; 1 + e^{-x} \right)
\]

where \( z F_1 \) is an hypergeometric function and \( b \) and \( d \) are expressed in terms of \( \epsilon_1 \) and \( \epsilon_3 \) as:

\[
b \equiv i \sqrt{\epsilon_1}, \quad d \equiv \sqrt{\frac{1}{4} + \epsilon_3}, \quad \text{and} \quad \epsilon_i \equiv k^2 \epsilon_{i_{\text{phys}}}^2.
\]

This solution is valid only for de Sitter space-time with \( x = k\eta/k_{\text{phys}} \) \((\eta < 0, \eta_k < 0)\). As already mentioned, this is due to the fact that, with the reformulated dispersion introduced above, the linear dependence of \( x \) in the conformal time is lost for other scale factors. However similar solutions for other metrics\(^1\) can be obtained if the dispersion relation is tuned to the power-law evolution of the scale factor, i.e. if the parameter \( x \) remains linear in \( \eta \) (possibly at the expense of linearity of \( \omega_{\text{phys}} \) in the small \( k_{\text{phys}} \) limit, see above). Equation (3) has in fact two independent solutions (see below) and the choice (5) represents only one branch of the solution, which is moreover written on the branch cut of the hypergeometric function \( z F_1 \). At early times \((\eta \to -\infty, \text{i.e. } x \to +\infty)\), this solution (5) does not oscillate and it blows up\(^2\).

It is more convenient to write the general solution to the field equation, with \( \xi = 1/6 \), as \( \mu_k = C_1 \mu_k^{(1)} + C_2 \mu_k^{(2)} \), where \( \mu_k^{(1)} \) and \( \mu_k^{(2)} \) are two independent solutions given by:

\[
\mu_k^{(1)}(\eta) = e^{b\eta}(1 + e^{-\theta})^{2d+1/2} z F_1 \left( b + d + \frac{1}{2}, -b + d + \frac{1}{2}; 2b + 1; -e^{\eta} \right)
\]

\[
\mu_k^{(2)}(\eta) = e^{-b\eta}(1 + e^{\theta})^{2d+1/2} z F_1 \left( -b + d + \frac{1}{2}, b + d + \frac{1}{2}; -2b + 1; e^{\eta} \right).
\]

\(^1\) Equation (25) in Ref. [1] contains a misprint that has been corrected in the following equation

\(^2\) A hypergeometric function of the form \( z F_1 (a, \beta; \alpha + \beta; z) \) is singular at \( z = 1 \). One can also solve Eq. (3) for \( \xi = 1/6 \) and \( \beta = 1 \) in the limit \( \eta \to -\infty \). In this case, the equation reduces to \( \mu_k^{(1)} + k^2 (\epsilon_1 + \epsilon_3) e^{-\eta} \mu_k = 0 \) and the solution can be written as:

\[
\mu_k(\eta) = \frac{1}{\sqrt{2}k} \left[ A_1(k) J_0(\epsilon^x) + A_2(k) Y_0(\epsilon^x) \right],
\]

where \( J_0 \) and \( Y_0 \) are Bessel functions and where \( \eta \equiv -\eta/2 + \ln(4\pi^2 (\epsilon_1 + \epsilon_3)/A^2)/2 \). The Neumann function diverges in the limit \( \eta \to -\infty \) (\( y \to -\infty \)). In the tail, the corresponding behavior for the scalar field itself is given by \( \phi \propto \eta \) and \( \phi \propto \eta^2 \).
Since $b$ is pure imaginary, and $d$ is real, one concludes easily that $\mu_k^{(2)} = \mu_k^{(1)*}$. The Wronskian of these two solutions is non-zero, and can be used to relate the coefficients $C_1$ and $C_2$ so as to obtain canonical commutation relations for the field operator and its adjoint. Since only one branch of the solution was given in Ref. [1], the canonical commutation relations for the field and its adjoint could not be satisfied. More precisely, it can be checked that the solution given in Eq. (5) is real. This is due to the fact that it involves a hypergeometric function of the form \[ _{0}F_{1}(a; a'; z) \] with $z = 1 + e^{\pi i}$ and \( a \equiv b + d + 1/2 \) in that case and \( _{1}F_{0}(a; b; c; z) = _{0}F_{1}(0, a; z) \) in the other. Therefore, one has \( _{0}F_{1}(a, a' + 1 + e^{-\pi i}) = _{0}F_{1}(a, a' + 1) \) and the mode function is indeed real.

It follows that the Wronskian of the solution considered in Ref. [1] vanishes: \( W(\mu, \mu^*) \equiv \mu_k \mu_k^{(1)} - \mu_k^{(2)} \mu_k = 0 \). Using the properties of hypergeometric functions, one can check that both independent solutions $\mu_{k}^{(1)}$ and $\mu_{k}^{(2)}$ behave as \( \propto x^{-\eta} \) in the limit $\eta \to -\infty$, i.e. these mode functions blow up. This result is consistent with Eq. (7) since, in the tail, the two branches $\mu_{k}^{(1)}$ and $\mu_{k}^{(2)}$ are linear combinations of the Bessel functions $J_\eta$ and $N_\eta$.

Therefore, we have shown that neither in the case $\xi \neq 1/6$ nor in the case $\xi = 1/6$ does the mode function behave as a plane wave in the tail. Thus the initial state of the field cannot reduce to the Bunch-Davies adiabatic vacuum, contrary to the claim [1]: “we show that there is no ambiguity in the correct choice of the initial vacuum state. The only initial vacuum is the adiabatic vacuum obtained by the solution to the mode equation”. The usual prescription to remove the ambiguity on the choice of vacuum state in curved spacetime, i.e. for constructing a vacuum state which is closest to the definition of vacuum in Minkowski, is indeed to rely on the WKB approximation to construct vacua of successively higher adiabatic order [7]. In this scenario [1, 2, 3], this construction cannot be performed for a simple reason: the WKB approximation, which quantifies the adiabaticity of the quantum mode evolution is violated at all times for modes contained in the tail, i.e. modes with $k_{\text{phys}} > k_c$ and $\omega_{\text{phys}} < H$. The WKB condition can be written in the form $|Q/\Omega_k^2| \ll 1$ [7], where $\Omega_k^2$ denotes the term in curly brackets in Eq. (3) as before, and $Q \equiv \Omega_k^2/2\Omega - (3/4)\Omega_k^2/\Omega_k^2$. The expression for $Q/\Omega_k^2$ is cumbersome, but since we are interested in the regime $k_{\text{phys}} \gg k_c$, we may use the limiting form of the dispersion relation:

$$\omega_{\text{phys}} \approx k_{\text{phys}} \sqrt{1 + \alpha e^{-k_{\text{phys}} l/(2k_c)}} \quad (k_{\text{phys}} \gg k_c),$$

(9)

If $\xi = 1/6$, then $\Omega_k = \alpha \omega_{\text{phys}}$ and $|Q/\Omega_k^2| \sim k_{\text{phys}}^2 H^2/16\alpha^2 \omega_{\text{phys}}^2$. In order to understand the behavior of $|Q/\Omega_k^2|$, it is convenient to introduce the physical wave-number $K_+ > k_c$ such that $\omega_{\text{phys}}(K_+) = \sqrt{(1 + \beta)/\beta H}$ [in the case $\xi = 0$, one has $\Omega_k(K_+) = 0$]. Using Eq. (9), one easily derives:

$$K_+ \approx 2k_c \ln \left[ \frac{(1 + e\xi)\beta 2k_c}{1 + \beta H} \right].$$

(10)

This formula is written to zeroth order in $\ln(k_c/H)/(k_c/H)$ but can be expanded to arbitrary order in a straightforward way. The meaning of the physical wave-number $K_+$ is the following (see Fig. 1 of Ref. [6]). If $k_{\text{phys}} \ll K_+$ but $k_{\text{phys}} > k_c$ (trans-Planckian region), the mode is outside of the tail with $\omega_{\text{phys}} \gg H$. If however $k_{\text{phys}} > K_+$, the mode is in the tail with $\omega_{\text{phys}} < H$. Then $k_{\text{phys}} \gg K_+$ means that $\omega_{\text{phys}} \ll H^2$ which implies in turn $|Q/\Omega_k^2| \gg k_{\text{phys}}^2/(16\Omega_k^2) \gg 1$, hence the WKB approximation is violated at all times in the tail. Note that outside of the tail, i.e. for $k_{\text{phys}} \ll K_+$ and $k_{\text{phys}} \gg k_c$, the WKB approximation becomes valid. In effect $k_{\text{phys}}/\omega_{\text{phys}} \propto \exp(k_{\text{phys}}/2k_c) \ll \ln(k_c/H)$, hence $|Q/\Omega_k^2| \ll 1$. If $\xi \neq 1/6$, then for $\omega_{\text{phys}} \ll H$, the dominant term is $\propto \alpha l/a$ in the expression of $\Omega_k^2$, namely $\Omega_k^2 \sim -(1 - 6\xi)(1 + \beta)\alpha^2 H^2/\beta$, hence $|Q/\Omega_k^2| \sim \left[ (1 - 6\xi)(1 + \beta)\alpha^2 H^2/\beta \right]^{-1}$, which for $\beta = 1$ (de Sitter) and $\xi = 0$ (minimal coupling) reduces to $1/8$. In this case, it can be shown that the WKB approximation does not give the right behavior for the mode function even though $|Q/\Omega_k^2|$ is smaller than unity [11], and that the WKB approximation is not valid either. Again, note that outside of the tail, the WKB approximation is valid. The calculation is the same as in the previous paragraph, since for $k_{\text{phys}} \ll K_+$ and $k_{\text{phys}} \gg k_c$, one has $\Omega_k^2 \sim a^2 \omega_{\text{phys}}^2$ since $\omega_{\text{phys}} \gg H$. Thus one finds $|Q/\Omega_k^2| \ll 1$ outside the tail even for $\xi \neq 1/6$.

To summarize this discussion the WKB condition is violated by the present dispersion relation in the tail at all times and an initial vacuum state cannot be constructed unambiguously. Outside of the tail, for $\omega_{\text{phys}} > H$ or $k_{\text{phys}} > K_+$, the WKB approximation is a good approximation. One can also verify that the construction of an initial vacuum state by minimization of the energy content does not work in this case, see Ref. [6]. This point is one major obstacle to the scenario proposed in Ref. [1]. Since there is no preferred initial vacuum state, all cosmological conclusions drawn depend directly on the particular choice of the initial state, hence on the choice of initial data. At the very best, one has to fine-tune the initial conditions to obtain a given amount of energy in a given part of the spectrum.

The standard calculation of the amount of energy contained at late times in a given co-moving wave-number mode is done by decomposing the solution at late times (outside the tail) in terms of positive and negative frequency plane waves, as
\[ \mu_k = \frac{\alpha_k}{\sqrt{2\omega_k}} e^{-i\omega_k \eta} + \frac{\beta_k}{\sqrt{2\omega_k}} \phi_k \, e^{-i\omega_k \eta}. \] (11)

The squared modulus of the Bogoliubov coefficient \( \beta_k \) then will give the occupation number of quanta produced in the mode of co-moving wave-number \( k \). Note that, in principle, the coefficients \( \alpha_k \) and \( \beta_k \) can be slowly varying functions of time, and the above expression implicitly involves a WKB approximation to first order in which the time evolution of \( \alpha_k \) and \( \beta_k \) is neglected. The corresponding vacuum is an adiabatic vacuum to first order.

In Ref. [1] \( \beta_k \) is calculated in the limit \( \eta \to +\infty \) as \( \omega^{\text{out}} \to \sqrt{\beta} k \). However the limit \( \eta \to +\infty \) does not hold in an inflationary universe with \( a \propto |\eta|^{-\beta} \) and \( \beta \geq 1 \) since \( a \) is singular as \( \eta \to 0^+ \). One needs to match the background evolution to a decelerated Universe as \( \eta \to 0^- \). Furthermore, as explained above, the solution to the field equation given in Ref. [1] [see Eq. (5)] describes only one branch of the solution. Also, the solution given in terms of the hypergeometric function is not valid at late times in the radiation dominated or matter dominated eras unless the parameter \( \kappa \) of the dispersion relation is tuned to the evolution of the scale factor, but the dispersion relation would become pathological as we saw before for \( x = (k_{\text{phys}}/k_c)^{1/\beta} \) with \( \beta < 0 \). Finally, one cannot compute \( \beta_k \) for modes still contained in the “tail” at late times in the above way, since for those modes, the WKB approximation is never valid so that the out solution cannot be decomposed in a sum of plane waves. Thus the calculation of the Bogoliubov coefficient \( \beta_k \) performed in Ref. [1] cannot apply to modes contained in the “tail” today.

III. THE TAIL ENERGY DENSITY

In this Section we calculate the amount of energy density created in quanta that redshift out of the “tail”, and show that it leads to a severe back-reaction problem. In Ref. [1] the energy density contained in the tail is calculated as

\[ \langle \rho_{\text{tail}} \rangle = \frac{1}{2\pi} k_{\text{phys}} dK_{\text{phys}} \int \omega_{\text{phys}} d\omega_{\text{phys}} |\beta_{k_{\text{phys}}}|^2, \] (12)

where \( k_H \) is the physical wave-number such that \( \omega_H \equiv \omega_{\text{phys}}(k_{\text{phys}}) = H_0 \) today. This expression for \( \langle \rho_{\text{tail}} \rangle \) is ill-defined due to the double integration element \( dK_{\text{phys}} \) in the absence of a Dirac function on the mass shell. The total energy density \( \langle \rho_{\text{total}} \rangle \) is defined analogously but the lower bound is extended to \( k \geq 0 \). Then, it is argued that \( \langle \rho_{\text{tail}} \rangle / \langle \rho_{\text{total}} \rangle \simeq 10^{-15} \) during inflation. Note that if \( \rho_{\text{tail}} \simeq 10^{-15} M_{\text{Pl}}^4 \) and \( \langle \rho_{\text{tail}} \rangle \) is constant (corresponding to a vacuum-like equation of state as suggested) in order to account for the dark energy then the above statement yields \( \rho_{\text{total}} \sim M_{\text{Pl}}^4 \). If this holds during inflation, one faces a severe back-reaction problem since the background energy density during inflation is \( \sim 10 \) orders of magnitude below \( M_{\text{Pl}}^4 \), and the overall calculation framework (a test quantum field on a classical background) breaks down. As we argue in this Section, it is actually a generic prediction of this model that \( \rho_{\text{total}} \sim M_{\text{Pl}}^4 \) at all times. This result \( \rho_{\text{total}} \sim M_{\text{Pl}}^4 \) is in agreement with a recent work by Starobinsky [8], which showed that models with dispersion relations such that the WKB approximation is not valid in the far past when the physical wave-number \( k_{\text{phys}} \gg M_{\text{Pl}} \) implies a very substantial amount of particle production.

In the following we calculate the amount of energy density stored in modes with physical wave-number \( k_{\text{phys}} \sim M_{\text{Pl}} \). The calculation follows the line of thought indicated in the previous Section. Since in the range \( H \ll k_{\text{phys}} \ll K_{\text{phys}} \) the WKB approximation is valid, one can decompose the solution to the field equation in terms of plane waves as in Eq. (11) when modes enter this regime. As long as \( |Q/\Omega_0^2| \ll 1 \) one can neglect the time evolution of \( \beta_k \), and it is natural to interpret \( |\beta_k|^2 \) as the occupation number of particles in mode \( k \). As argued earlier this decomposition in plane waves cannot be made for modes that are still contained in the tail.

One can then calculate the amount of energy density \( d\rho_\omega/d\ln(k_{\text{phys}}) \) stored in the log interval around the physical wave-number \( k_{\text{phys}} \) and the corresponding fractional density parameter \( d\Omega_\omega/d\ln(k_{\text{phys}}) \) in units of the background energy density:

\[ \frac{d\Omega_\omega}{d\ln k_{\text{phys}}} = \frac{4 k_{\text{phys}}^3 \omega_{\text{phys}} |\beta_k|^2}{3\pi H^2 M_{\text{Pl}}^4}, \] (13)

using \( d\rho_\omega/d\ln(k_{\text{phys}}) = 4 k_{\text{phys}}^3 \omega_{\text{phys}} |\beta_k|^2 / 2\pi^2 \). The fractional density parameter must be smaller than unity at all times and for all physical wave-numbers, otherwise back-reaction is significant and all semi-classical first order calculations are unreliable. In the following we calculate this quantity \( d\Omega_\omega/d\ln(k_{\text{phys}}) \) for a physical wave-number \( k_{\text{phys}} \sim k_c \), i.e., once the wavelength becomes larger than the fundamental scale. It can be expressed via \( \beta_k \) in terms of the constants
that parametrize the choice of initial data. Our goal here is to study the dependence of the amount of energy density created in modes of physical momenta \( \sim M_{\text{Pl}} \) on the initial data, for which there is no definite prescription as we argued in the previous Section.

A. Conformal coupling: \( \xi = 1/6 \)

In the case of conformal coupling \( \xi = 1/6 \) there exists an exact solution to the field equation written in terms of the two independent solutions \( \mu^{(1)}_k \) and \( \mu^{(2)}_k \) in Eq. (8). One can then calculate the Bogoliubov coefficient deep in the region where the WKB approximation is valid, for instance around \( k_{\text{phys}} \sim k_c \) by decomposing this exact solution in plane waves. However the coefficients of the hypergeometric function in term of which the exact solution hence \( \beta_k \) are written are of order \( k_c/H \gg 1 \). For values of these coefficients that are relevant for our cosmological applications (i.e. \( k_c/H \sim 10^6 \) during inflation), the numerical calculation of the hypergeometric function turns out to be too involved and we have been unable to calculate \( \beta_k \) in a reasonable amount of time for \( k_c/H \gtrsim 10^3 \). Therefore we take a different approach and approximate the exact solution in the tail \( k_{\text{phys}} > k_+ \) by the solution derived in terms of Bessel functions in Eq. (7), and that in the region \( k_c < k_{\text{phys}} < k_+ \) by the plane wave solution. The Bogoliubov coefficient \( \beta_k \) of the plane wave solution is obtained by matching the two solutions and their first derivatives at the transition point \( k_{\text{phys}} = k_+ \). Of course, it gives an approximation to \( \beta_k \), but as we show in the following the deviation from the overall behavior of \( \beta_k \) away from its minimum and on its behavior around its minimum are negligible. We thus proceed as follows: in the following Section, we calculate the Bogoliubov coefficient denoted \( \beta_k^{(\text{approx})} \) by solving for the Bessel functions in the remote past and performing the matching at \( k_+ \). In the subsequent Section, we calculate the Bogoliubov coefficient \( \beta_k^{(\text{exact})} \) analytically using the exact solution and demonstrate that \( \beta_k^{(\text{approx})} \) is a good approximation for values of \( k_c/H \) as high as \( \simeq 10^6 \). Finally we examine the behavior of \( \beta_k^{(\text{approx})} \) and evaluate the amount of energy density produced by the non-adiabatic evolution of modes in the “tail” for realistic values of \( k_c/H \). This calculation is entirely analytical, only the verification of the accuracy of the approximation is numerical.

1. Approximate calculation of the Bogoliubov coefficient

As already mentioned above, see Eq. (7), the mode function in the tail can be approximatively expressed in terms of the Bessel and Neumann functions \( J_\nu \) and \( N_\nu \) as

\[
\mu_k(\eta) \approx \frac{1}{\sqrt{2k}} \left[ A_1(k)J_\nu(\eta^2) + A_2(k)N_\nu(\eta^2) \right], \quad \eta \equiv \frac{H}{2k_c}k\eta + \frac{1}{2} \ln \left[ \frac{1}{2} \left( 1 - \frac{\epsilon_1}{4} \right) \right].
\]

This solution is valid if the scale factor is that of the de Sitter space-time: \( a(\eta) = -1/(H \eta) \). The mode function must satisfy the relation \( W \equiv \mu_1 \mu_2 - \mu_2 \mu_1 = i \). Using the above equation, one finds that the Wronskian is equal to \( W = -H(A_2A_1^* - A_1A_2^*)/(2\pi k_c) \). As a consequence, if one represents the coefficient \( A_2 \) in polar form, \( A_2 \equiv re^{i\Phi} \), one has \( A_1 = -\pi k_c/(H r \sin \Phi) \), where we have chosen \( A_1 \) to be real. The parameters \( r \) and \( \Phi \) will characterize in the following the choice of initial data.

In the region where the WKB approximation is valid, i.e. for \( \omega_{\text{phys}} \gg H \), one has

\[
\mu_k(\eta) \approx \frac{a_k}{\sqrt{2\omega(k, \eta)}}e^{-i\Omega} + \frac{\beta_k}{\sqrt{2\omega(k, \eta)}}e^{i\Omega},
\]

where \( \Omega \equiv \int_0^\infty d\tau \omega(k, \tau) \). In order to express the Bogoliubov coefficient \( |\beta_k| \) in terms of the constants parameterizing the choice of the initial data in the tail, \( A_1(k) \) and \( A_2(k) \), we use the continuity of the mode function \( \mu_k \) and of its derivative at the transition between the two regions at \( y = y_m \), for which \( \omega_{\text{phys}}(y_m) = \sqrt{2}H \). The result reads

\[
\beta_k^{(\text{approx})} = \frac{ie^{-i\Omega}}{4\sqrt{2}\omega(k, y_m)} \left\{ A_1(k) \left[ -\gamma_k(y_m)J_0(\eta^2) + \frac{H}{2k_c}e^{i\eta^2}J_1(\eta^2) \right] + A_2(k) \left[ -\gamma_k N_0(\eta^2) + \frac{H}{2k_c}e^{i\eta^2}N_1(\eta^2) \right] \right\}
\]

where \( \gamma_k \equiv \omega/(2\omega) + i\omega \). Working out this last expression, one obtains

\[
\beta_k^{(\text{approx})} = \frac{ie^{-i\Omega}}{4\sqrt{2}H} \left( \frac{H}{k_+} \right)^{1/2} \left\{ A_1(k) \left[ -\left( \frac{K_+}{4k_c} + i\sqrt{2} \right) J_0(\eta^2) + \sqrt{2}J_1(\eta^2) \right] + A_2(k) \left[ -\left( \frac{K_+}{4k_c} + i\sqrt{2} \right) N_0(\eta^2) + \sqrt{2}N_1(\eta^2) \right] \right\}.
\]
We are now in a position where we can compute the \( |\beta_h^{\text{approx}}|^2 \) using the parametrization of the coefficients \( A_1 \) and \( A_2 \) introduced above. The final result reads

\[
|\beta_h^{\text{approx}}|^2(\rho, \Phi) = \frac{\pi^2}{4\sqrt{2}} \frac{J}{\rho^2 \sin \Phi} + \frac{N \rho^2}{4\sqrt{2}} - \frac{\pi K}{2\sqrt{2}} \cot \Phi - \frac{1}{2},
\]

where we have defined the rescaled variable \( \rho \) by \( \rho \equiv r \sqrt{H / K} \) and where the coefficients \( J, N \) and \( K \) can be expressed as

\[
J = \frac{1}{16} J_0^2 - \frac{1}{2} \frac{k^2}{K^2} J_0 J_1 + 2 \frac{k^2}{K^2} (J_0^2 + J_1^2), \quad N = \frac{K_+}{16k_c^2} N_0^2 + \frac{1}{2} \frac{K}{k_c} N_0 N_1 + 2(N_0^2 + N_1^2),
\]

\[
K = \frac{K_+}{16k_c} J_0 N_0 + \sqrt{\frac{2}{3}} \left( J_0 N_1 + J_1 N_0 \right) + 2 \frac{k^2}{K} \left( J_0 N_0 + J_1 N_1 \right).
\]

The Bessel and Neumann functions are evaluated at the matching point for which their argument reads \( e^{\beta m} = 2\sqrt{2} k_c / K_0 \). A direct calculation shows that \( J N - K^2 = 2 / \pi^2 \). The Bogoliubov coefficient \( |\beta_h^{\text{approx}}|^2 \) can be viewed as a two-dimensional surface parametrized by the polar coordinates \((\rho, \Phi)\).

### 2. Test of the method of approximation

Before studying the above Bogoliubov coefficient in greater detail, one must check that the approximation is well-controlled. For this purpose, it is interesting to calculate the Bogoliubov coefficient using the exact solution expressed in terms of hypergeometric functions

\[
\mu_k(\eta) = \frac{1}{\sqrt{2k}} \left[ C_1(k) \mu_{\text{ex}}^{(1)}(\eta) + C_2(k) \mu_{\text{ex}}^{(2)}(\eta) \right],
\]

where the functions \( \mu_{\text{ex}}^{(1)} \) and \( \mu_{\text{ex}}^{(2)} \) have been defined in Eq. (8) above, and the (dimensionless) functions \( C_1(k) \) and \( C_2(k) \) are related to each other by the Wronskian normalization condition [12]:

\[
\mu_k \mu_k^{*} - \mu_{\text{ex}}^{(1)} \mu_{\text{ex}}^{(2)*} = \left[ |C_1(k)|^2 - |C_2(k)|^2 \right] e^{2\pi i b} \frac{1 - e^{2\pi i b}}{2k_c |\eta|} \mathcal{F},
\]

with the numerical factor \( \mathcal{F} \) is written in terms of the parameters \( b \) and \( d \), as:

\[
\mathcal{F} = \frac{(b + d + \frac{1}{2})^2}{2b + 1} 2F_1 \left( \begin{array}{c} b + d + \frac{3}{2} \end{array} , \begin{array}{c} b - d + \frac{1}{2} \end{array} ; 2b + 2 ; -1 \right) 2F_1 \left( \begin{array}{c} -b - d + \frac{1}{2} \end{array} , \begin{array}{c} -b + d + \frac{1}{2} \end{array} ; -2b + 1 ; -1 \right) + \\
2b_2 F_1 \left( \begin{array}{c} b + d + \frac{1}{2} \end{array} , \begin{array}{c} b - d + \frac{1}{2} \end{array} ; 2b + 1 ; -1 \right) 2F_1 \left( \begin{array}{c} -b - d + \frac{1}{2} \end{array} , \begin{array}{c} -b + d + \frac{1}{2} \end{array} ; -2b + 1 ; -1 \right) + \\
\frac{(-b + d + \frac{1}{2})^2}{2b - 1} 2F_1 \left( \begin{array}{c} b + d + \frac{1}{2} \end{array} , \begin{array}{c} b - d + \frac{1}{2} \end{array} ; 2b + 1 ; -1 \right) 2F_1 \left( \begin{array}{c} -b - d + \frac{1}{2} \end{array} , \begin{array}{c} -b + d + \frac{1}{2} \end{array} ; -2b + 2 ; -1 \right).
\]

This solution is valid at all times since it is an exact solution of the field equation. In this case, one can calculate the Bogoliubov coefficient at any time provided the WKB approximation is then valid, using:

\[
|\beta_h^{\text{exact}}|^2 \equiv \frac{1}{\sqrt{2\omega \mu_k}} \left[ \frac{\omega}{2\omega} + i \omega \right] \mu_k,
\]

where, in the last expression, \( \mu_k \) is given by Eq. (21). Notice that this procedure differs from the previous calculation of the Bogoliubov coefficient. Here, we do not perform a matching at the transition between the tail and the WKB region but rather use the exact solution (21) all the way through and calculate its “overlap” with the WKB solution deep in the WKB region. The initial conditions enter this expression via the two constants \( C_1(k) \) and \( C_2(k) \).

We need to compare \( |\beta_h^{\text{exact}}|^2 \) with \( |\beta_h^{\text{approx}}|^2 \) for the same initial conditions. Since \( |\beta_h^{\text{approx}}|^2 \) is expressed in terms of the constants \( A_1(k) \) and \( A_2(k) \), one needs to re-express \( A_1(k) \) and \( A_2(k) \) in terms of \( C_1(k) \) and \( C_2(k) \). This can be done by matching the asymptotic behaviors of the two solutions deep in the tail, i.e. in the limit \( \eta \to -\infty \). There, the approximate solution given by Eq. (14) reduces to

\[
\mu_k(\eta) \simeq \frac{1}{\sqrt{2k}} \left( A_1 - A_2 \frac{H}{\pi k_c} \right) |\eta| \left[ -2A_2 \ln 2 + 2A_2 \gamma_E + A_2 \ln \left( \frac{4k_c^2(4 - c_t)}{H^2} \right) \right],
\]
where $\gamma_E$ is the Euler constant, $\gamma_E \simeq 0.5772$. On the other hand, the exact solution of Eq. (21) can be written as

$$\mu_k(\eta) \simeq \frac{1}{\sqrt{2\pi k_c}} \left\{ (G C_1 + G^* C_2) - G C_1 \left[ 2\gamma_E + \Psi \left( \frac{b + d + \frac{1}{2}}{2} \right) + \Psi \left( \frac{b - d + \frac{1}{2}}{2} \right) \right] 
- G^* C_2 \left[ 2\gamma_E + \Psi \left( -\frac{b + d + \frac{1}{2}}{2} \right) + \Psi \left( -\frac{b - d + \frac{1}{2}}{2} \right) \right] \right\},$$

(26)

where the coefficient $C_i$ is given in terms of the Euler Beta function as: $G = 1/B(b + d + 1/2, b - d + 1/2) = \Gamma(2b + 1)/[\Gamma(b + d + 1/2) \Gamma(b - d + 1/2)]$, and satisfies, since $b$ is pure imaginary and $d$ is real, $G^* \equiv G(b \leftrightarrow -b)$. This relation stems from the asymptotic behavior of the hypergeometric functions for large values of their argument given by Eq. (15.3.13) of Ref. [9]. The di-Gamma function $\Psi(x)$ function is defined by $\Psi(x) \equiv d \ln \Gamma(x)/dx$. If one identifies the constant term and the linear term in conformal time of the two previous relations, we obtain:

$$A_1(k) = G C_1(k) \left\{ \ln \left[ \frac{k_c^2(4 - c_1)}{H^2} \right] - \Psi \left( \frac{b + d + \frac{1}{2}}{2} \right) - \Psi \left( \frac{b - d + \frac{1}{2}}{2} \right) \right\},$$

$$A_2(k) = -\pi \left[ G C_1(k) + G^* C_2(k) \right].$$

(27)

(28)

Then, it is sufficient to use the above relations in Eq. (17) to obtain $|\beta_k^{(\text{approx})}|$ in terms of $C_1(k)$ and $C_2(k)$ and compare it to $|\beta_k^{(\text{exact})}|$. In order to characterize the accuracy with which the Bogoliubov coefficient is calculated, we plot the following quantity

$$A \equiv 2 \left| \frac{\beta_k^{(\text{approx})}}{\beta_k^{(\text{approx})}} - \frac{\beta_k^{(\text{exact})}}{\beta_k^{(\text{exact})}} \right|,$$

(29)

for various values of the coefficients $C_1(k)$ and $C_2(k)$. More precisely, we use a polynomial representation and take $C_2(k) = Re\delta'$ while $C_1(k)$ is real and calculated in terms of $C_2(k)$ using the Wronskian relation Eq. (22). In Fig. 1, we have plotted $A(\rho, \phi)$ for $k_c/H = 10^2$ and $k_c = M_{Pl}$. We see that the error for large values of $\rho$ is less than $\sim 40\%$ and is constant, i.e. the offset between the two Bogoliubov coefficients does not depend on $\rho$ and $\phi$ in a first approximation. For $\rho \sim 0$, the error increases to 1; this artefact comes from the fact that the minimum where the two Bogoliubov coefficients vanish are slightly offset one from the other. If one coefficient vanishes while the other remains finite and non-zero, then the value of $A$ is pushed toward one, $A \to 1$. This error however is of no consequence for what follows. Indeed we will not be interested in the location of the minimum but in the behavior of $\beta_k$ around the minimum and
far from the minimum. As is obvious from Fig. 1, these behaviors match closely in both cases and our approximation will be sufficient for our purposes. We have checked that the function $A$ remains the same for other values of $k_c/H$ which allow numerical calculations, i.e. $k_c/H \in [10, 10^3]$.

The situation is in fact very similar to the standard calculation of the power spectrum in an inflationary theory: in principle, one cannot match two different branches at a point where the approximation breaks down (for standard inflation this occurs at first horizon crossing). However, since the approximation is only violated in a small region one expects the corresponding result to be correct at leading order. This is indeed the case for inflation for which the amplitude of the spectrum is predicted up to a factor of order unity and the spectral slope is unchanged by the matching. Here we also find that $|\dot{\beta}_k^{\text{approx}}| = \mathcal{O}(1)|\dot{\beta}_k^{\text{exact}}|$.

3. Fine-tuning of the initial conditions

Since we have demonstrated that the approximation to $|\dot{\beta}_k|$ is quite reasonable, we now study the behavior of Eq. (18) for more realistic values of the ratio $k_c/H$. The Bogoliubov coefficient possesses an absolute minimum with $\beta_k = 0$ located at

$$
\rho_{\min} = \frac{\pi^2 J^2}{J N - K^2}, \quad \cos \Phi_{\min} = \frac{K}{\sqrt{J N}},
$$

using the notations defined previously. One should not be surprised to find a minimum with $\beta_k = 0$ since one can express the matching of the two branches of the solution and their first derivatives at $\eta = 0$ as two equations relating the coefficients $A_1(k)$ and $A_2(k)$ as a function of $\alpha_k$ and $\beta_k$ and find a solution with $\beta_k = 0$. The Wronskian normalization condition is always satisfied by both branches of the solution. This solution with initial conditions ($\rho_{\min}$, $\Phi_{\min}$) corresponds to a choice of initial data such that at late times, when modes have exited from the tail, their quantum state is that of an adiabatic vacuum. Note therefore that there is no naturalness in choosing these initial conditions since the adiabatic vacuum is only a late time consequence of such initial data. Furthermore one can show that for generic initial data, the state of the quantum field at late times is not an adiabatic vacuum, hence quanta have been produced.

Indeed the behavior of $|\dot{\beta}_k|^2$ around this absolute minimum can easily be established. From a Taylor expansion, one obtains

$$
|\dot{\beta}_k|^2 \simeq \frac{N}{\sqrt{2}} (\rho - \rho_{\min})^2, \quad |\dot{\beta}_k|^2 \simeq \frac{\pi^4}{16} N^2 (\Phi - \Phi_{\min})^2.
$$

For a crude order of magnitude estimate, one can develop the Bessel functions to first order in the small argument limit $e^{y_{min}} = 2 \sqrt{2} k_c/K_+ < 1$ (more exactly, for de Sitter inflation and $k_c = M_{Pl}$, one has $k_c/K_+ \approx 0.06$). This leads to $J \simeq 1/16 + \mathcal{O}(k_c^2/K_+^2)$ and $N \simeq \ln^2(\sqrt{2} k_c/K_+) / (4 \pi^2) (K_+^2/k_c^2) + \mathcal{O}(k_c^2/K_+^2)$. Thus in order to avoid a back-reaction problem, the initial conditions $\rho$ and $\Phi$ must not differ too much from $\rho_{\min}$ and $\Phi_{\min}$ which lead to $\beta_k = 0$ (hence a zero amount of energy density created). More precisely, the energy density produced is of the order of the background energy density, i.e. $d\Omega_e/d\ln(k_{phys}) = 1$, when $\rho$ or $\Phi$ respectively depart from the minimum by amounts $\delta \rho$ or $\delta \Phi$ given by:

$$
\delta \rho \simeq \mathcal{O} \left( \frac{H}{M_{Pl}} \ln^{-1} \left( \frac{M_{Pl}}{H} \right) \right), \quad \delta \Phi \simeq \mathcal{O} \left( \frac{H}{M_{Pl}} \ln^{-2} \left( \frac{M_{Pl}}{H} \right) \right).
$$

Here we have assumed $k_c = M_{Pl}$. Hence the corresponding fine-tuning of the initial conditions is of order $H/M_{Pl}$ (if one assumes a uniform measure in $\rho$ and $\Phi$ in parameter space).

One should note that the above constraint is valid for a given co-moving wave-number $k$ and has been calculated at a time when $k/a = k_c$. Since the fractional density parameter of quanta extracted out of the vacuum $d\Omega_e/d\ln(k_{phys})$ must be smaller at all times during inflation, i.e. for a range of co-moving wave-numbers $k$ since $k \ll H$ can be related by the above constraint $k/a = k_c$, the above constraint on $\rho_{\min}$ and $\Phi_{\min}$ rather applies to a continuum of values of co-moving wave-numbers. In other words one does not have to fine-tune two parameters characterizing the initial data but a whole continuum of parameters, i.e. the functions $\rho_{\min}(k)$ and $\Phi_{\min}(k)$ themselves. The dependence in $k$ of these functions is hidden in the argument of the Bessel and Neumann functions $e^{\rho \tau}$, since $\eta_{\min}$ depends on $k$.

B. Non conformal coupling: $\xi \neq 1/6$

One can also perform a similar calculation of the Bogoliubov coefficient when the coupling is no longer conformal $\xi \neq 1/6$. In this case the calculation can be performed analytically for all background scale factors. For the sake of
simplicity we choose minimal coupling $\xi = 0$ but this can be trivially expanded to various choices of the coupling to gravity, and does not modify the conclusions we derive below.

If $\xi = 0$, the evolution of the modes is dominated by $a''/a$ in the tail, i.e. when $\omega_{\text{phys}} \ll H \left( k_{\text{phys}} \gg k_c \right)$, and the solution can be written as:

$$\mu_k(\eta) = \frac{1}{\sqrt{2k}} \left[ C_+(k) \frac{a(\eta)}{a_0} - C_-(k) \frac{a_0}{a} \frac{d}{d\eta} \int^{\eta}_{\eta_0} \frac{d\tau}{\sqrt{a^2(\tau)}} \right],$$

where $C_+(k)$ and $C_-(k)$ are two dimensionless $k$-dependent constants, and $\eta$ is some initial conformal time. One can check that this solution and the power-laws given in Eq. (4) are the same. Here one cannot choose the time of matching $\eta_m$ to the WKB solution, since the matching has to be done when $\Omega_k = 0$, i.e. when $\omega_k^2 = \mu_k^2 = a''/a$ or $k_{\text{phys}} = K_+$. In the region $\omega_{\text{phys}} \gg H$, i.e. for $\eta \gg \eta_m$, the WKB approximation is valid as we have seen before, and the matching to the WKB form is justified at $\eta \equiv \eta_m(k)$. For a given wave-number $k$, we are free to set $\eta_m(\eta) = \eta_m$, since this amounts to a redefinition of the constants $C_+(k)$ and $C_-(k)$ by a function of $k$. The matching at $\eta_m$ then gives:

$$\alpha_k = \frac{i}{\sqrt{4k_\omega}} \left[ C_+ \left( \gamma_k + \frac{d'}{a} \right) - \frac{C_-}{\eta_m} \right], \quad \beta_k = -\frac{i}{\sqrt{4k_\omega}} \left[ C_+ \left( \gamma_k + \frac{d'}{a} \right) - \frac{C_-}{\eta_m} \right],$$

with the function $\gamma_k \equiv \omega_k^2/2 \omega + \omega_\omega$ as above [see Eq. (16)] and where all quantities in the above two equations are understood to be taken at $\eta = \eta_m(k)$. In particular, at time $\eta_m$, $\omega_k^2 = \mu_k^2 = (1 + \beta) a_m^2 H_m^2 / \beta$. Since $k_{\text{phys}} \gg k_c$ at $\eta_m$ one can use the limiting form of $\omega_{\text{phys}}$ given in Eq. (9), hence:

$$\gamma_k + \frac{d'}{a} \simeq a_m H_m \left( \frac{K_+}{4k_c} + 1 + i \sqrt{\frac{1 + \beta}{\beta}} \right).$$

The constants $C_+(k)$ and $C_-(k)$ are related to one another by the normalization of the mode functions: $\mu_k^2 - \mu_k^* \mu_k = i$, which gives:

$$C_+(k) = -\beta \frac{K_+}{H_m r \sin \Phi},$$

and as before we keep $r \equiv |C_+(k)|$ and $\Phi = \arg(C_+(k))$ as the two independent parameters characterizing the choice of initial data. One finally derives the squared modulus of the Bogoliubov coefficient $\beta_k$ as:

$$|\beta_k|^2 = \frac{1}{r^2 \sin^2 \Phi} \frac{\rho^2}{4} \sqrt{\frac{\beta}{\beta + 1 + H_m + 1}} \left( 1 + \frac{K_+}{4k_c} \right)^2 + \frac{1 + \beta}{\beta} + \frac{r^2}{4 \beta^2} \sqrt{\frac{\beta}{1 + \beta K_+} - 1} \frac{1}{2 \beta^{3 / 2}} \sqrt{1 + \frac{K_+}{4k_c}} \cot \Phi - \frac{1}{2} \left( \beta \right)^2 \left[ \frac{1}{2} \right]$$

Since $K_+ / H_m$ is a large number, in the following we use the rescaled variable $\rho \equiv r \sqrt{H / K_+}$ instead of $r$. Equation (37) above is particularly attractive because it has exactly the same functional shape as Eq. (18). It allows us to understand analytically the behavior of the amount of energy density produced in modes with $k_{\text{phys}} \sim k_c$ as a function of the initial data. The occupation number $|\beta_k|^2$ has an absolute minimum located at:

$$\rho_{\text{min}} = |\beta| \left( \frac{\beta}{1 + \beta} \right)^{1 / 4} \sqrt{\frac{1 + K_+ / (4k_c)}{1 + \frac{K_+ / (4k_c)}{1 + \beta}}} \left[ \sqrt{1 + \frac{K_+ / (4k_c)}{1 + K_+ / (4k_c)}} + (1 + \beta) / \beta \right], \quad \cos \Phi_{\text{min}} = \frac{1 + K_+ / (4k_c)}{\sqrt{1 + K_+ / (4k_c)}^2 + (1 + \beta) / \beta},$$

As before the occupation number vanishes exactly at this minimum, but the back-reaction problem cannot be avoided for generic initial conditions. In the present case it is not possible to make a sensible contour plot of $d\Omega_k / d \ln(k_{\text{phys}})$ since this function changes by many orders of magnitude over very small intervals of $\rho$, $\Phi$. Therefore, we take a conservative approach in which we calculate $|\beta_k|^2$ as a function of $\rho$ for the values of $\Phi$ that minimize this quantity at each $\rho$. We also evaluate $|\beta_k(\Phi)|^2$ as a function of $\Phi$ for the values of $\rho$ that minimize this quantity at each $\Phi$. In other words, we solve $\partial_\rho |\beta_k|^2 = 0$ for $\Phi$ as a function of $\rho$ and $\partial_\Phi |\beta_k|^2 = 0$ for $\rho$ as a function of $\Phi$:

$$\Phi_{\text{min}}(\rho) = \tan^{-1} \left( \frac{\beta^2 \left[ 1 + K_+ / (4k_c) \right]^2 + (1 + \beta) / \beta}{\rho^2 \left[ 1 + K_+ / (4k_c) \right]^2 + (1 + \beta) / \beta} \right), \quad \rho_{\text{min}}(\Phi) = \frac{\beta^2}{\sin^2 \Phi} \left[ \left( 1 + \frac{K_+}{4k_c} \right)^2 + (1 + \beta) / \beta \right],$$
FIG. 2: Left panel: the solid line represents \( \log_{10} \left( \left| \beta_k \right|^2 / \beta_{\text{max}}^2 \right) \equiv \log_{10} \left( \partial \Omega / \partial \ln(k_{\text{phy}}) \right) \) plotted as a function of the rescaled variable \( \rho \) characterizing the initial data. This plot corresponds to an inflationary era with a de Sitter metric, and Hubble parameter \( \sim 10^{-6} M_{\text{Pl}} \). The other parameter of initial data is \( \Phi = \Phi_{\text{min}} \). Allowed regions correspond to \( \log_{10} \left( \left| \beta_k \right|^2 / \beta_{\text{max}}^2 \right) < 0 \), and are peaked around a particular value of \( \rho \). The minimum is in fact \( \left| \beta_k \right|^2 = 0 \), corresponding to \( \log_{10} \left( \left| \beta_k \right|^2 / \beta_{\text{max}}^2 \right) = -\infty \), but cannot be seen in the figure due to insufficient resolution. The dotted line provides a continuation of the numerical result to the analytical value at that point. In most of parameter space, the energy density is too large by \( \sim 10 \) orders of magnitude. Right panel: Same as left panel for \( k_0 \sim 10^{-6} M_{\text{Pl}} \) in a matter dominated era. In nearly all of parameter space the energy density is too large by \( \sim 122 \) orders of magnitude.

FIG. 3: Left panel: \( \log_{10} \left( \left| \beta_k \right|^2 / \beta_{\text{max}}^2 \right) \equiv \log_{10} \left( \partial \Omega / \partial \ln(k_{\text{phy}}) \right) \) plotted as a function of the phase characterizing the initial data. This plot corresponds to an inflationary era with a de Sitter metric, and Hubble parameter \( \sim 10^{-6} M_{\text{Pl}} \). As indicated in the text, the other parameter of initial data (a modulus) has been eliminated for the phase \( \Phi \) by minimizing \( \left| \beta_k \right|^2 \). Allowed regions correspond to \( \log_{10} \left( \left| \beta_k \right|^2 / \beta_{\text{max}}^2 \right) < 0 \), and are seen to be peaked around a particular value of \( \Phi \). The plot is symmetric in the interchange \( \Phi \leftrightarrow \Phi + \pi \). Note that the ordinate scale is in \( \log_{10} \); in most of parameter space, the energy density is too large by \( \sim 10 \) orders of magnitude. Right panel: Same as left panel for \( k_0 \sim 10^{-6} M_{\text{Pl}} \) in a matter dominated era. In nearly all of parameter space the energy density is too large by \( \sim 122 \) orders of magnitude, i.e. \( \partial \Omega / \partial \ln(k_{\text{phy}}) \sim M_{\text{Pl}}^4 \).

and plot \( \partial \Omega / \partial \ln(k_{\text{phy}}) \) for \( \beta_k[\rho, \Phi_{\text{min}}(\rho)] \) and \( \beta_k[\rho_{\text{min}}(\Phi), \Phi] \) in Figs. 2 and 3 (for respectively de Sitter inflation and today). One clearly sees from these figures that \( \partial \Omega / \partial \ln(k_{\text{phy}}) \sim M_{\text{Pl}}^4 / H^2 \) for most values of \( \Phi \), which corresponds to our previous expectations, i.e. the amount of energy density created in quanta with \( k_{\text{phy}} \sim k_{\text{c}} \sim M_{\text{Pl}} \) is of order \( M_{\text{Pl}}^4 \). The behavior of \( \left| \beta_k \right|^2 \) around the local minimum can be studied in the same way as before and one
obtains:
\[
|\beta_k|^2(\rho) \simeq \frac{1}{\beta^2} \left( \frac{\beta}{1 + \beta} \right)^{1/2} (\rho - \rho_{\min})^2, \quad (\rho - \rho_{\min}),
\]
\[
|\beta_k|^2(\Phi) \simeq \frac{1}{4} \left( \frac{\beta}{1 + \beta} \right)^2 \left[ \left( 1 + \frac{K_+}{4c_0^2} \right) + \frac{1 + \beta}{\beta} \right]^2 (\Phi - \Phi_{\min})^2, \quad (\Phi - \Phi_{\min}).
\]
If we write the value $|\beta_k|^2_{\text{max}}$ such that $d\Omega_\omega / d\ln(k_{\text{phys}}) = 1$ (i.e. similar amount of energy created in quanta with $k_{\text{phys}} \sim k_c$ than in the background), then this value is reached if $\rho$ and $\Phi$ depart from $\rho_{\min}$ and $\Phi_{\min}$ by an amount $\delta \rho$, $\delta \Phi$, with:
\[
\delta \rho \sim \mathcal{O} \left( \frac{H}{M_{\text{Pl}}} \right), \quad \delta \Phi \sim \mathcal{O} \left( \frac{H}{M_{\text{Pl}}} \ln^{-2} \left( \frac{M_{\text{Pl}}}{H} \right) \right).
\]
In other words, the fine-tuning necessary to avoid back-reaction is of order $\sim \delta \Phi \sim H/M_{\text{Pl}}$ (neglecting the $\ln$ term). During inflation $H/M_{\text{Pl}} \sim 10^{-6}$, and today $H/M_{\text{Pl}} \sim 10^{-11}$. Although the fine-tuning can be considered as not too severe during inflation, we see that, during the matter epoch, it is as severe as the usual fine-tuning of the cosmological constant problem. Therefore, the scenario of Refs. [1, 2, 3] does not improve the situation.

At this point, it should be emphasized that the above problem is a generic feature of the dispersion relation used in Refs. [1, 2, 3]. In order for the trans-Planckian effects to modify the power spectrum of the fluctuations, the physical modes of interest must spend some time in a region where the WKB approximation is violated. As already mentioned, this implies production of particles and the energy density associated to these particles must not exceed the background energy density. This implies some constraints on the occupation number $|\beta_k|^2$. It has been shown in Ref. [8] that these constraints are quite stringent if the production is taking place today. Usually, these tight constraints can be avoided by requiring that the violation only occurs during inflation where the problem is less severe. This can be achieved if the dispersion relation is such that $\omega_{\text{phys}} \gg H$ for all trans-Planckian wave-numbers, with $H$ the Hubble constant some unspecified time after inflation. Such an example is provided in Fig. 2 of Ref. [8]. Then as was argued in the discussion following Eq. (9) the WKB approximation should be valid at all times after inflation and adiabaticity restored. Here however, since $\omega_{\text{phys}} \to 0$ as $k_{\text{phys}} \to +\infty$, there is at all times a region where the WKB is violated. Therefore the class of dispersion relations used in Refs. [1, 2, 3] suffer in a generic way from the problem discussed in Ref. [8].

Furthermore, we also note as previously that the above calculation of the fine-tuning holds for a given co-moving wave-number $k$. Similar constraints apply for other wave-numbers but the values of $\Phi_{\min}$ and $\rho_{\min}$ are shifted from the above [notably because of the choice $\eta_\rho = \eta_{\min}(k)$]. Therefore one must not only pick one right value of $\Phi$ to one part in $\sim M_{\text{Pl}}/H$, but a whole continuum of values of $\Phi(k)$ such that back-reaction is avoided for all of these values.

\section{Equation of State}

Up to now we have argued that: (i) in the scenario proposed in Refs. [1, 2, 3] there is no preferred initial vacuum state, hence all conclusions drawn depend on the ad-hoc choice of initial data; (ii) for arbitrary values of the two parameters characterizing this choice of initial data one finds that energy in excess of the background energy density is produced due to the non adiabatic evolution of modes in the tail.

In a separate publication, we have constructed an effective energy-momentum tensor for scalar field theories with non-linear dispersion relations, and we have shown that dispersion relations of the form of that proposed in Refs. [1, 2, 3] generically lead to the wrong equation of state. This finding has been challenged by Basto-Geil and Mersini recently, who argued that the energy-momentum tensor we have constructed is ill-defined as it (supposedly) is not divergenceless.

Explicitly, in Ref. [6], the vev for the energy density and pressure are given by
\[
\langle \rho \rangle = \frac{1}{4\pi^2a^2} \int dk k^2 \left[ a^2 \left( \frac{\mu_k}{a} \right)^2 + \omega^2(k) \left| \mu_k \right|^2 \right],
\]
\[
\langle p \rangle = \frac{1}{4\pi^2a^2} \int dk k^2 \left[ \frac{\mu_k}{a} \right]^2 + \frac{k^2 \omega \delta \omega^2}{3} - \omega^2 \left| \mu_k \right|^2 ,
\]
and the integrals extend from 0 to $+\infty$. Reference [3] claims that $\langle \rho \rangle + 3H(\rho + p) \neq 0$, with $H \equiv a^\prime/a$. However, noting that the co-moving frequency $\omega(k) = \omega_{\text{phys}}(k_{\text{phys}})$, a straightforward calculation gives:
\[ \langle \rho \rangle \equiv \frac{1}{4\pi a^3} \int d^3k k^3 \left\{ \left[ \mu_k^\prime - \mathcal{H} \mu_k + \left( \mathcal{H}^2 - \frac{d}{a} \right) \mu_k \right] \left( \mu_k^\prime - \mathcal{H} \mu_k \right) + \left( \mu_k - \mathcal{H} \mu_k \right) \left[ \mu_k^\prime - \mathcal{H} \mu_k + \left( \mathcal{H}^2 - \frac{d}{a} \right) \mu_k \right] 
\right. 
- 4\mathcal{H} \left[ \mu_k^\prime - \mathcal{H} \mu_k \right] + \omega^2(k) (\mu_k^\prime + \mu_k^\prime) - 2\mathcal{H} k^2 \frac{d^2}{dk^2} \left| \mu_k \right|^2 \right\}^2 
\left. \frac{1}{4\pi a^3} \int d^3k k^3 \left\{ \left( \mu_k^\prime - \frac{d}{a} \mu_k \right) (\mu_k^\prime - \mathcal{H} \mu_k^\prime) + \left( \mu_k + \mathcal{H} \mu_k \right) \left( \mu_k^\prime - \frac{d}{a} \mu_k \right) \right. 
- 6\mathcal{H} \left| \mu_k \right|^2 - 2\mathcal{H} k^2 \frac{d^2}{dk^2} \left| \mu_k \right|^2 \left. \right\}^2 
\right. 
= \frac{1}{4\pi a^3} \int d^3k k^3 \left( -6\mathcal{H} \left| \mu_k \right|^2 - 2\mathcal{H} k^2 \frac{d^2}{dk^2} \left| \mu_k \right|^2 \right). \] 
\[ (45) \]
where the field equation \( \mu_k^\prime - (d/a) \mu_k = -\omega^2(k) \mu_k \) has been used in the last step. Since
\[ 3\mathcal{H} (\rho + p) = \frac{1}{4\pi a^3} \int d^3k k^3 \left( 6\mathcal{H} \left| \mu_k \right|^2 + 2\mathcal{H} k^2 \frac{d^2}{dk^2} \left| \mu_k \right|^2 \right), \] 
the energy conservation condition \( \langle \rho \rangle \equiv 3\mathcal{H} (\rho + p) = 0 \) is trivially satisfied.

We take advantage of this Section to point out that the energy-momentum tensor we have constructed in a previous publication [6] is well behaved and the construction is entirely consistent. In effect we constructed a generally covariant Lagrangian for a scalar field with a non-linear dispersion relation. The breaking of the Lorentz invariance is explicit in introducing a dynamical four-vector \( \gamma^\mu \) in the Lagrangian whose role is to define the preferred rest frame while preserving general covariance at the same time, following previous work by Jacobson and Mattingly [13] (see also references therein). The energy-momentum tensor derived by varying the action with respect to the metric is the sum of the energy-momentum tensor of the scalar field and that of \( \gamma^\mu \). We have restricted ourselves to FLRW space-times by choosing \( \gamma^\mu = (-1, 0, 0, 0) \). This approach is consistent as \( \gamma^\mu \) satisfies its field equation [see Eq. (15) in Ref. [6]], and the scalar field also satisfies its field equation. In FLRW space-time, the energy-momentum tensor of the four-vector (i.e. the part which depends only on \( \gamma^\mu \)) vanishes since \( \gamma^\mu \) is constant [6], hence the remainder is the scalar field energy-momentum tensor, which is conserved as shown above. We consider this scalar field as a test field, which we quantize on the curved background. The dynamics of the background can be specified by adding matter fields to the action without modifying our approach. Indeed the matter field energy-momentum tensor will be separately conserved. Therefore our earlier criticisms on the equation of state of the trans-Planckian modes apply.

We also note that a recent paper by Frampton [10] argues that by adding higher order terms of the form \( \mu_{2n} u^2 u_{\phi} \) in the effective Lagrangian, one can obtain a correct equation of state, i.e. similar to that of vacuum. Here \( D^2 \) denotes a three-dimensional Laplacian expressed in a generally covariant way on space-like hyper-surfaces (see Refs. [6, 13] for explicit definitions). The choice \( u^\mu = (-1, 0, 0, 0) \) then reduces this term to the usual three-dimensional Laplacian on constant time hyper-surfaces for FLRW space-times. However, as should be obvious, the only terms that can enter the energy-momentum tensor via this new term always contain derivatives of the form \( D^2 u^\mu \) when \( n \geq 2 \). Those terms vanish when one picks \( u^\mu \) to be constant as above. For \( n = 1 \), i.e. the lowest order term of the expansion, one can check that the variation of \( D^2 u^\mu \) with respect to the metric or its first derivative always induces a term proportional to a derivative of \( u^\mu \) that is either spatial or temporal. This calculation can be found in Eqs. (B3) and (B4) of Ref. [6] (the substitution \( \phi \to u^\mu \) in this calculation can be made without changing the result since the equation is written in a generally covariant way). Therefore we conclude that the addition of terms proposed by Frampton cannot lead to any difference with respect to the energy-momentum tensor we proposed earlier. Hence such terms cannot account straightforwardly for a vacuum-like equation of state in this scenario.

The difference with the conclusion of Ref. [10] probably lies in the choice of normalization of \( u^\mu \): Ref. [10] claims that in a cosmological context, \( u^\mu u_\mu - a^{-2} = 0 \) (\( a \) is the scale factor). However, \( u^\mu u_\mu \) is a scalar, while \( a^{-2} \) is the component of a rank two tensor so that this choice of normalization is manifestly not covariant. Of course, it can be made covariant at the expense of the introduction of a second vector field but this does not seem to be the case in Ref. [10]. This implies that the Lagrangian proposed by Frampton is generally not covariant. Of course the correct normalization for \( u^\mu \) in a cosmological context is always \( u^\mu u_\mu - 1 = 0 \) (up to the sign convention), and \( u^\mu \) can be written as \((-1/a, 0, 0, 0)\) if the FLRW metric is written in terms of conformal time, or \((-1, 0, 0, 0)\) if the metric is written in terms of cosmic time.

To finish, let us also note that it is claimed in Ref. [1] that “the tail modes are still frozen at present... Thus the energy of the tail is a contribution to the dark energy of the Universe: up to the present it has the equation of state of a cosmological constant term”. Here, that the tail modes are frozen means \( (\mu_k/a)^\prime \sim 0 \), the solution to the
field equation when the term $d^2/a$ dominates (for $\xi = 0$). But there is no logical relationship between these modes being frozen and the equation of state being that of a cosmological constant. As a matter of fact the equation of state of the modes of wavelengths larger than the horizon size for a scalar field with a linear dispersion relation takes the form $p = -\rho/3$. The derivation of the equation of state requires to define correctly a stress-energy tensor when such non-linear dispersion relations are taken into account (and thus when Lorentz invariance is broken) as we did in Ref. [6]. Note also that there is a contradiction between the above claim of Ref. [1] and the whole content of the recent paper by Bastero-Gil and Mersini [3] which calculates the equation of state of the trans-Planckian modes and finds that it is not the equation of state of the vacuum but that it approaches it only at late times.

V. CONCLUSIONS

In this article, we have examined in detail the scenario proposed in Refs. [1, 2, 3] which attributes the dark energy to the properties of a scalar field with a dispersion relation that decreases exponentially with trans-Planckian wave-number $k_{\text{phys}} \gg k_c \sim M_P$. We have demonstrated that this mechanism does not work, mainly for two reasons. (i) The mode function of the scalar field does not behave as a plane wave as $\eta \to -\infty$ in the so-called “tail” (i.e., the part of the non-linear dispersion relation where $\omega_{\text{phys}} \ll H$ and $k_{\text{phys}} \gg M_P$). This implies that there is no definite prescription for constructing a well-defined initial vacuum state, hence that the choice of initial data is entirely arbitrary. This situation is similar to the problem of setting initial data for cosmological perturbations in the absence of an accelerated expansion era (but with linear dispersion relations), in which case the data would have to be specified on super-horizon scales where the mode function does not oscillate and is frozen by the expansion. In the scenario of Refs. [1, 2, 3] the WKB approximation is not valid at all times for modes in the tail. This explains that the notion of adiabatic vacuum cannot be used to set up initial data and the initial state proposed in Ref. [1] is thus ad-hoc. It also brings us to the second objection against this scenario: (ii) since all modes originate in the tail of the dispersion relation, the breakdown of the WKB approximation for a given physical wave-number at early times implies the continuous production of a substantial number of quanta with physical wave-numbers $\geq k_c$. The breakdown of the WKB approximation can indeed be seen as the signal of a strongly non-adiabatic evolution which is generically associated with particle production in expanding space-times. We have calculated the amount of energy density produced in modes of physical wave-number $k_c$ as a function of the two parameters that characterize the (arbitrary) choice of initial data. We have shown that this energy density is generically of order $M_P^3$. The production of energy density in quanta with wave-numbers $\sim M_P$ in excess of the background energy density $\sim H^2 M_P^3$ implies the breakdown of the perturbative semi-classical framework used for the calculation, and renders all claims irrelevant. There is a small region of parameter space in which this energy density can be tuned down to zero, but the fine-tuning in the choice of initial data is of order $H/M_P$. Such a fine-tuning at the time of inflation is of order $\sim 10^{-6}$ and is probably acceptable, but today, it is of the same order as the celebrated fine-tuning of the cosmological constant problem.