Algebra of chiral currents on the physical surface

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Abstract
Using a particular structure for the Lagrangian action in a one-dimensional Thirring model and performing the Dirac’s procedure, we are able to obtain the algebra for chiral currents which is entirely defined on the constraint surface in the corresponding hamiltonian description of the theory.

1 Introduction

The Thirring model is a well known model because it is exactly solvable in (1+1) dimensions [1]. If this model is treated in a Hamiltonian way in terms of the currents, with interaction between them, it can be solved in an easy way by executing a Bogoliubov transformation. This is, concerning to find the fermionic correlators using a bosonization procedure [2].

An extensive investigation of its current algebra has been performed. Dell’Antonio et. al. [3], by means of an exact expansion in bilocal operators products, they solved the model entirely in terms of the currents. They defined the term of Schwinger in function of a c-number and then they determined their form assuring that a spinors transformation law was fulfilled. Takahashi and Ogura [4] also obtain a very similar Schwinger term in their formulation of Thirring model with a regularization for the currents.

A particulary interesting description of the current algebra for this model is given by Gomes et. al. [5]. Following a Dirac’s procedure, they are able to calculate the currents algebra that live in the constraint surface, but the inclusion in that surface of the Hamiltonian which govern the dynamics and participates in the algebra it is not assured in their procedure.

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The purpose of this paper was, taking advantage of the special Lagrangian action introduced by Floreanini and Jackiw for self-dual fields [6] (used also by Gomes et. al. [5, 7, 8]), we shall to analyze the current algebra in a massless Thirring model where the proposed Lagrangian action for the currents generates a Hamiltonian that is defined also in the constraint surface. In this way, the current algebra obtained by the Dirac’s procedure [9] entirely lives in the same physical surface.

The plan of this paper is the following. In section 2, we review the known current algebra in the Dirac formalism [5], we also obtain the corresponding Hamiltonian on the constraint surface, which coincides with the usual form for the free theory. In section 3, we repeat the procedure that was successful in the free case, obtaining an complete interacting current algebra defined on the physical surface. Finally in section 4, we examine for a formal extension of this procedure for the case of N fermion currents with interaction, and in order to illustrate, we solve the complete algebra for the $N = 2$ case, which exhibits all the interesting features claimed before.

2 Non-interaction Kac-Moody $U(1)$ algebra of chiral currents on the physical surface

Based on the treatment for fermion field in term of currents explored by Dashen and Sharp [10], Sugawara [11] and Sommerfield [12], let us start with the following Lagrangian for the right ($J_R$) and left ($J_L$) currents of the model

$$L = \frac{1}{2} \int dx dy \left[ J_R(x) \varepsilon(x - y) \partial_+ J_R(y) - J_L(x) \varepsilon(x - y) \partial_+ J_L(y) \right], \quad (1)$$

here $J_{R,L}$ are the right and left currents and $\varepsilon$ is the Heaviside function ($\varepsilon(0) \equiv 0$ is assumed). This Lagrangian is similar to one used in [6] and [5], but here the canonical Hamiltonian is zero. Later a new one will be obtained on the constraint surface, which satisfy the same motion equations in it.

The constraints for this system are

$$\chi_1(x) = \Pi_R(x) - \frac{1}{2} \int dy J_R(y) \varepsilon(y - x), \quad (2)$$
$$\chi_2(x) = \Pi_L(x) + \frac{1}{2} \int dy J_L(y) \varepsilon(y - x). \quad (3)$$

These are second class constraints and they are automatically conserved because the canonical Hamiltonian vanishes.
Following the Dirac’s procedure we will calculate the Dirac’s brackets defined as
\[
\{A, B\}_D = \{A, B\} - \int dzd\omega \{A, \chi_i(z)\} C_{i,j}^{-1}(z, \omega) \{\chi_j(\omega), B\},
\]
where \(C_{i,j}^{-1}\) being the constraint Poisson brackets matrix
\[
C_{i,j}(z, \omega) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \epsilon(z - \omega),
\]
with
\[
C_{i,j}^{-1}(z, \omega) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta'(z - \omega),
\]
and \(\chi_i\) denote second class constraints, \(\{\cdot, \cdot\}\) set for usual Poisson’s brackets. After simple calculation we obtain the algebra of currents in this formalism
\[
\{J_L(x), J_L(y)\}_D = \delta'(x - y),
\]
\[
\{J_R(x), J_R(y)\}_D = -\delta'(x - y),
\]
\[
\{J_L(x), J_R(y)\}_D = 0.
\]

This is the result given by Gomes et. al. in [5]. Now we want to obtain a similar Hamiltonian to their, but assuring by construction [9, 13], that it entirely lives on the constraint surface, and that it governs the dynamics of currents involved in the algebra (6)-(8).

To obtain the Hamiltonian we will introduce the following definition
\[
Z^\alpha = \{\Pi_L; \Pi_R; J_L; J_R\},
\]
on the constraint surface. Constraints themselves are strong identities, then we can parameterize the momenta in terms of currents. Furthermore, in order to retrieve Hamiltonian relations between our new variables we must define a new Hamiltonian on that surface

\[ K = H_0 + F, \] (10)

where \( H_0 \) is the initial canonical Hamiltonian and \( F \) obeys the relation

\[ \frac{\delta F(x)}{\delta u^i} = \int dwdy \left( \frac{\partial Z^\alpha(x)}{\partial u^i(w)} W_{\alpha\beta}(x,y) \frac{\partial Z^\beta(y)}{\partial w} \right), \] (11)

where \( W_{\alpha\beta} \) is the initial symplectic matrix, \( u^i \) are \( J_L \) or \( J_R \), and \( Z^\alpha \) is a component of \( Z \) defined in (9). It looks like

\[ W_{\alpha\beta}(x-y) = \begin{pmatrix}
0 & 0 & \delta(x-y) & 0 \\
0 & 0 & 0 & \delta(x-y) \\
-\delta(x-y) & 0 & 0 & 0 \\
0 & -\delta(x-y) & 0 & 0
\end{pmatrix}. \] (12)

The consistency of this procedure is ensured because \( W \) is time independent. In order to obtain \( F \), we must observe its variations along \( J_L \) and \( J_R \)

\[ \int dx \delta F(x) = \int dxdz \left( \{ F(x), J_L(z) \} \delta J_L(z) + \{ F(x), J_R(z) \} \delta J_R(z) \right) \]

\[ = \int dxdzdw \left( \frac{\partial F(x)}{\partial J_L(w)} \{ J_L(w), J_L(z) \} \delta J_L(z) + \frac{\partial F(x)}{\partial J_R(w)} \{ J_R(w), J_R(z) \} \delta J_R(z) \right) \]

\[ = -\int dxdz \left( \frac{\partial F(x)}{\partial J_R(z)} \delta J_R(z) - \frac{\partial F(x)}{\partial J_L(z)} \delta J_L(z) \right) \]

\[ = \int dx \left( \frac{\partial F(x)}{\partial J_L(z)} \delta J_L(z) - \frac{\partial F(x)}{\partial J_R(z)} \delta J_R(z) \right) \] (13)

note that from (12) and (13) is easy to calculate \( \partial F/\partial J_L \) and \( \partial F/\partial J_R \), then
\[ \int dx \delta F(x) = 2 \int dx [J_L(x) \delta J_L(x) - J_R(x) \delta J_R(x)], \quad (14) \]

due to the canonical Hamiltonian vanishes, the modified Hamiltonian is
\[ K = \int dx (J_L^2(x) + J_R^2(x)). \quad (15) \]

Finally
\[ \{K, J_{L,R}(x)\}_D = \pm 2 J'_{L,R}(x). \quad (16) \]

The advantage of this procedure is that the Hamiltonian that participates in the Kac-Moody algebra is defined on the constraint surface i.e. on the physical surface, which guarantees that the solution of motion equations are in complete consistency with the algebra. On the other hand it is very important because it assures that in the quantization procedure of the theory, only gauge degree of freedom be counted.

3 Chiral current algebra on the physical surface for Thirring model

Now consider the Lagrangian
\[ L = \frac{1}{2} \int dxdy \left[ (J_R(x) - gJ_L(x))\varepsilon(x - y)\partial_+ J_R(y) - (J_L(x) + gJ_R(x))\varepsilon(x - y)\partial_+ J_L(y) \right] \quad (17) \]

g being the coupling constant, this one admit the same considerations made in the previous section.

Now, our second-class conserved constraints are
\[ \chi_1 = \Pi_R(x) - \frac{1}{2} \int dy (J_R(y) - gJ_L(y))\varepsilon(y - x), \quad (18) \]
\[ \chi_2 = \Pi_L(x) + \frac{1}{2} \int dy (J_L(y) + gJ_R(y))\varepsilon(y - x). \quad (19) \]
These are again second class constraints and, as in the non-interacting case, they are conserved (in this sense, it is a closed system).

Now we can follow the Dirac’s procedure and calculate the Dirac’s brackets on the constraint surface. To do this is necessary to build $W$ (that is the matrix that resume the former bracket relations).

$$C(z) = \begin{pmatrix} 1 & -g \\ -g & -1 \end{pmatrix} \varepsilon(z). \quad (20)$$

Note that the matrix is symmetric. In order to calculate Dirac’s brackets, first we need to calculate the inverse of $C$

$$C^{-1}(z) = \frac{1}{(1 + g^2)} \begin{pmatrix} 1 & -g \\ -g & -1 \end{pmatrix} \partial z \delta(z). \quad (21)$$

The Dirac’s brackets between currents on the constraint surface, turn out to be

$$\{J_L(x), J_L(y)\}_D = \frac{1}{(1 + g^2)} \delta'(x - y), \quad (22)$$

$$\{J_R(x), J_R(y)\}_D = \frac{-1}{(1 + g^2)} \delta'(x - y), \quad (23)$$

$$\{J_L(x), J_R(y)\}_D = \frac{-g}{(1 + g^2)} \delta'(x - y), \quad (24)$$

$$\{J_R(x), J_L(y)\}_D = \frac{-g}{(1 + g^2)} \delta'(x - y). \quad (25)$$

Now, if we calculate the new Hamiltonian on the physical surface it can be obtained in the same way as before, it becomes

$$K = \int dx[J_L^2(x) + J_R^2(x) + 2gJ_L(x)J_R(x)]. \quad (26)$$

So,
We have obtained a modification of the Schwinger term as [3, 4]. It is also remarkable that this result involves the last section one, which is reached directly vanishing the coupling constant $g$.

4 An extension of the model

The previous results can be extended straightforwardly to the case of $N$ currents with interaction among all them. Let us to consider the following Lagrangian

$$
L = \frac{1}{2} \int dx dy \sum_{i=1}^{N} \left[ J_i^L(x) - \sum_{j=1}^{N} g_{ij} J_j^R(x) \right] \varepsilon(x - y) \partial_+ J_i^L(y) $$

$$
- \frac{1}{2} \int dx dy \sum_{i=1}^{N} \left[ J_i^R(x) + \sum_{j=1}^{N} g_{ij} J_j^L(x) \right] \varepsilon(x - y) \partial_+ J_i^R(y),
$$

where $g_{ii} \equiv g$ and $g_{ij} \equiv h \ (i \neq j)$. The currents are defined by

$$
J_{i}^{L,R}(x) = \psi_{i}^{L,R\dagger}(x) \psi_{i}^{L,R}(x). \tag{29}
$$

and, the new constraint set is

$$
\chi_k^L(z) = \Pi_k^L(z) - \frac{1}{2} \int dx \left[ J_i^L(x) - \sum_{j=1}^{N} g_{kj} J_j^R(x) \right] \varepsilon(x - z), \tag{30}
$$

$$
\chi_k^R(z) = \Pi_k^R(z) + \frac{1}{2} \int dx \left[ J_i^R(x) + \sum_{j=1}^{N} g_{kj} J_j^L(x) \right] \varepsilon(x - z), \tag{31}
$$

where, $J$ and $\Pi$ are canonical fields, therefore
\[ \{ J_i^L(x), \Pi_k^R(y) \} = \delta(x-y)\delta_{ik}. \] (32)

For this case, the whole Kac-Moody algebra in Dirac’s formalism is

\[ \{ J^a_k(x), J^b_m(y) \}_D = \{ J^a_k(x), J^b_m(y) \} - \int dzd\omega \{ J^a_k(x), \chi_\mu(z) \} C^\mu\nu(z-\omega) \{ \chi_\nu(\omega), J^b_m(y) \}, \] (33)

where \( C_{\mu\nu} = \| \{ \chi_\mu, \chi_\nu \} \| \) and \( \int dzC_{\mu\nu}(x-z)C^{\tau\nu}(z-y) = \delta(x-y)\delta^\nu_\mu. \)

The current Hamiltonian on the current surface for this model is

\[ H = \int dx \left\{ \sum_{i=1}^N [\{ J_i^L(x) \}^2 + \{ J_i^R(x) \}^2] + \sum_{i,j=1}^N g_{ij} J_i^L(x) J_j^R(x) \right\}. \] (34)

To give a compact expression for the inverse of a matrix \((N \times N)\) in the more general case is a very complicated task, furthermore it is not very illustrative for our main objective. So, let us consider the case \( N = 2 \), which allows to illustrate all desired features of this extension.

In this case we have

\[ C_{\alpha\beta}(z) = \begin{pmatrix} 1 & -g & 0 & -h \\ -g & -1 & -h & 0 \\ 0 & -h & 1 & -g \\ -h & 0 & -g & -1 \end{pmatrix} \varepsilon(z), \] (35)

and, after some calculations, we obtain

\[ \{ J_k^L(x), J_m^L(y) \}_D = f\delta'(x-y) \left[ (1 + g^2 + h^2) (\delta_{k1}\delta_{1m} + \delta_{k2}\delta_{2m}) -2gh (\delta_{k1}\delta_{2m} + \delta_{k2}\delta_{1m}) \right], \] (36)

\[ \{ J_k^R(x), J_m^R(y) \}_D = -f\delta'(x-y) \left[ (1 + g^2 + h^2) (\delta_{k1}\delta_{1m} + \delta_{k2}\delta_{2m}) -2gh (\delta_{k1}\delta_{2m} + \delta_{k2}\delta_{1m}) \right], \] (37)
\[
\begin{align*}
\{J^L_k(x), J^R_m(y)\}_D &= -f (x - y) \left[ (g + g^3 - gh^2) \left( \delta_{k1} \delta_{1m} + \delta_{k2} \delta_{2m} \right) \\
&\quad + (h + h^3 - g^2 h) \left( \delta_{k1} \delta_{2m} + \delta_{k2} \delta_{1m} \right) \right], \\
(38) \\
\{J^R_k(x), J^L_m(y)\}_D &= -f (x - y) \left[ (g + g^3 - gh^2) \left( \delta_{k1} \delta_{1m} + \delta_{k2} \delta_{2m} \right) \\
&\quad + (h + h^3 - g^2 h) \left( \delta_{k1} \delta_{2m} + \delta_{k2} \delta_{1m} \right) \right], \\
(39)
\end{align*}
\]

where

\[
f = \frac{1}{(1 + (g - h)^2) (1 + (g + h)^2)}.
(40)
\]

This is the whole current algebra for \( N = 2 \) case. Also is direct to obtain the previous results by vanishing either only \( h \), for the section 3 case, or both \( h \) and \( g \) for the section 2 case. The Hamiltonian on the constraint surface is

\[
K = \int dx \left\{ \sum_{i=1}^{2} \left[ (J^L_i(x))^2 + (J^R_i(x))^2 \right] + \sum_{i,j=1}^{2} g_{ij} J^L_i(x) J^R_j(x) \right\},
(41)
\]

and

\[
\left\{ K, J^L_m, J^R_n(x) \right\}_D = \pm 2f \left\{ (1 + g^2 + h^2) \left[ J^L_1 R(x) + g J^R_1 L(x) + h J^R_2 L(x) \right] \\
- 2gh \left[ J^R_2 L(x) + g J^R_2 R(x) + h J^R_1 L(x) \right] \\
\mp (g + g^3 - gh^2) h \left[ J^R_2 L(x) + g J^R_2 R(x) + h J^R_1 L(x) \right] \right\} \delta_{1m} \\
\pm 2f \left\{ (1 + g^2 + h^2) \left[ J^R_1 L(x) + g J^R_1 R(x) + h J^R_2 L(x) \right] \\
- 2gh \left[ J^R_2 L(x) + g J^R_2 R(x) + h J^R_1 L(x) \right] \\
\mp (g + g^3 - gh^2) h \left[ J^R_2 L(x) + g J^R_2 R(x) + h J^R_1 L(x) \right] \right\} \delta_{2m}
\]

Also from here we can get back to the previous results, by turning off sequentially the constants \( h \) and \( g \) and setting \( m = 1 \) (sections 3 and 2 respectively). So, in this sense, the consistency of this extension is fulfilled.
5 Conclusions

In this contribution, a Lagrangian formulation was proposed for the Thirring chiral currents model. It allows to give a treatment for the current algebra entirely on the physical surface, and this algebra have the usual features reported by other authors. We also give a first generalization of our results for $N$ currents with interaction between them. The remarkable advantage of this procedure is that the Hamiltonian as well as the others elements that participates in the Kac-Moody algebra are defined on the constraint surface. The fact that the Hamiltonian lives on the constraint surface ensure that the equation of motion of the theory have a full consistency with the Kac-Moody algebra that satisfies the currents. We consider this result so important because the corresponding quantized theory ("a la Dirac", for instance) is a gauge theory. The complete theory living on the physical surface asserts that the gauge symmetry will be guarantied.

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References

