Two–loop Neutrino Mass Generation
and its Experimental Consequences

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Abstract

If neutrino masses have a radiative origin, their smallness can be naturally understood even when lepton number violation occurs near the weak scale. We analyze a specific model of this type wherein the neutrino masses arise as two–loop radiative corrections. We show that the model admits the near bimaximal mixing pattern suggested by the current neutrino oscillation data. Unlike the conventional seesaw models, these two–loop models can be directly tested in lepton flavor violating decays $\tau \to 3\mu$ and $\mu \to e + \gamma$ as well as at colliders by the direct observation of charged scalars needed for the mass generation. It is shown that consistency with the neutrino oscillation data requires that the leptonic rare decays should be within reach of forthcoming experiments and that the charged scalars are likely to be within reach of the LHC.
I. INTRODUCTION

Recent neutrino oscillation experiments, notably Super-Kamiokande and SNO, have provided a wealth of information on neutrino masses and mixing pattern. First and foremost, the solar [1] and the atmospheric neutrino data [2] are incompatible with the hypothesis of zero neutrino mass. The inferred values of neutrino masses are in the sub–eV range. The mixing angle $\theta_{12}$ relevant for solar neutrino oscillations appears to be large ($\tan^2\theta_{12} \sim 0.4$) [3], while the one for atmospheric neutrino oscillations $\theta_{23} \sim \pi/4$ is near maximal. Reactor neutrino experiments [4] constrain the third mixing angle $\theta_{13}$ to be relatively small, $\theta_{13} \leq 0.16$. It will be of great interest to seek a theoretical understanding of these observations.

The seesaw mechanism [5] has been the most popular explanation of the small neutrino masses. It is elegant and simple, and relies only on dimensional analysis for the required new physics. Current neutrino data points to a seesaw scale of $M_R \sim 10^{10} - 10^{15}$ GeV, where lepton number violation occurs through the Majorana masses of the right–handed neutrinos. With such a high scale, effects of lepton flavor violation in processes other than neutrino oscillation itself becomes extremely small. For example, the branching ratio for the decay $\mu \to e + \gamma$ is of order $10^{-50}$ within the seesaw extension of the Standard Model.

An alternative to the seesaw mechanism which also explains the smallness of neutrino masses naturally is the radiative mass generation mechanism [6–14]. In this approach, neutrino masses are zero at the tree–level, and are induced only as finite radiative corrections. These radiative corrections are typically proportional to the square of the charged lepton (or quark) masses divided by the scale of new physics $\Lambda$. The neutrino masses can then be in the sub–eV range even for $\Lambda \sim$ TeV. In this case, lepton flavor violation in processes other than neutrino oscillations may become experimentally accessible.

In this paper we propose to analyze in detail the experimental consequences of a specific model wherein the neutrino masses arise as two–loop radiative corrections [8,9]. The magnitude of these induced masses are of order $m_\nu \sim [(f^2 h)/(16\pi^2)^2](m^2/\Lambda)$, where $f, h$ are dimensionless Yukawa couplings and $\Lambda$ is the scale of new physics. For $f \sim h \sim 0.1$ and $\Lambda \sim$ 1 TeV, the neutrino mass will be of order 0.1 eV, which is compatible with the oscillation data. Within this model, it is required that $f, h \geq 0.1$ and $\Lambda \leq$ 1 TeV, or else the induced neutrino mass will be too small to be relevant experimentally. These limits, along with the requirement of large solar and atmospheric neutrino oscillation angles, imply that lepton flavor violation in processes such as $\tau \to 3\mu$ and $\mu \to e + \gamma$ cannot be suppressed arbitrarily. We shall see that these decays are within reach of the next round of rare decay experiments. Furthermore, the upper limit on $\Lambda$ implies that it is very likely that the new scalars predicted by the model for neutrino mass generation will be accessible for direct observation at the LHC and perhaps even at Run II of the Tevatron.

A variety of radiative neutrino mass models exist in the literature. The Zee model of neutrino mass [6] is a popular example. In this model, neutrino masses arise as one–loop radiative corrections. This model appears to be incompatible with the large mixing angle MSW solution for the solar neutrino problem as it predicts the relevant mixing angle to be very close to $\pi/4$ [15]. In any case, one would expect rare leptonic processes to be more suppressed in the Zee model compared to the two–loop neutrino mass model [8,9]. This is because the neutrino masses in the Zee model, being of one–loop origin, are of order...
Another class of neutrino mass models that has been widely studied is supersymmetric models with explicit $R$–parity violation [10]. In this case, it turns out that only one neutrino acquires a tree–level mass, the other two obtain masses as one–loop radiative corrections. The phenomenology of such models, especially when $R$–parity violation is soft, arising only through bilinear terms, has been thoroughly investigated [13].

In Ref. [14] a general effective operator approach for small Majorana neutrino masses has been presented. It follows from that analysis that if lepton number violation resides only in the leptonic sector and does not involve the quarks, then the only interesting model for the current neutrino oscillation data that also predicts observable lepton flavor violation signals is the two–loop model of Ref. [8,9]. This is the main reason for revisiting the two–loop model in the present paper.

The outline of the paper is as follows. In Section II we will have a quick review of the two–loop neutrino mass model. In Section III we analyze quantitatively the constraints on the model arising from neutrino oscillation data. Here we also show how the model can accommodate the near bi–maximal mixing pattern preferred by the current data. In Section IV, we analyze the constraints arising from rare lepton decays. Here we present bounds on the masses and couplings of the charged scalars present in the model. Section V is devoted to the collider implications of the model. We provide our conclusions in Section VI.

II. DESCRIPTION OF THE MODEL

We start by reviewing the essential features of the two–loop neutrino mass model [9,8]. The gauge group of the model is the same as the Standard Model (SM). In addition to the SM particles the model introduces two charged scalars $h^+$ and $k^{++}$ which are both singlets of $SU(3)_C$ and $SU(2)_L$. These scalars have Yukawa couplings to the leptons, which can be parametrized in terms of two complex matrices $f$ and $h$:

$$L_Y = f_{ab} (\psi_{aL}^T C \psi_{bL}^L) \epsilon_{ij} h^+ + h'_{ab} (l_{aR}^T C l_{bR}) k^{++} + \text{h.c.}$$

Here $\psi_L$ stands for the left-handed lepton doublet, and $l_R$ for the right-handed lepton singlet. $C$ is the charge conjugation matrix. $i, j$ are $SU(2)_L$ indices, while $a, b$ are generation indices. The matrix $f$ is antisymmetric ($f_{ab} = -f_{ba}$) due to Fermi statistics and antisymmetry in the $SU(2)_L$ indices, while $h'$ is a symmetric matrix ($h'_{ab} = h'_{ba}$). The interaction Lagrangian in terms of the component fields will then be

$$L_Y = 2 \left[ f_{e\mu} (\overline{\nu}_e^\mu \mu_L - \overline{\nu}_e^\mu \epsilon_L) + f_{e\tau} (\overline{\nu}_e^\tau \tau_L - \overline{\nu}_e^\tau \epsilon_L) + f_{\mu\tau} (\overline{\nu}_\mu^\tau \tau_L - \overline{\nu}_e^\tau \mu_L) \right] h^+ + \left[ h_{ee} \overline{\epsilon}_e e_R + h_{\mu\mu} \overline{\epsilon}_\mu \mu_R + h_{\tau\tau} \overline{\epsilon}_\tau \tau_R + h_{e\mu} \overline{\epsilon}_e \mu_R + h_{e\tau} \overline{\epsilon}_e \tau_R + h_{\mu\tau} \overline{\epsilon}_\mu \tau_R \right] k^{++} + \text{h.c.}$$

where we have defined $h_{aa} = h'_{aa}$, $h_{ab} = 2h'_{ab}$ for $a \neq b$.

The Yukawa interactions of Eq. (1) conserves lepton number ($L$), as can be seen by assigning two units of $L$ to the $h^+$ and $k^{++}$ fields. The scalar piece of the Lagrangian contains a term
\[ \mathcal{L}_{h-k} = -\mu h^+ h^- + h.c. \] (3)

which would then violate \( L \). In fact, one sees that the combination of Eqs. (1) and (3) explicitly breaks lepton number. This would lead to the generation of Majorana neutrino masses at the two–loop level. The relevant diagram is shown in Fig. 1. The induced neutrino mass can be calculated to be

\[ (\mathcal{M}_\nu)_{ab} = 8\mu f_{ac} h^c m_d f_{db} I_{cd} \] (4)

where \( m_c, m_d \) are the charged lepton masses and \( I_{cd} \) is the two loop integral function given by

\[
I_{cd} = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(k^2 - m_c^2)} \frac{1}{(q^2 - m_d^2)} \frac{1}{(k - q)^2 - m_k^2}.
\] (5)

In order to evaluate the above integral, we may neglect the lepton masses in the denominator, since these masses are much smaller than the charged scalar masses \( m_h \) and \( m_k \). Then

\[
I_{cd} \approx I = \frac{1}{(16\pi^2)^2} \frac{1}{m_h^2} \tilde{I}\left(\frac{m_k^2}{m_h^2}\right).
\] (6)

The dimensionless quantity \( \tilde{I} \) defined as

\[
\tilde{I}(r) = -\int_0^1 dx \int_0^{1-x} dy \frac{1-y}{x+(r-1)y+y^2} \log y \frac{1-y}{x+ry}
\] (7)

is plotted in Fig. 2 for \( r = m_k^2/m_h^2 < 100 \). The asymptotic behavior for small \( m_k \) and \( m_k \gg m_h \) is given by:

\[
\tilde{I}(r) \to \begin{cases} \log^2 r + \pi^2/3 - 1 & \text{for } r \gg 1 \\ \pi^2/3 & \text{for } r \to 0 \end{cases}
\] (8)

Before we estimate the induced neutrino masses and mixings, we wish to make some remarks on the expected magnitude of the relevant coupling parameters \( f_{ab}, h_{ab} \) and \( \mu \). First,
in order for perturbative expansion to be sensible, the Yukawa couplings \( f_{ab}, h_{ab} \) have to be of order unity or smaller. For concreteness we shall take \( f_{ab} < 1, h_{ab} < 2 \) (the \( f_{ab} \) couplings appear with a factor of 2 in Eq. (2)) (see further remarks at the end of this section). The dimensional parameter \( \mu \) can be constrained as follows. The Lagrangian terms in Eq. (3) lead to an effective quartic interaction for the \( h^+ \) and \( k^{++} \) fields given by

\[
-\mathcal{L}_{\text{eff}} = \lambda_{\text{eff}} (h^+)^2 (h^-)^2 + \lambda'_{\text{eff}} (k^{++})^2 (k^{--})^2 + \lambda''_{\text{eff}} (h^+ h^-) (k^{--} k^{++}).
\]  

Evaluating the diagrams in Fig. 3, we obtain:

\[
\lambda_{\text{eff}} = -\frac{1}{2\pi^2} \frac{\mu^4}{(m_k^2 - m_h^2)^2} \left[ \frac{m_k^2 + m_h^2}{m_k^2 - m_h^2} \log \left( \frac{m_k^2}{m_h^2} \right) - 2 \right],
\]

\[
\lambda'_{\text{eff}} = -\frac{1}{4\pi^2} \frac{\mu^4}{6m_h^4},
\]

\[
\lambda''_{\text{eff}} = -\frac{1}{\pi^2} \frac{\mu^4 m_k^2}{2(m_k^2 - m_h^2)^3} \left[ \frac{m_k^2}{m_h^2} - \frac{m_h^2}{m_k^2} - 2\log \left( \frac{m_k^2}{m_h^2} \right) \right].
\]

In the limit where the two masses are almost equal \( m_k \approx m_h \)

\[
2\lambda_{\text{eff}} \approx \lambda''_{\text{eff}} \approx 4\lambda'_{\text{eff}} = -\frac{1}{\pi^2} \frac{\mu^4}{6m_h^4}.
\]

Since the effective couplings are negative, it follows that there must exist in the tree–level Lagrangian terms similar to those in Eq. (9) whose couplings (call them \( \lambda, \lambda', \lambda'' \)) should be positive and greater in absolute value than \( \lambda_{\text{eff}}, \lambda'_{\text{eff}}, \lambda''_{\text{eff}} \) (otherwise the vacuum will be unstable). Requiring that the theory be in the perturbative regime with respect to these interactions \( \lambda, \lambda', \lambda'' < 1 \) imposes the following constraints on the \( \mu \) parameter:

\[
\mu < \begin{cases}
    m_h \times (6\pi^2)^{1/4} & \text{if } m_k \approx m_h \\
    m_h \times (2\pi^2)^{1/4} & \text{if } m_k \ll m_h \\
    m_h \times (24\pi^2)^{1/4} & \text{if } m_k \gg m_h.
\end{cases}
\]
FIG. 3. Diagrams giving rise to the effective quartic interaction terms for the $k$ and $h$ fields.

The upper limits on the Yukawa couplings $f_{ab}$ and $h_{ab}$ can be made more precise by resorting to the renormalization group equations (RGE) and by demanding that the couplings remain perturbative to a momentum scale at least an order of magnitude higher than the weak scale. For example, consider the RGE evolution of the coupling $f_{\mu\tau}$:

$$\frac{df_{\mu\tau}}{dt} = \left[ \frac{3}{(4\pi^2)} \right] f_{\mu\tau}^3 + \ldots (t \equiv \ln \mu).$$

From this it follows that an initial value of $f_{\mu\tau} = 1$ will increase to about 1.5 at a scale an order of magnitude higher. Any larger initial value of $f_{\mu\tau}$ will signal non-perturbative effects. Similarly, the coupling $\lambda_{eff}$ of Eq. (9) will receive a correction proportional to $f_{\mu\tau}^4$, obtained from the RGE $d\lambda/dt = 2f_{\mu\tau}^4/\pi^2 + \ldots$. An initial value of $f_{\mu\tau} = 1.5$ will result in $\lambda_{eff} = 2.4$ at a momentum scale an order of magnitude higher, which will not be in the perturbative regime. An initial value of $f_{\mu\tau} = 1$ will lead to $\lambda_{eff} = 0.47$ at a scale ten times higher, which is within the perturbation theory. These arguments suggest that $f_{ab}$ should be smaller than about 1 for perturbation theory to be valid. Similar arguments suggest that $h_{ab} \leq 2$.

III. FITTING THE NEUTRINO OSCILLATION DATA

In this section, we first show how the two-loop neutrino mass model fits the current neutrino oscillation data consistently and then derive the constraints imposed by the data on the model parameters.

We use the standard parametrization of the neutrino mixing matrix (the MNS matrix) in terms of three angles and a phase [16]:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(12)

where $c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij}$, and $\theta_{ij}$ is the mixing angle between the flavor eigenstates labeled by indices $i$ and $j$. The transformation

$$U^T \mathcal{M}_\nu U = \mathcal{M}_{diag}$$

(13)
diagonalizes the neutrino mass matrix of Eq. (4).

Before going further, we shall make some remarks on the possible values of the neutrino masses. An interesting feature of the two–loop neutrino mass model is that, owing to the antisymmetric nature of the coupling matrix \( f \), the determinant of the mass matrix \( M_\nu \) is zero [9]. Thus one of the neutrino mass eigenvalues will be zero within the model. Although this result does not prevail when three–loop and higher order corrections are included, the induced mass at these higher loops will be extremely small and can be neglected for all practical purposes.

If one uses the standard parametrization of \( U \) shown in Eq. (12), the neutrino mass eigenvalues should be allowed to be complex in general. Since an overall phase can be removed by field redefinitions, there are two relative (Majorana) phases in the mass eigenvalues in general. However, since one of the mass eigenvalues is zero in the model under discussion, there is a single Majorana phase, the relative phase of the two non–zero mass eigenvalues. Note that with the choice of \( U \) as in Eq. (12), the elements of the matrix \( f \) cannot be made real.

Experimental results indicate at least two mass hierarchies (we shall not consider the LSND experiment here): from the atmospheric oscillation data \( (\nu_\mu \leftrightarrow \nu_\tau) \), \( \Delta m^2_{\text{atm}} \simeq 2.5 \times 10^{-3} \text{ eV}^2 \) [2], and from the solar neutrinos oscillation \( (\nu_e \leftrightarrow \nu_\mu \text{ or } \nu_\tau) \) \( \Delta m^2_{\text{LMA}} \lesssim 10^{-4} \text{ eV}^2 \) (the Large Mixing Angle MSW solution) or even smaller [1]. As a consequence, there are two possibilities for the diagonal mass matrix \( M_\text{diag} \) in Eq. (13): the hierarchical form

\[
M_\text{diag} = \text{diag}(0, m, M) \quad \text{with } |m| \ll |M| \quad (14)
\]

or the inverted hierarchy form:

\[
M'_{\text{diag}_\pm} = \text{diag}(M, \pm M + m, 0) \quad \text{with } |m| \ll |M| . \quad (15)
\]

We have allowed two possible signs in the inverted hierarchy case. From the point of view of fitting the neutrino oscillation data alone, the + sign will be identical to the hierarchical form (Eq. (14)) since the matrices differ only by an identity matrix. However, the implications for the model parameters are different, and thus the predictions for rare processes will be different. Note also that the model does not admit the case of three–fold degenerate neutrino spectrum.

The atmospheric oscillation data implies that \( |M| \) is about 0.05 eV. This already can give some information on the magnitude of the parameters of the model. From Eq. (4), we have

\[
\frac{8\mu}{(16\pi^2)^2} \frac{|f|^2 |\Omega|}{m_h^2} \tilde{I} \approx |M| \approx 5 \times 10^{-12} \text{ GeV} \quad (16)
\]

where \( |f| = \max(f_{ab}) \), \( |\Omega| = \max(\omega_{ab} = m_a h_{a\nu_b}^* m_b) \). Taking \( \tilde{I} \) to be of order unity and \( \mu \) to have its maximum value of about \( 3m_h \), the above equation becomes

\[
\frac{|f|^2 |\Omega|}{m_h} \approx 0.5 \times 10^{-8} \text{ GeV} . \quad (17)
\]

Since \( m_h \) has to be at least of order 100 GeV (otherwise \( h^+ \) would have already been seen in collider experiments), the larger of the couplings \( f_{ab} \), \( \omega_{ab}/\text{GeV}^2 \) have to be at least of the
order $10^{-2}$. This raises the possibility that lepton flavor violation will be observable in rare decays. There may even be potential conflict with the current limits. We shall discuss these constraints quantitatively in the next section. Our conclusion there is that presently there is no conflict with the data, but the rare decays should be accessible to the next round of experiments.

In the rest of this section we shall perform a more detailed analysis of the neutrino mixings in the two–loop neutrino mass model. Our first aim is to reproduce the neutrino mass hierarchy and the three neutrino mixing angles. For the case of the atmospheric oscillations, the SuperKamiokande data indicates that the mixing is nearly maximal: $|\theta_{23}| \simeq \pi/4$. On the other hand, the $\nu_e \leftrightarrow \nu_\tau$ mixing angle is close to zero; the reactor neutrino experiments \cite{4} have set a limit $\sin \theta_{13} < 0.16$. For the solar neutrino oscillations, the SNO and SuperKamiokande results indicate that while the mixing is not exactly maximal, the mixing angle $\theta_{12}$ is of order one, with a preferred value $\tan^2 \theta_{12} \approx 0.4$.

In a first approximation let us take $\theta_{23} = \theta_{12} = \pi/4$ and $\theta_{13} = 0$. This is the exact bimaximal mixing limit. Then the neutrino mass matrix in the flavor basis should have the form

$$\hat{M}_\nu = U M_{\text{diag}} U^T = \frac{1}{2} \begin{pmatrix} m & m/\sqrt{2} & -m/\sqrt{2} \\ m/\sqrt{2} & M + m/2 & M - m/2 \\ -m/\sqrt{2} & M - m/2 & M + m/2 \end{pmatrix}$$

(18)

for the hierarchical case, or

$$\hat{M}'_{\nu+} = U M'_{\text{diag+}} U^T = \frac{1}{2} \begin{pmatrix} 2M + m & m/\sqrt{2} & -m/\sqrt{2} \\ m/\sqrt{2} & M + m/2 & -M - m/2 \\ -m/\sqrt{2} & -M - m/2 & M + m/2 \end{pmatrix}$$

(19)

$$\hat{M}'_{\nu-} = U M'_{\text{diag-}} U^T = \frac{1}{4} \begin{pmatrix} 2m & (-2M + m)/\sqrt{2} & (2M - m)/\sqrt{2} \\ (-2M + m)/\sqrt{2} & m & -m \\ (2M - m)/\sqrt{2} & -m & m \end{pmatrix}$$

(20)

when the diagonal mass matrix has an inverted hierarchy form. The mass matrices above will get small corrections due to the deviation of the mixing angles from the exact bimaximal limit.

On the other hand, by expanding Eq. (4), the neutrino mass matrix from the two loop model can be written in the form

$$M_\nu = \zeta \times \begin{pmatrix} e^2 \omega_{\tau\tau} + 2e \epsilon' \omega_{\mu\tau} + e^2 \omega_{\mu\mu}, & e \omega_{\tau\tau} + \epsilon' \omega_{\mu\tau} - e \epsilon' \omega_{e\tau} - e \omega_{\mu\tau} - \epsilon' \omega_{\mu\mu} - e^2 \omega_{e\tau} \\ -e^2 \omega_{e\mu}, & -\epsilon' \omega_{e\mu} \\ \omega_{\tau\tau} - 2e \omega_{e\tau} + e^2 \omega_{ee}, & \omega_{\mu\tau} - e \omega_{e\tau} + \epsilon' \omega_{ee} + \epsilon \omega_{e\mu} \\ \omega_{\mu\mu} + 2e \omega_{ee} & e \omega_{ee} \end{pmatrix}$$

(21)

where we have defined
\[ \zeta \equiv \frac{8 \mu}{(16 \pi^2)^2} \frac{f_{\mu \tau}^2}{m_h^2} \tilde{I}, \quad \omega_{ab} \equiv m_a h_{ab}^* m_b, \]
\[ \epsilon \equiv f_{e\tau}/f_{\mu\tau}, \quad \epsilon' \equiv f_{e\mu}/f_{\mu\tau}. \]  
(22)

(We have written explicitly just the elements above the diagonal in Eq. (21), since the matrix is symmetric). Let us assume for the moment that there is not a strong hierarchy among the \( f_{ab}, h_{ab} \) couplings; in this case, the hierarchy among the lepton masses \( (m_e \ll m_\mu, m_\tau) \) allows us to neglect the \( \epsilon \omega_{ea}, \epsilon' \omega_{ea} \) quantities with respect to \( \omega_{\mu\mu}, \omega_{\mu\tau}, \omega_{\tau\tau} \). With this simplification, the neutrino mass matrix will become

\[ M_\nu \simeq \zeta \epsilon^2 \omega_{\tau\tau} + 2 \epsilon' \omega_{\mu\tau} + \epsilon^2 \omega_{\mu\mu}, \quad \epsilon \omega_{\tau\tau} + \epsilon' \omega_{\mu\tau} - \epsilon \omega_{\mu\mu} - \epsilon' \omega_{\mu\mu} \]

\[ = \begin{pmatrix}
\epsilon^2 \omega_{\tau\tau} + 2 \epsilon' \omega_{\mu\tau} + \epsilon^2 \omega_{\mu\mu} & \epsilon \omega_{\tau\tau} + \epsilon' \omega_{\mu\tau} - \epsilon \omega_{\mu\mu} \\
\epsilon \omega_{\tau\tau} + \epsilon' \omega_{\mu\tau} - \epsilon \omega_{\mu\mu} & -\omega_{\mu\tau} \\
-\omega_{\mu\tau} & \omega_{\mu\mu}.
\end{pmatrix} \]  
(23)

Comparing with Eq. (18), we see that a choice

\[ \omega_{\mu\mu} \approx -\omega_{\mu\tau} \approx \omega_{\tau\tau}, \quad \epsilon \approx \epsilon' \approx 1 \]  
(24)

gives a good fit to the hierarchical neutrino mass matrix (the mass hierarchy in this case being \( m/M \) of order \( \Delta \omega/\omega \)). For the inverse mass hierarchy case, the choice \( \omega_{\mu\mu} \approx \omega_{\mu\tau} \approx \omega_{\tau\tau} \) would fit the lower right hand side \( 2 \times 2 \) block of the neutrino mass matrices in Eq. (19), but the fact that there are large elements in the first row of these matrices will require that there is a large hierarchy among the \( f \) couplings; that is, \( \epsilon, \epsilon' \) should be of order \( M/m \) or larger. In this case, though, the simplification made to reach the form of Eq. (23) may not be entirely justified.

The question arises then if it is possible to solve for the parameters \( \epsilon, \omega \) exactly, without any approximation, in terms of the elements of \( \hat{M}_\nu \). Let’s consider the general equation

\[ \mathcal{M}_\nu \equiv \zeta \frac{f_{\tau\mu}^2}{f_{\mu\tau}} f \omega f^T = \hat{M} \]  
(25)

and try to solve for the parameters \( \epsilon, \epsilon', \omega_{ab} \) in terms of the \( m_{ij} \) elements of the general matrix \( \hat{M} \) (the only constraints on this matrix are that it be symmetric and of zero determinant). At first glance, this task may seem impossible; there are five independent equations, each of them of order three in the unknown parameters. However, the particular form of the two loop mass matrix \( \mathcal{M}_\nu \sim f \omega f^T \), allows for considerable simplification of the problem.

Note that \( f \), having zero determinant (as a consequence of antisymmetry), has an eigenvector with zero as an eigenvalue:

\[ v_0^T = (1, -\epsilon, \epsilon'); \quad f v_0 = 0. \]  
(26)

Moreover, \( v_0 \) is also an eigenvector of the \( \mathcal{M}_\nu \) mass matrix; multiplying both sides of the matrix equation (25) by \( v_0 \) we obtain

\[ \hat{M} v_0 = 0. \]  
(27)

The vector equation above allows solving for the parameters \( \epsilon, \epsilon' \) in terms of the elements of the matrix \( \hat{M} \) (denoted by \( m_{ij} \)):
\[ \epsilon = \frac{m_{12}m_{33} - m_{13}m_{23}}{m_{22}m_{33} - m_{23}^2}, \quad \epsilon' = \frac{m_{12}m_{23} - m_{13}m_{22}}{m_{22}m_{33} - m_{23}^2}. \]  

(28)

The surprising result is that the \( \epsilon, \epsilon' \) parameters are completely determined by the mass matrix elements \( m_{ij} \), regardless of what the \( \omega \) parameters might be.

Once \( \epsilon \) and \( \epsilon' \) are determined, there are six \( \omega \) parameters left, and three independent equations. These equations are linear in \( \omega \), and therefore easy to solve. For example, we could use the 22, 23 and 33 matrix equalities

\[ \zeta (\omega_{\tau\tau} - 2\epsilon' \omega_{ee} + \epsilon'^2 \omega_{ee}) = m_{22} \]
\[ \zeta (-\omega_{\mu\tau} - \epsilon \omega_{ee} + \epsilon' \omega_{e\mu} + \epsilon \omega_{ee}) = m_{23} \]
\[ \zeta (\omega_{\mu\mu} + 2\epsilon \omega_{e\mu} + \epsilon^2 \omega_{ee}) = m_{33} \]

(29)

to eliminate any three of the \( \omega \) parameters in terms of the other three.

In the following, we will apply this general analysis to the two situations resulting from the two possible hierarchies (Eqs. (14,15)) in the neutrino masses.

- In the case of the hierarchical neutrino mass matrix, Eqs. (28) read as

\[ \epsilon = \tan \theta_{12} \frac{\cos \theta_{23}}{\cos \theta_{13}} + \tan \theta_{13} \sin \theta_{23} e^{-i\delta} \]  

(30)
\[ \epsilon' = \tan \theta_{12} \frac{\sin \theta_{23}}{\cos \theta_{13}} - \tan \theta_{13} \cos \theta_{23} e^{-i\delta} \]

for general mixing angles. Taking \( \theta_{23} \approx \pi/4, \theta_{13} \approx 0 \), we have

\[ \epsilon \approx \epsilon' \approx \frac{\tan \theta_{12}}{\sqrt{2}}. \]  

(31)

Then, for the LMA and LOW solutions to the solar neutrino problem, the \( \epsilon \) and \( \epsilon' \) parameters are in the range 0.4 – 0.5, with the difference between them being of order 2\( \theta_{13} \).

As mentioned above, for the case when \( \epsilon, \epsilon' \) are of order unity (or smaller) the \( \epsilon \omega_{ee} \) terms can be neglected with respect to \( \omega_{\mu\mu}, \omega_{\mu\tau}, \omega_{\tau\tau} \). Eq. (29) becomes

\[ \zeta \omega_{\tau\tau} = M/2 + m \cos^2 \theta_{12} (1 - \theta_{13} e^{i\delta} \tan \theta_{12})/2 \]
\[ \zeta \omega_{\mu\tau} = -M/2 + m \cos^2 \theta_{12}/2 \]
\[ \zeta \omega_{\mu\mu} = M/2 + m \cos^2 \theta_{12} (1 + \theta_{13} e^{i\delta} \tan \theta_{12})/2 \]

(32)
in the \( \theta_{23} = \pi/4, \) small \( \theta_{13} \) limit. The relative difference between the magnitudes of the large \( \omega \) parameters is \( |\omega_{\mu\mu} - \omega_{\mu\tau}|/|\omega_{\mu\mu}| \approx 2(m/M)\cos^2 \theta_{12} \), about 0.1 for the LMA solution, and 0.01 for the LOW solution.

The SMA solution to the solar neutrino oscillations, although strongly disfavored by experimental data, can also be accommodated in this model; in this case, \( \epsilon, \epsilon' \) would be of order \( \max(\theta_{12}, \theta_{13}) < 1/10 \). The relations (32) still hold.

Before going further, let us consider the reverse problem, that is, determining the masses and mixing angles in terms of the \( \epsilon \) and \( \omega \) parameters. Eq. (30) can inverted for two angles in terms of a third:
\[ \tan \theta_{13} = e^{i \delta} (\epsilon \sin \theta_{23} - \epsilon' \cos \theta_{23}) \]  
\[ \tan \theta_{12} = \cos \theta_{13} (\epsilon \sin \theta_{23} + \epsilon' \cos \theta_{23}) \].

Thus, we see that, provided that \( \theta_{23} \approx \pi/4 \), \( \epsilon \) and \( \epsilon' \) being almost equal implies that the mixing angle \( \theta_{13} \) is close to zero.

- For the inverted mass hierarchy case, the relations (28) read as
  \[ \epsilon = -\sin \theta_{23} \cot \theta_{13} e^{-i \delta} \]  
  \[ \epsilon' = \cos \theta_{23} \cot \theta_{13} e^{-i \delta} \].

Note that here \( \theta_{13} = 0 \) would require \( f_{\mu \tau} = 0 \). Barring this singular case, we see that the parameters \( \epsilon, \epsilon' \) are both of order \( 0.7/\theta_{13} \gtrsim 5 \). Then, neglecting the \( \epsilon \omega_{ea} \) terms is not necessarily well justified. While a general analysis is possible, for the sake of simplicity we will restrict ourselves to the case when such terms are small.

The equations determining the \( h_{ab} \) coupling constants are:

\[ \zeta \omega_{\tau \tau} = M/2 + m \cos^2 \theta_{12} (1 - \theta_{13} e^{i \delta} \tan \theta_{12})/2 \]  
\[ \zeta \omega_{\mu \tau} = M/2 + m \cos^2 \theta_{12}/2 \]  
\[ \zeta \omega_{\mu \mu} = M/2 + m \cos^2 \theta_{12} (1 + \theta_{13} e^{i \delta} \tan \theta_{12})/2 \]

for the \(+\) sign in Eq. (15), and

\[ \zeta \omega_{\tau \tau} = (m \cos^2 \theta_{12})/2 + M \cos 2 \theta_{12} (1 - 4 \theta_{13} e^{i \delta} \tan 2 \theta_{12})/2 \]  
\[ \zeta \omega_{\mu \tau} = -(m \cos^2 \theta_{12})/2 + (M \cos 2 \theta_{12})/2 \]  
\[ \zeta \omega_{\mu \mu} = (m \cos^2 \theta_{12})/2 + M \cos 2 \theta_{12} (1 + 4 \theta_{13} e^{i \delta} \tan 2 \theta_{12})/2 \]

for the \(-\) sign. In this latter case, since \( m/M \simeq 10^{-2} \) for the LMA solution (or smaller for the other solutions to the solar neutrino oscillations), and the 1-2 mixing is not quite maximal \((\cos^2 \theta_{12} \simeq 0.44 \text{ for LMA again})\), the scale in the lower right-hand corner of the neutrino mass matrix is set by the the terms proportional to the large mass \( M \). This means that the 2-3 family symmetry visible in the exact bimaximal form of the neutrino mass matrix Eq. (20) does not necessarily survive the small corrections in the mixing angles required by experimental results. Also, the relation (24) between the \( \omega \) parameters is somewhat broken; we have

\[ \omega_{\tau \tau} : \omega_{\mu \tau} : \omega_{\mu \mu} \simeq (1 - 4 \theta_{13} \tan 2 \theta_{12}) : 1 : (1 + 4 \theta_{13} \tan 2 \theta_{12}). \]

However, even in this case the \( h_{\mu \mu} \) coupling is the largest one.

In summary, we have found that both the hierarchical neutrino spectrum and the inverted hierarchical spectrum with near bi-maximal mixings can be accommodated in the two-loop neutrino mass model without difficulty. The parameters of the model are then mostly determined. We now turn to other experimental signatures of the model.

**IV. EXPERIMENTAL CONSTRAINTS**

Besides neutrino masses and mixings, the Lagrangian in Eq. (1) leads to non-standard lepton flavor violating processes. For example, at tree level, the second line in Eq. (2) will
allow lepton number violating decays, such as $\mu^- \rightarrow e^+ e^- e^-$ and $\tau^- \rightarrow \mu^- \mu^+ \mu^-$, while the first line will give extra contributions to the standard decay of the leptons. At one loop level, the $f_{ab}$ and $h_{ab}$ couplings will contribute to the anomalous magnetic momentum of the $e$ and the $\mu$, and will allow decays such as $\mu \rightarrow e\gamma$. Experimental constraints will therefore impose limits on the parameters of the model.

In the following, we analyze these processes in turn.

- **Lepton family number violating $\mu$ and $\tau$ decays.**

  The partial widths for these decay $l_a^- \rightarrow l_b^+ l_c^- l_d^-$ is given by

  \[
  \Gamma(l_a^- \rightarrow l_b^+ l_c^- l_d^-) = \frac{1}{8 \sqrt{2} \pi} \frac{m_a}{192 \pi^3} \left| h_{ab} h_{cd}^* \right|^2 \]

  in the limit when the masses of the decay products are neglected with respect to the mass of the decaying particle. The limits experimental constraints [16] set on the $h_{ab} h_{cd}$ combinations are summarized in Table 1.

- **Muonium-antimuonium oscillations.**

  The last line in Table 1 is the limit arising from muonium-antimuonium conversion process $\mu^+ e^- \rightarrow \mu^- e^+ [17]$, mediated by $t$-channel $k^{++}$ exchange. Using Fierz rearrangement, the amplitude for this process can be written as

  \[
  \frac{1}{2 m_k^2} \bar{u}(\mu^+) \gamma^\alpha P_R u(e^+) \bar{u}(\mu^-) \gamma_\alpha P_R u(e^-) (h_{ee} h_{\mu\mu}^*)
  \]

  giving rise to an effective Lagrangian coupling coefficient [18]

  \[
  G_{MM} = \frac{\sqrt{2}}{8} \frac{h_{ee} h_{\mu\mu}^*}{m_k^2}.
  \]

<table>
<thead>
<tr>
<th>Process</th>
<th>Exp. bound</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^- \rightarrow e^+ e^- e^-$</td>
<td>Br. $&lt; 1. \times 10^{-12}$</td>
<td>$</td>
</tr>
<tr>
<td>$\tau^- \rightarrow e^+ e^- e^-$</td>
<td>Br. $&lt; 2.9 \times 10^{-6}$</td>
<td>$</td>
</tr>
<tr>
<td>$\tau^- \rightarrow e^+ \mu^- \mu^-$</td>
<td>Br. $&lt; 1.5 \times 10^{-6}$</td>
<td>$</td>
</tr>
<tr>
<td>$\tau^- \rightarrow e^- \mu^- e^-$</td>
<td>Br. $&lt; 1.7 \times 10^{-6}$</td>
<td>$</td>
</tr>
<tr>
<td>$\tau^- \rightarrow \mu^+ \mu^- e^-$</td>
<td>Br. $&lt; 1.5 \times 10^{-6}$</td>
<td>$</td>
</tr>
<tr>
<td>$\tau^- \rightarrow \mu^+ e^- \mu^-$</td>
<td>Br. $&lt; 1.9 \times 10^{-6}$</td>
<td>$</td>
</tr>
<tr>
<td>$\tau^- \rightarrow \mu^+ e^- e^-$</td>
<td>Br. $&lt; 1.8 \times 10^{-6}$</td>
<td>$</td>
</tr>
<tr>
<td>$\mu^+ e^- \rightarrow \mu^- e^+$</td>
<td>$G_{MM} &lt; 0.003 \ G_F$</td>
<td>$</td>
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</tbody>
</table>

**TABLE I.** Constraints from lepton family number violating processes.
Anomalous magnetic momenta and $\mu \to e\gamma$ -type processes.

At the one–loop level, both $h^+$ and $k^{++}$ exchange contribute to the processes $l_a \to l_b\gamma$. For the case of same leptons in the initial and final states ($l_a \equiv l_b$) this leads to change in the anomalous magnetic moment of the lepton:

\[
\delta(g - 2)_a = \frac{4m_a^2}{96\pi^2} \left[ -\frac{(f^+f)_{aa}}{m_h^2} + \frac{(h^+h)_{aa}}{m_k^2} \right]. \tag{39}
\]

For different leptons in the initial and final states, the decay width for $l_a \to l_b\gamma$ (with $m_a > m_b$) is:

\[
\Gamma(l_a \to l_b\gamma) = 2\alpha m_a^3 \left(\frac{m_a}{96\pi^2}\right)^2 \left[ \left(\frac{f^+f}{m_h^2}\right)^2 + \left(\frac{h^+h}{m_k^2}\right)^2 \right]. \tag{40}
\]

where $\alpha$ is the fine structure constant. The limits experimental results on anomalous magnetic moments \(^1\) and the $l_a \to l_b\gamma$ processes \([16]\) impose on the $f$ and $h$ couplings are summarized in Table 2.

$l_a \to l_b\nu\bar{\nu}$ decays.

The exchange of a $h^+$ boson will contribute to the semileptonic decays of the $\mu$ and $\tau$ leptons. Using Fierz rearrangement of spinors, the amplitude of the $h^+$ exchange diagram for the process $l_a \to \nu_a l_b\bar{\nu}_b$ can be shown to be proportional to the SM diagram:

\[
A_{h^+} = A_{SM} \times 4\left|\frac{f_{ab}}{m_h^2}\right|^2 \frac{M_W^2}{g^2}. \tag{41}
\]

Therefore, the angular distribution of the decay products is not affected, while the total decay rate is. For the case of $\mu$ decay, this implies the redefinition of the Fermi decay constant (which is extracted from the muon lifetime measurements):

\[
G_{\mu} = G_F \left(1 + 4\left|\frac{f_{e\mu}}{m_h^2}\right|^2 \frac{M_W^2}{g^2}\right)^2 = 1.16639(1) \times 10^{-5}\text{GeV}^{-2} \tag{42}
\]

where $G_F$ refers to the SM Fermi constant, which is related to other well–measured quantities in the Standard Model as (in the on-shell scheme)

\[
G_F = \frac{\pi\alpha}{\sqrt{2}M_W^2(1 - M_W^2/M_Z^2)(1 - \Delta r)}. \tag{43}
\]

\(^1\)For the anomalous magnetic moment of the muon, the bound on the non-SM contribution is obtained by adding the experimental error in the measurement of $\delta(g - 2)_{\mu}$ \([19]\), the theoretical error in evaluating the SM prediction for this quantity \([20]\), and the difference between the experiment and theory values of $\delta(g - 2)_{\mu}$.
Here $\Delta r$ encodes the effect of radiative corrections, which depends on the top quark and the Higgs boson masses. The redefinition of $G_F$ from Eq. (42) can be interpreted then as a redefinition of the $\Delta r$ parameter; $\Delta r \rightarrow \Delta r + \delta \Delta r$, with

$$
\delta \Delta r = -8 \frac{|f_{\mu e}|^2 M_W^2}{m_{h^0}^2 g^2}
$$

(44)

Analyses indicate that variations in $\Delta r$ of order 0.002 are acceptable in the Standard Model [21]. This would translate into a constraint on $f_{\mu e}$ which can be found on the first line in Table 3.

It is interesting to note that the $h^+$ exchange adds constructively to the muon decay rate. The neutron and the nuclear beta decay rates, on the other hand, are unchanged compared to the Standard Model. The value of the CKM matrix element $V_{ud}$ extracted from beta decay measurements will have to be re-interpreted in the present model. We find an upward shift in $V_{ud}$ compared to the SM by an amount given by $|\delta \Delta r|$ of Eq. (44). Currently there is a 2.2 sigma anomaly in the unitarity of the first row of the CKM matrix: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2$ is smaller than by about 2.2 sigma [16]. With $|\delta \Delta r| = 0.002$, the upward shift in the value of $V_{ud}$ can nicely reconcile this anomaly within this model.

Table 3 also contains constraints on the $f$ couplings coming from the widths of semileptonic $\tau$ decays [16](lines 2,3). Since the redefined Fermi constant is used to compute these decays, the constraints are on the differences between the $f$ couplings. Line 4 contains the constraint coming from $e - \mu$ universality in $\tau$ decay [22].

<table>
<thead>
<tr>
<th>Process</th>
<th>Exp. bound</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e \rightarrow e\gamma$</td>
<td>$\delta(g-2)_e &lt; 8 \times 10^{-12}$</td>
<td>$</td>
</tr>
<tr>
<td>$\mu \rightarrow \mu\gamma$</td>
<td>$\delta(g-2)_\mu &lt; 1.2 \times 10^{-8}$</td>
<td>$</td>
</tr>
<tr>
<td>$\mu \rightarrow e\gamma$</td>
<td>Br. $&lt; 1.2 \times 10^{-11}$</td>
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<tr>
<td>$\tau \rightarrow e\gamma$</td>
<td>Br. $&lt; 2.7 \times 10^{-6}$</td>
<td>$</td>
</tr>
<tr>
<td>$\tau \rightarrow \mu\gamma$</td>
<td>Br. $&lt; 1.1 \times 10^{-6}$</td>
<td>$</td>
</tr>
</tbody>
</table>

TABLE II. Constraints from $l_a \rightarrow l_b\gamma$ type processes.
Process \[ \rightarrow \] \text{Exp. bound} \quad \text{Constraint}

\[ \mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e \quad |\delta \Delta r| < 0.002 \quad |f_{e\mu}|^2/m_\nu^2 < 1.6 \times 10^{-8} \text{ GeV}^{-2} \]

\[ \tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e \quad \text{Br} = 17.83 \pm 0.06 \% \quad \frac{|f_{e\tau}|^2 - |f_{e\mu}|^2}{m_\nu^2} < 3.4 \times 10^{-8} \text{ GeV}^{-2} \]

\[ \tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu \quad \text{Br} = 17.37 \pm 0.07 \% \quad \frac{|f_{\mu\tau}|^2 - |f_{e\mu}|^2}{m_\nu^2} < 4. \times 10^{-8} \text{ GeV}^{-2} \]

e/\mu \text{ universality} \quad \frac{G_{e\nu}}{G_{\tau\mu}} = 0.999 \pm 0.003 \quad \frac{|f_{e\tau}|^2 - |f_{e\mu}|^2}{m_\nu^2} < 2.5 \times 10^{-8} \text{ GeV}^{-2} 

\begin{tabular}{ |l|l|l| } 
\hline
Process & Exp. bound & Constraint \\
\hline
$\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e$ & $|\delta \Delta r| < 0.002$ & $|f_{e\mu}|^2/m_\nu^2 < 1.6 \times 10^{-8} \text{ GeV}^{-2}$ \\
$\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e$ & $\text{Br} = 17.83 \pm 0.06 \%$ & $|f_{e\tau}|^2 - |f_{e\mu}|^2 < 3.4 \times 10^{-8} \text{ GeV}^{-2}$ \\
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e/\mu \text{ universality} & $G_{e\nu}/G_{\tau\mu} = 0.999 \pm 0.003$ & $|f_{e\tau}|^2 - |f_{e\mu}|^2 < 2.5 \times 10^{-8} \text{ GeV}^{-2}$ \\
\hline
\end{tabular}

\textbf{TABLE III.} Constraints from $\mu$ decay and semileptonic $\tau$ decays.

- \textit{Neutrinoless double beta decay.}

From the approximate forms of the neutrino mass matrix given in Eqs. (18)-(20), it follows that the effective neutrino mass relevant for neutrinoless double beta decay is approximately $m(\beta\beta0\nu) \sim m/2 \sim 10^{-3} \text{ eV}$ in the hierarchical case as well as in the inverted hierarchical case of Eq. (20). This will be difficult to observe in the near future. On the other hand, in the inverted mass hierarchy of Eq. (19), the effective mass is $m(\beta\beta0\nu) \sim M \simeq 0.05 \text{ eV}$, which should be observable in the next round of experiments.

Now, let us look at these constraints in light of the relations among the $f_{ab}$ and $h_{ab}$ couplings imposed by neutrino masses and mixings results of Sec. III. Consider first the $f$ parameters. In the case of hierarchical form for the neutrino mass matrix, from Eqs. (30), (31) we have

$$f_{e\mu} \approx f_{e\tau} \approx \frac{f_{\mu\tau}}{2}.$$  \hfill (45)

The strongest constraint on the $f$ couplings comes then from the $\mu \rightarrow e\gamma$ process. From Table 2 we have

$$\frac{f_{\mu\tau}^2}{2m_\nu^2} < 0.4 \times 10^{-8} \text{ GeV}^{-2}.$$  \hfill (46)

On the other hand, from the last of Eq. (32), we have

$$\frac{8\mu}{(16\pi^2)^2} \frac{f_{\mu\tau}^2}{m_\nu^2} \bar{I} m_\mu^2 h_{\mu\mu} \approx \frac{M}{2}.$$  \hfill (47)
with \( M \approx \sqrt{2.5 \times 10^{-5}} \text{ eV} = 5 \times 10^{-11} \text{GeV} \) (here we have neglected the contribution of the terms proportional to \( m_e \)). Plugging in the numbers we find

\[
m_h \simeq 10^5 \left( \frac{\mu}{m_h} \right) \frac{f_{\mu\tau}^2 h_{\mu\mu}}{I} \text{ GeV}.
\]  

(48)

Assume for now that \( m_k \) is smaller or of the same order as \( m_h \); then \( I \approx 2 \). Using the upper limits on the \( h \) couplings and \( \mu/m_h \) ratio derived in Sect. II, we get the following allowable range for the \( h^+ \) scalar mass:

\[
10^4 f_{\mu\tau} \text{ GeV} < m_h < 10^6 f_{\mu\tau}^2 \text{ GeV}.
\]  

(49)

Here the lower limit comes from the \( \mu \to e\gamma \) constraint Eq. (46), while the upper limit comes from the neutrino mass equations above. We see then that when the \( f_{\mu\tau} \) coupling takes its maximum value (of order unity), the \( h^+ \) scalar will be out of reach of future colliders. However, this is the most unfavorable case; if the \( h, f, \) and \( \mu \) couplings take values smaller than the highest values admissible, the upper bound on \( m_h \) moves to lower values. For example, if \( f_{\mu\tau} \approx 0.1, m_h \) has to be smaller than 10 TeV. Note also that Eqs. (46), (48) impose a lower limit on the strength of the \( f_{\mu\tau}, h_{\mu\mu} \) couplings:

\[
f_{\mu\tau} > \frac{0.1}{h_{\mu\mu} I} \left( \frac{\mu}{m_h} \right)^{-1} \gtrsim 1 \times 10^{-2}
\]  

(50)

\[
h_{\mu\mu} > \frac{0.1}{f_{\mu\tau} I} \left( \frac{\mu}{m_h} \right)^{-1} \gtrsim 1.7 \times 10^{-2}.
\]

This requires that \( m_h \) has to be greater than about 100 GeV. Also, the lower bound on \( h_{\mu\mu} \) justifies neglecting the \( \omega_{ee}, \omega_{e\mu} \) terms in the neutrino mass matrix; indeed, for the \( \omega_{e\mu} \) term to be of the same order of magnitude as \( \omega_{\mu\mu} \), we would need \( h_{e\mu} \approx 200 \times h_{\mu\mu} \) which will move the theory into the nonperturbative regime.

We conclude from the preceding discussions that the decay \( \mu \to e+\gamma \) should be accessible to the next round of rare decay experiments. Some of the model parameters are already excluded by current limits. For example, if \( m_h \sim 1 \text{ TeV} \), any value of \( f_{\mu\tau} \geq 2 \times 10^{-3} \) will be inconsistent with current limits. There are plans to improve the present limit on \( \mu \to e+\gamma \) by several orders of magnitude in the near future [24]. This decay is predicted to be within reach of these improved experiments.

The leptonic phenomenology constraints on the \( h_{ab} \) coupling constants and the mass of the doubly charged scalar \( m_k \) are somewhat weaker. All constraints from processes involving the electron can be made to go away by choosing \( h_{ee}, h_{e\mu} \) and/or \( h_{e\tau} \) to be close to zero (which is allowed by the neutrino masses and mixings pattern). Then, the strongest constraint on the \( h_{ab}/m_k \) ratio comes from the \( \tau^- \to \mu^+\mu^-\mu^- \) process; using Eq. (24) to relate \( h_{\mu\tau} \) and \( h_{\mu\mu} \), this constraint reads as

\[
\frac{h_{\mu\mu}}{m_k} < 1.3 \times 10^{-3} \text{ GeV}^{-1}
\]

which can be easily satisfied without violating the other constraints discussed previously. Note also that it is not required that \( h_{ee}, h_{e\mu}, \text{ and } h_{e\tau} \) be close to zero; independent constrains
on these quantities from Table 1 and Table 2 are:

\[ \frac{|h_{ee}h_{\mu\mu}^*|}{m_k^2} < 4 \times 10^{-7}, \quad \frac{|h_{ep}h_{\mu\mu}^*|}{m_k^2} < 4 \times 10^{-9}, \quad \frac{|h_{ep}h_{\mu\mu}^*|}{m_k^2} < 9 \times 10^{-8}, \]

allowing any of them to be of the same order of magnitude as \( h_{\mu\mu} \) (although not both \( h_{ee} \) and \( h_{ep} \) at the same time).

The bounds on the parameters of this model are more stringent for the inverted mass hierarchy case. In this case \( |f_{e\mu}| \simeq |f_{e\tau}| \simeq |f_{\mu\tau}|/\sin \theta_{13} \) are the larger \( f \) couplings. In terms of \( f_{e\mu} \) the bounds on \( m_h \) can be written:

\[
2 \sqrt{\sin \theta_{13}} \times 10^4 \, f_{e\mu} \, \text{GeV} < m_h \simeq 2 \sin^2 \theta_{13} \times 10^5 \left( \frac{\mu}{m_h} \right) f_{e\mu}^2 h_{\mu\mu} \tilde{I} \, \text{GeV} \quad (51)
\]

with \( f_{e\mu} < 1 \). The first inequality comes from the \( \mu \to e\gamma \) process in Table II, while the last one comes from the neutrino mass equations (35) (we considered the + case here). Note that the allowed interval is narrower and shifted toward lower values of \( m_h \). The lower bounds on the \( f_{e\mu} \) and and \( h_{\mu\mu} \) in Eqs. (50) are increased by a factor \( 1/(\sin \theta_{13})^{3/2} \). This also imposes a lower bound on the value of the \( \theta_{13} \) mixing angle:

\[
\sin \theta_{13} > 0.046 \quad (52)
\]

which should be testable in neutrino oscillation experiments. These constraints are relaxed by a factor \( \cos 2\theta_{12}(1 + 4\theta_{13}\tan 2\theta_{12}) \) (\( \simeq 0.44(1 + 8\theta_{13}) \) for \( \tan^2 \theta_{12} = 0.4 \) for the + sign in the inverted mass hierarchy case (using Eqs. (36) for the neutrino mass matrix elements).
We now turn to the possible signals coming from the production of the charged scalars at colliders. Consider first the case of hadron colliders. Since the $h^+$ and $k^{++}$ scalars do not couple directly to the quarks, the production of these particles will proceed through the $s$-channel processes: $q\bar{q} \to \gamma^*, Z^* \to h^+h^-, k^{++}k^{--}$. The cross-section for these processes, at the 2 TeV Tevatron and at the LHC, are presented in Fig. 4, as a function of the mass of these particles. As can be seen from Fig. 4, the cross-section for the production of the doubly charged scalar $k^{++}$ is about four times larger than the one for the production of the singly charged scalar $h^+$ (for equal masses). Experimentally, the $k^{++}$ is also much easier to see; the hierarchy between the $h$ couplings $h_{\mu\mu} : h_{\mu\tau} : h_{\tau\tau} \simeq 1 : m_\mu/m_\tau : (m_\mu/m_\tau)^2$ (Eqs. (24,37)) implies that the $k^{++}$ will decay predominantly to a same sign muon pair (or electron pair, if $h_{ee}$ is of the same order of magnitude as $h_{\mu\mu}$). In any case, the experimental signature of 4-lepton final state will be striking. The SM background from $ZZ$ production can be greatly reduced by imposing high $p_T$ cuts on the transverse momenta of the leptons, and by requiring that the invariant mass of opposite sign lepton pairs be different from the $Z$ mass. In Ref. [23], it has been estimated that as few as 10 events are enough for the discovery of a doubly charged boson which decays mostly to $e\mu$ or $\mu\mu$ pairs. In this case, the Tevatron Run II will be able to probe up to about 250 GeV with an integrated luminosity of 15 fb$^{-1}$, while the LHC reach will be about 800 GeV with 100 fb$^{-1}$, or 1 TeV with 1 ab$^{-1}$ integrated luminosity.

At a hadron collider, the $s$-channel production of a $h^+h^-$ pair will be much harder to detect experimentally. The final state of two leptons + missing energy (associated with the neutrinos coming from $h^+$ decay) has SM backgrounds coming from $ZZ, ZZ$ (with one lepton lost) and $WW$ production, as well as Drell-Yan production of two leptons with mis-measured energy. $p_T$ cuts can be used to reduce the background in this case too, but a more detailed analysis is needed to obtain the hadron collider reach for $h^+$ production in this mode. Another interesting possibility would be the production of a single $h^+$ through radiation from the $\nu$ line in the process $q\bar{q}' \to l\bar{\nu}_l$. The final state in this case will be three leptons plus missing $E_T$. High $p_T$ cuts and cuts on the invariant mass of lepton pairs can be used here too to reduce the SM background (which will come mostly from $WZ$ production). However, the production cross section in this case is proportional to the $f$ couplings squared. For $h^+$ boson masses below 1 TeV, the $\mu \to e\gamma$ constraint Eq. (46) requires that $f$ be in the 0.01 - 0.1 range; as a consequence, there will be at most of order tens of single $h^+$ events produced at the LHC for any $h^+$ mass in the TeV range.

At $e^+e^-$ colliders, $s$-channel pair production of singly charged or doubly charged bosons will be limited by the energy of the machine. However, due to the cleanliness of the environment, they will be easy to see, if kinematically accessible. The reach in mass at these machines will therefore be roughly $\sqrt{s}/2$. Provided that the $h_{ee}$ coupling is large, $k^{++}$ can also be singly produced at an $e^-e^-$ collider. More interesting, though, would be the study of the doubly charged scalar at a muon collider, since the $h_{\mu\mu}$ coupling of the $k^{++}$ has to be large in this model.
VI. CONCLUSIONS

In this paper we have performed a detailed analysis of a specific two–loop neutrino mass model. We have shown that the model can accommodate both the solar and the atmospheric neutrino data, while at the same time satisfying all bounds arising from leptonic phenomenology. While at present there is no conflict with any limit, the model predicts that the rare decays $\tau \to 3\mu$ and $\mu \to e + \gamma$ should be within reach of forthcoming experiments.

Although the model contains two independent coupling matrices, one symmetric and the other antisymmetric, they are well constrained by the current neutrino oscillation data. We have found a simple way of relating these matrices to neutrino oscillation parameters. In particular, we have shown that the relative ratios between the three elements of the antisymmetric coupling matrix $f$ are completely determined by the neutrino mixing angles, regardless of the other parameters of the model. Since the contribution of the symmetric couplings $h_{ab}$ to the neutrino mass matrix is proportional to the product of the masses of the $a$ and $b$ charged leptons, only $h_{\mu\mu}, h_{\mu\tau}$ and $h_{\tau\tau}$ are relevant for neutrino oscillations. The facts that the atmospheric mixing angle $\theta_{23}$ is close to $\pi/4$ and that there is a hierarchy in the solar and atmospheric neutrino mass–splittings fixes the relative ratio between these three parameters to be $h_{\mu\mu} : h_{\mu\tau} : h_{\tau\tau} \simeq 1 : m_{\mu}/m_{\tau} : (m_{\mu}/m_{\tau})^2$. The fact that $h_{\mu\mu}$ is the larger of these may have implications on the search for the doubly charged scalar $k^{++}$ in collider experiments.

From the experimental result on the atmospheric neutrino mass splitting and the constraints on the anomalous process $\mu \to e\gamma$, a lower bound of 0.01 on the largest of the $f$ and $h$ coupling constants has been derived. There are upper bounds on the masses of the charged scalars $h^+$ and $k^{++}$ as well. For non-maximal values of the Yukawa couplings (for example, $f_{\mu\tau} \approx h_{\mu\mu} \approx 0.1$) the predicted values for the scalar masses are in the TeV range, which should be probed at the LHC, and perhaps at Run II of the Tevatron.

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