Spin-1 gravitational waves

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Abstract

Exact solutions of Einstein equations invariant for a 2-dimensional Lie algebra of Killing fields \([X, Y] = Y\), with \(Y\) of light type, are analyzed. The conditions for them to represent gravitational waves are established and the definition of energy and polarization is addressed. PACS numbers: 04.20.-q, 04.20.Gz, 04.20.Jb

Introduction

Gravitational waves, that is a propagating warpage of space time generated from compact concentrations of energy like neutron stars and black holes, have not yet been detected directly, although their indirect influence has been seen and measured with great accuracy. Presently there are, worldwide, many efforts to detect gravitational radiation, not only because a direct confirmation of their existence is interesting per se but also because new insights on the nature of gravity and of the Universe itself could be gained. For these reasons exact solutions of the Einstein field equations deserve special attention when they are of propagative nature. The need of taking into full account the nonlinearity of Einstein’s equations when studying the generation of gravitational waves from strong sources is generally recognized [20]. Moreover, despite the great distance of the sources from Earth (where most of the experimental devices, laser interferometers and resonant antennas, are located) there are situations where the non linear effects cannot be neglected. This is the case when the source is a binary coalescence: indeed it has been shown [4] that a secondary wave, called the Christodoulou memory is generated via the non linearity of Einstein’s field equations. The memory seems to be too weak to be detected from the present generation of interferometers [20] (even if its frequency is in the optimal band for the LIGO/VIRGO interferometers) but of the same order as the linear effects related to the same source, thus stressing the relevance of the nonlinearity of the Einstein’s equations also (soon) from an experimental point of view. On the theoretical side, starting from the seventy’s new powerful
mathematical methods have been developed to deal with nonlinear evolution equations. For instance, a suitable generalization of the Inverse Scattering Transform allows to integrate [3] Einstein field equations for a metric of the form

\[ g = f(z, t) \left( dt^2 - dz^2 \right) + h_{11}(z, t) \, dx^2 + h_{22}(z, t) \, dy^2 + 2h_{12}(z, t) \, dxdy. \]

Indeed, the corresponding vacuum Einstein field equations reduce essentially * to

\[ \left( \alpha H^{-1} H_\xi \right)_\eta + \left( \alpha H^{-1} H_\eta \right)_\xi = 0, \]

where \( H \equiv \| h_{ab} \|, \quad \xi = (t + z) / \sqrt{2}, \quad \eta = (t - z) / \sqrt{2}, \quad \alpha = \sqrt{|\det H|}. \) This is a non-linear differential equation whose form is typical of two-dimensional integrable systems. Its solution through the Inverse Scattering Transform, yields solitary waves solutions. The 1-soliton solution has the form

\[ g = C_1 z e^{2q} \cosh(qr + C_2) \left( dt^2 - dz^2 \right) + \frac{\cosh(s_1 r + C_2)}{\cosh(qr + C_2)} z^{2s_1} \, dx^2 + \frac{\cosh(s_2 r - C_2)}{\cosh(qr + C_2)} z^{2s_2} \, dy^2 - \frac{2 \sinh(r/2)}{\cosh(qr + C_2)} zdxdy, \]

with \( t^2 \geq z^2 \) and where \( s_1 \) and \( s_2 \) are constants satisfying the condition \( s_1 + s_2 = 1 \), so that they can be expressed, in terms of one arbitrary constant parameter \( q \), as \( s_1 = 1/2 + q, \quad s_2 = 1/2 - q. \) The function \( r \) is defined by:

\[ \exp r = 2z^{-2}t^2 - 1 - 2 \left( z^{-2}t^2 (z^{-2}t^2 - 1) \right)^{1/2}. \]

It can be easily verified that for any \( t \) the extremum of \( g_{11} \) with respect to the spacelike coordinate \( z \), corresponds to the same constant value \( r_0 \) of the function \( r \). Then, the world line of the extremum obeys the equation \( t = z \cosh(r_0/2) \), and therefore the speed of this localized disturbance is smaller than the light velocity (for a review see [22]). A geometric inspection of the metric above shows that it is invariant under translations along the \( x, y \)-axes, i.e. it admits two Killing fields, \( \partial_z \) and \( \partial_y \), closing on an Abelian† two-dimensional Lie algebra \( A_2 \). Moreover, the distribution \( D \), generated by \( \partial_x \) and \( \partial_y \), is 2-dimensional and the distribution \( D^\perp \) orthogonal to \( D \) is integrable and transversal to \( D \). Thus, it has been natural to consider [19] the general problem of characterizing all gravitational fields \( g \) admitting a Lie algebra \( G \) of Killing fields such that: I. the distribution \( D \), generated by the vector fields of \( G \), is two-dimensional. II. the distribution \( D^\perp \) orthogonal to \( D \) is integrable and transversal to \( D \).

The aim of this article is to study, among those solutions, the ones which show a propagative nature. A preliminary account of our results is given in [5].

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*The function \( f \) can be obtained by quadratures in terms of the matrix \( H \).
†The study of metrics invariant for a Abelian 2-dimensional Killing Lie algebra goes back to Einstein and Rosen [7, 17], Kompaneyets [9].
where simple cases are analysed. The article is organized as follows. In section 1 gravitational fields invariant for a two dimensional Lie algebra are characterized. In section 2 the Einstein equations for such metrics are reduced, by using the symmetry, to the so called $\mu$-deformed Laplace equation. Harmonic coordinates are also introduced. Section 3 is devoted to the analysis of the wave-like character of the solutions through the Zel’manov and the Pirani criterions. In section 4 the canonical and the Landau energy-momentum pseudo-tensors are introduced and a comparison with the linearised theory is performed. Eventually, section 5 is devoted to the analysis of the polarization of the waves. In the following, $\text{Kil}(g)$ will denote the Lie algebra of all Killing fields of a metric $g$ while $\text{Killing algebra}$ will denote a sub-algebra of $\text{Kil}(g)$. Moreover, an integral (two-dimensional) submanifold of $\mathcal{D}$ will be called a $\text{Killing leaf}$, and an integral (two-dimensional) submanifold of $\mathcal{D}^\perp$ $\text{orthogonal leaf}$.

1 Invariant metrics

Let $g$ be a metric on the space-time $M$ and $\mathcal{G}_2$ one of its Killing algebras whose generators $X,Y$ satisfy the commutation relation

$$[X,Y] = \sigma Y, \quad \sigma = 0, 1. \quad (1)$$

The Frobenius distribution $\mathcal{D}$ is two-dimensional and a coordinate system $(x^\mu)$, $\mu = 1, 2, 3, 4$ exists such that

$$X = \frac{\partial}{\partial x^3}, \quad Y = \exp(\sigma x^3) \frac{\partial}{\partial x^4}.$$ 

Such a coordinate system is called [19] semiadapted (to the Killing fields). Condition II of the previous section allows to construct special semi-adapted charts, with coordinates $(y^1, y^2, y^3, y^4)$, such that the fields $e_1 = \partial/\partial y^1$, $e_2 = \partial/\partial y^2$, belong to $\mathcal{D}^\perp$. In such a chart, called from now on adapted, the most general $\mathcal{G}_2$-invariant metric has the form [19]

$$g = g_{ij} dy^i dy^j + \left(\sigma^2 \lambda (y^4)^2 - 2\sigma \mu y^4 + \nu\right) dy^3 dy^3 + 2(\mu - \sigma \lambda y^4) dy^3 dy^4 + \lambda dy^4 dy^4, \quad i, j = 1, 2$$

where $g_{ij}$, $\lambda$, $\mu$, $\nu$ are arbitrary functions of $(y^1, y^2)$.

2 Einstein metrics

If the Killing field $Y$ is not of light type, i.e. $g(Y,Y) \neq 0$, condition II of section 1 follows automatically from I. The local structure of this class of Einstein metrics has been explicitly described; it turns out [19] that a third Killing field $Z$ exists such that $(X,Y,Z)$ span the Killing algebra $so(2,1)$ and all invariant metrics are static and locally diffeomorphic to the pseudo-Schwarzschild metric.
This metric was also found in the context of warped solutions [8, 23]. In the following the Killing field \( Y \) will be assumed to be of light type. In this case the general solution of vacuum Einstein equations, in the adapted coordinates \((y^1, y^2, y^3, y^4)\), is given by

\[
g = 2f(dy^1dy^1 + dy^2dy^2) + \mu[(w(y^1, y^2) - 2\sigma y^4)dy^3dy^3 + 2dy^3dy^4],
\]

where \( \mu = D\Phi + B \); \( D, B \in R \), \( \Phi \) is a non constant harmonic function of \( y^1 \) and \( y^2 \), \( f = (\nabla\Phi)^2 \sqrt{|\mu|}/\mu \), and \( w(y^1, y^2) \) is a solution of the \( \mu \)-deformed Laplace equation:

\[
\Delta w + (\partial_{y^1} \ln |\mu|) \partial_{y^1} w + (\partial_{y^2} \ln |\mu|) \partial_{y^2} w = 0.
\]

The solutions of the \( \mu \)-deformed Laplace equation will be also called \( \mu \)-harmonic functions. When \( w \) is constant the above family of Einstein metrics admits the time-like Killing vector field

\[
T = \exp (-y^3) \frac{\partial}{\partial y^3} + \frac{1}{2} \left[(w - 2\sigma y^4) \exp (-y^3) + \exp (y^3)\right] \frac{\partial}{\partial y^4}
\]

which is hyper-surfaces orthogonal. This means that these Einstein metrics are just static gravitational fields. As it will be argued in the next sections, when \( w \) is not constant the above family of Einstein metrics may represent propagative gravitational fields. Since the distribution \( D^\perp \) is assumed to be transversal to \( D \), the restriction of \( g \) to any Killing leaf, say \( S \), is non-degenerate. So, \((S, g|_S)\) is a homogeneous two-dimensional Riemannian manifold. In particular, the Gauss curvature \( K(S) \) of the Killing leaves is constant. An explicit computation shows that \( K(S) \) vanishes. Thus, the space-time \( M \) has a fiber bundle structure

\[
\pi : M \rightarrow W,
\]

whose basis \( W \) is diffeomorphic to the orthogonal leaves and whose fibers are the Killing leaves and as such are flat two-dimensional Riemann manifolds. The coordinates \((y^3, y^4)\) on the Killing leaves \( S \) have a clear geometric meaning but are of difficult physical interpretation. Fortunately, being the Killing leaves flat manifolds, it is possible to introduce coordinates \((\tilde{z}, \tilde{t})\) diagonalizing the metric

\[
g|_S = \tilde{\mu}[(\tilde{w} - 2\sigma y^4)dy^3dy^3 + 2dy^3dy^4],
\]

where \( \tilde{\mu} \) and \( \tilde{w} \), being the restriction of the functions \( \mu \) and \( w \) to the Killing leaves, are constant.

### 2.1 The harmonic coordinates in the Abelian case.

In the case \( \sigma = 0 \), the Killing algebra (1) is Abelian and the metrics (2) restricted to the Killing leaves \( S \) are diagonalized by the transformation

\[
\begin{align*}
\tilde{z} - \tilde{t} &= \sqrt{2}y^3 \\
\tilde{z} + \tilde{t} &= \sqrt{2}y^4 + \frac{1}{\sqrt{2}}\tilde{w}y^3,
\end{align*}
\]
so that it becomes
\[
g|_S = \tilde{\mu} (dz^2 - dt^2).
\]

The coordinate system \((x, y, z, t)\), where
\[
\begin{align*}
x &= w_1(y^1, y^2) \\
y &= w_2(y^1, y^2) \\
z &= \frac{1}{2} \sqrt{2} [y^3 + y^4 + w(y^1, y^2)] \\
t &= \frac{1}{2} \left[ \sqrt{2} (y^3 - y^4) + \frac{1}{2} w(y^1, y^2) \right],
\end{align*}
\]
is harmonic, \(w_1(y^1, y^2)\) and \(w_2(y^1, y^2)\) denoting any two independent \(\mu\)-harmonic functions.

In these coordinates metrics (2), with \(\sigma = 0\), become:
\[
g = 2\sqrt{\mu} |(\nabla \Phi)|^2 J^{-2} \left[ (\nabla y)^2 dx^2 + (\nabla x)^2 dy^2 - 2 \nabla x \nabla y dx dy \right] + \mu [dz^2 - dt^2 + (z - t) d(z - t) dw],
\]
where \(J = \partial_{y_1} w_1 \partial_{y_2} w_2 - \partial_{y_2} w_1 \partial_{y_1} w_2\) is the Jacobian determinant of the map \((y^1, y^2) \rightarrow (x, y)\).

### 2.2 The harmonic coordinates in the non Abelian case.

For \(\sigma = 1\), the Killing algebra is non-Abelian and the metrics (2 ) restricted to the Killing leaves \(S\) are diagonalized by the transformation
\[
\begin{align*}
\bar{z} - \bar{t} &= \exp (-y^3) \\
\bar{z} + \bar{t} &= - (\bar{w} - 2y^4) \exp (y^3),
\end{align*}
\]
so that
\[
g|_S = \tilde{\mu} (d\bar{z}^2 - d\bar{t}^2).
\]

Then, the coordinate system \((x, y, z, t)\), where for \(z > t\)
\[
\begin{align*}
x &= w_1(y^1, y^2) \\
y &= w_2(y^1, y^2) \\
z &= \frac{1}{2} \left[ (2y^4 - w(y^1, y^2)) \exp (y^3) + \exp (-y^3) \right] \\
t &= \frac{1}{2} \left[ (2y^4 - w(y^1, y^2)) \exp (y^3) - \exp (-y^3) \right],
\end{align*}
\]
is harmonic (Appendix A), \(w_1(y^1, y^2)\) and \(w_2(y^1, y^2)\) denoting, as before, any two independent \(\mu\)-harmonic functions. The generic Killing leaf \(S\) is mapped onto the half-plane \(z > t\), the line \(z = t\) representing the points with \(y^3 = +\infty\). In these coordinates, the metrics (2) take the form
\[
g = 2\sqrt{\mu} |(\nabla \Phi)|^2 J^{-2} \left[ (\nabla y)^2 dx^2 + (\nabla x)^2 dy^2 - 2 \nabla x \nabla y dx dy \right] + \mu [dz^2 - dt^2 + dwd \ln (z - t)],
\]
where $J = \partial_y^1 w_1 \partial_y^2 w_2 - \partial_y^2 w_1 \partial_y^1 w_2$ is the Jacobian determinant of the map $(y^1, y^2) \rightarrow (x, y)$. The coordinate $t$ plays the role of time and $z > t$. Despite the non-linear nature of general relativity, the gravitational fields (2) obey to two superposition laws. Indeed, with two harmonic functions $\Phi_1$ and $\Phi_2$ we can associate three gravitational fields (in facts a whole two-parameters family), that is, $g_1$, $g_2$ and $g_1 \Phi_1 + g_2 \Phi_2$; the last one, which is associated with the linear combination of $\Phi_1$ and $\Phi_2$, may be regarded as the superposition of the two associated solutions $g_1$ and $g_2$. The second superposition law follows from the linearity of the $\mu$-deformed Laplace equation, so that with two $\mu$-harmonic functions $w_1$ and $w_2$ we can associate three gravitational fields $g_{w_1}$, $g_{w_2}$ and their superposition $g_{w_1 + w_2} \equiv (g_{2w_1} + g_{2w_2})/2$. From a physical point of view a hint that the above Einstein metrics might be representing gravitational waves comes from the form of the solutions in the harmonic coordinates. Indeed, Eq (4) shows that the wave-like components decrease very slowly when $|z - t| \rightarrow \infty$.

It is, however, important to check the wave behaviour of these exact solutions in a rigorous way. This problem will be addressed in the forthcoming subsection where a generally covariant criterium due to Zel’manov and Zakharov will be discussed and applied. In the case $\mu = \text{const}$, the $\mu$-deformed Laplace equation reduces to the Laplace equation and $w_1$, $w_2$ reduce to be just harmonic functions. Thus, it is possible to choose $x = y^1$, $y = y^2$ so that in the harmonic coordinates $(x, y, z, t)$, and for $\mu = 1$, the above Einstein metrics take the particularly simple form

$$g = 2f(dx^2 + dy^2) + dz^2 - dt^2 + dw \ln |z - t|.$$  \hspace{1cm} (5)

This coordinates system explicitly shows that, when $w$ is constant, the Einstein metrics given by Eq. (5) are static and, under the further assumption $\Phi = x\sqrt{2}$, they reduce to the Minkowski one. Moreover, when $w$ is not constant, gravitational fields (5) look like a disturbance moving, along the $z$ direction on the Killing leaves, at light velocity. However, the last observation is neither rigorous nor covariant. Since the propagation direction is the most important ingredient in the study of the polarization, in the following sections a detailed analysis will be devoted to this question. In the following we will assume that $w$ is not constant.

3 Wave character of the field

The second step to attempt a physical interpretation of the solutions we are considering is the study of their wave character. To check if the Einstein metrics given by Eq. (2) can be classified as gravitational waves we will apply a Zakharov generalization of the Zel’manov criterion [26] which states that a vacuum solution of the Einstein equations is a gravitational wave if the components $R_{\mu \nu \lambda \sigma}$ of the corresponding Riemann tensor field $R$, satisfy a hyperbolic equation of the form

$$g^{\alpha \beta} \nabla_\alpha \nabla_\beta R_{\mu \nu \lambda \sigma} = N_{\mu \nu \lambda \sigma}.$$  \hspace{1cm} (6)
where $\nabla_\beta$ denotes the Levi-Civita covariant derivative of the metric and $N_{\mu\nu\lambda\sigma}$ denote the components of a tensor field $N$ depending at most on first derivatives of the Riemann tensor itself. For symmetric manifolds Eq. (6) with $N = 0$ is an identity because the Riemann tensor is covariantly constant, but it may become an identity also in the case of Einstein manifolds ($R_{\alpha\beta} = \kappa g_{\alpha\beta}$), for special choices of the tensor field $N$. Hence, to exclude a priori these situations, the original Zel’manov criterion is formulated in the more restrictive assumptions:

- $R_{\alpha\beta\gamma\delta}$ not covariantly constant\(^1\);
- $g^{\alpha\beta}\nabla_\alpha \nabla_\beta R_{\mu\nu\lambda\sigma} = 0$.

The metrics in Eq. (2) certainly do not define symmetric or Einstein manifolds, as can be checked from the components of the Ricci tensor given below. Hence, the first hypothesis is certainly satisfied while the second one, *i.e.* $N = 0$, which ensures the applicability of the criterion to Einstein manifolds too, is not needed. Concerning the physical meaning of this criterion, it can be shown [26] that the characteristic hypersurface of the system of equations (6) is identical with the characteristic hypersurface of the Einstein and Maxwell equations in curved space-time. Consequently Eqs. (6) describe the propagation of the discontinuities of the second derivatives of the Riemann tensor. This links the Zel’manov criterion to the intuitive concept of *local wave of curvature*. The criterion is independent on the explicit form of $N_{\mu\nu\lambda\sigma}$; in fact, the characteristic hypersurface of a system of equations is determined only by the highest derivative term. Then we will not fix an explicit form of $N_{\mu\nu\lambda\sigma}$ but just require that $N_{\mu\nu\lambda\sigma}$ be a tensor containing at most first derivatives of the Riemann tensor. This clearly corresponds to a covariant criterion [26]. Then a sufficient condition is

$$g^{\alpha\beta}\partial_\alpha \partial_\beta R_{\mu\nu\lambda\sigma} = 0 \quad (7)$$

where $\partial_\beta$ are the usual partial derivatives. In fact, if this is the case then $N_{\mu\nu\lambda\sigma}$ is a tensor containing at most first derivatives of the Riemann tensor. To start with let us verify that gravitational fields (5) effectively satisfy Eqs. (7). In the harmonic coordinates system the only nonvanishing components of the Riemann and Ricci tensor fields corresponding to metrics (5) are proportional to one of the following

$$R_{txxx} = \frac{(2f_{,xx}w_{,xx} + f_{,yy}w_{,yy} - f_{,xx}w_{,xx})}{4f(z-t)^2}$$
$$R_{txzy} = \frac{(2f_{,xy}w_{,xy} - f_{,yy}w_{,yy} + f_{,xx}w_{,xx})}{4f(z-t)^2}$$
$$R_{tyzy} = \frac{(2f_{,yy}w_{,yy} + f_{,xx}w_{,xx})}{4f(z-t)^2}$$
$$R_{xyxy} = \frac{f_{,y}^2 + f_{,x}^2 - f(f_{,xx} + f_{,yy})}{f} \quad (8)$$

\(^1\)That is the manifold is not symmetric.
and

\[ R_{tt} = \frac{\Delta w}{2f(z-t)^2}, \quad R_{xx} = \frac{f_y^2 + f_x^2 - f(f_{xx} + f_{yy})}{2f^2}, \]

respectively. Moreover, the harmonicity condition for \( \Phi \) implies that the last components of the Riemann and Ricci tensors both vanish. In fact, when \( \mu = \text{const}, \Delta \Phi = 0 \) implies for \( f \) that

\[ f \Delta f - (\nabla f)^2 = 0. \tag{9} \]

- When \( f \) is a constant function, Eqs. (8) reduce to

\[ R_{txzx} = w, \quad R_{txzy} = w, \quad R_{tyzy} = w, \tag{10} \]

which, \( w(x, y) \) being a harmonic function, are all harmonic functions of \( x, y \). As a consequence the generalized Zel’manov criterion in the form (7) is straightforward to check, and satisfied [5]. For instance, this is easily seen for the component \( R_{tyzy} \)

\[ g_{\alpha\beta} \partial_\alpha \partial_\beta R_{tyzy} = \frac{\partial^4 w(x, y)}{4(z-t)^2f} = \frac{\partial^2 \Delta w(x, y)}{4(z-t)^2f} = 0. \]

- When \( f \) is not a constant function, the generalized Zel’manov criterion is still satisfied in the form (7) thanks to a non trivial combination of the harmonicity condition for \( w \) and Eq. (9). Let us see the calculation in detail for one of the components of the Riemann tensor (8), say \( R_{txzx} \).

\[
\begin{align*}
g^{\alpha\beta} \partial_\alpha \partial_\beta R_{txzx} &= \frac{1}{f^2(t-z)^2} \left\{ f^2 (2f \partial_x^2 w + \partial_y f \partial_y w - \partial_x f \partial_x w)(\Delta w) \\
&- [2(\partial_z w \partial_x f - \partial_y w \partial_y f) - (\partial_z w \partial_x f - \partial_y w \partial_y f)](\nabla f)^2 - f \Delta f \\
&- 2f \partial^2 w[(\partial_x f)^2 - f \partial^2_x f] + 2f \partial^2 w[(\partial_y f)^2 - f \partial^2_y f] \right\} = 0.
\end{align*}
\]

Thus, the class of solutions represented by Eq.(5) certainly represent gravitational waves.

- In the general case when \( f \) and \( \mu \) are not constant functions the Zel’manov criterion is satisfied in the form expressed by Eq.(6). This may be more conveniently checked in the adapted coordinates \( y^\mu \). On using the harmonicity condition for \( \Phi \) and the expression of \( \mu \) and \( f \) in terms of \( \Phi \), for the Riemann component

\[ R_{3434} = \frac{1}{8} D^2 B + D\Phi(y^1, y^2), \]

it turns out

\[ g^{\alpha\beta} \partial_\alpha \partial_\beta R_{3434} = -\frac{D^7}{64^2 (R_{3434})^2}, \]

so that \( g^{\alpha\beta} \nabla_\alpha \nabla_\beta R_{\mu
u\lambda\sigma} \) depend at most on first derivatives of the Riemann tensor itself. Analogous results obtain for the other components.
The Zel’manov criterion, even if it is covariant and allows a clear physical interpretation in terms of local waves of curvature, does not determine the propagation direction of the waves, that is the most important ingredient in the study of their polarization. In the next sections we will overcome this drawback of the Zel’manov criterion by using a suitable energy-momentum pseudo-tensor. Besides the Zel’manov-Zakharov criterion, the Pirani algebraic criterion, which is based on the Petrov classification, is satisfied. First of all, let us recall that a vacuum solution of the Einstein equations is a gravitational wave according to Pirani if its Riemann tensor is of type II, N or III in the Petrov classification [13]. Then, in light-cone coordinates \((u = (z - t) / \sqrt{2}, \ v = (z + t) / \sqrt{2})\), where the metrics given by Eq.\((5)\) read

\[ g = 2f(dx^2 + dy^2) + 2dudv + dw \, d\ln |u|, \]

the vector fields \(\partial_u\) and \(\partial_v\) are both isotropic. Moreover, it is trivial to show that the only non vanishing components of the Riemann tensor are

\[ R_{uinj} = \pm \frac{1}{2u^2} \partial_j^2 w \]

and this clearly corresponds to a type-N Riemann tensor in the Petrov classification. Furthermore, it follows from the natural interpretation of the Pirani criterion [14] that the gravitational wave propagates along the null vector field \(\partial_u\), or, in other words, the gravitational wave \((5)\) propagates along the \(z\)-axis with velocity \(c = 1\). Thus, the Pirani criterion, even if with a less clear physical interpretation, allows an easy and covariant determination of the propagation direction. It will be an important self-consistency check for our calculations to discover the same results by means of the energy-momentum pseudo-tensors.

4 The energy-momentum pseudo-tensors

The definition of momentum and energy associated with a gravitational field is an intrinsically controversial problem because these quantities are connected to the space-time translation invariance, whereas the group of invariance of general relativity is much bigger. With this cautionary remark in mind, various definitions are available which attain to different physical situations. When dealing with the solutions of the linearised Einstein equations in the vacuum (plane gravitational waves) a commonly accepted definition is based on the canonical energy-momentum pseudo-tensor \(( [21, 6, 25])\):

\[ \tau^\nu_\mu = \frac{\partial L}{\partial (\partial_\nu g_{\alpha\beta})} \partial_\mu g_{\alpha\beta} - g^\nu_\mu L \]

where

\[ L/\sqrt{|g|} = g^{\mu\nu} \left[ \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\sigma} - \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\sigma} \right] \]

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is the Ricci scalar deprived of terms containing the second derivative of the metric. Then, for the wave solutions of the linearised Einstein equations the energy density $\tau^0_0$ is expressed [6] as the sum of squares of derivatives of some metric components which do represent the physical degrees of freedom of the metric. For instance, if a metric has the form

$$g = \eta + h$$

where $\eta$ is the Minkowski metric and $h$ is a small perturbation propagating in the $z$ direction with components only in the ($x, y$) plane (if the usual transverse-traceless gauge is chosen), then

$$\tau^0_0 \sim c_1 (h_{xx,x})^2 + c_2 (h_{xy,y})^2$$

(14)

where $c_1$ and $c_2$ are positive numerical constants and the coordinates are chosen to be harmonic. Under a transformation preserving the propagation direction and the harmonic character of the coordinates system, in particular a rotation in the ($x, y$) plane, the physical components of the metric transform like a spin-2 field. It is well known that in general $\tau^{\mu}_{\nu}$ in Eq. (12) is not a tensor field but it does transform as a tensor field under those transformations which preserve the character of the field of consisting only of waves moving in the $z$ direction, so that the $g_{\mu\nu}$ remain functions of the single variable $z - t$.

Thus, within the linearised theory, the canonical energy-momentum pseudo-tensor is a good tool to study the physical properties of the gravitational waves. Moreover, other methods to study the physical properties of the gravitational waves lead to the same results ([25]). The exact gravitational wave

$$g = dx^2 + dy^2 + dz^2 - dt^2 + dwd\ln |z - t|,$$

(15)

given by Eq. (5) for $\mu = 1, f = 1/2$, has the physically interesting form of a perturbed Minkowski metric with

$$h = dwd\ln |z - t| = \frac{1}{z - t} [w_x dxdz + w_y dydz - w_x dxdt - w_y dydt].$$

Moreover, besides being an exact solution of the Einstein equations, it is a solution of the linearised Einstein equations on a flat background too:

$$\eta^{\mu\nu} \partial_\mu \partial_\nu h = 0.$$  

Then, to study its energy and polarization, the standard tools of the linearised theory and in particular the canonical energy-momentum pseudo-tensor, could be used. Nevertheless, from Eq. (13) it follows that

$$L = -\frac{1}{2} \left[ (\partial_t + \partial_z) g_{tx} \right]^2 + (\partial_t + \partial_z) g_{ty}^2 \right],$$

and, once the explicit form of $g$ is replaced, it is seen that the $\tau^0_0$ component of the canonical energy-momentum tensor vanishes. This is due to the fact that the components of the tensor $h$ cannot be expressed in the transverse-traceless gauge since $h$ has only one index in the plane transversal to the propagation direction. This point is discussed in more detail in the next section.
4.1 Comparison with the linearised theory

To analyse the role played by the usual transverse gauge in the linearised vacuum Einstein equations let us consider a generic metric of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta$ is the Minkowski metric and $h$ is a perturbation. The inverse metric has the form $g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}$ with $k = \sum_n k(n)$ and $k(n)$ is of order $h^n$ and $k(1) = -\eta h \eta$. Then we have for the Christoffel symbols

$$\Gamma^{\lambda}_{\rho\mu\nu} = \frac{1}{2}(h_{\rho\mu,.\nu} + h_{\nu\rho,.\mu} - h_{\mu\nu,.\rho})$$

and

$$\Gamma^{\lambda}_{\mu\nu} = \sum_n \Gamma^{(n)}_{\mu\nu}$$

where $\Gamma^{(n)}_{\mu\nu} = k^{(n-1)\lambda\rho} \Gamma_{\rho\mu\nu}$. Thus the Ricci tensor may be written as

$$R_{\mu\nu} = \sum_n R^{(n)}_{\mu\nu},$$

with

$$R^{(n)}_{\mu\nu} = \Gamma^{(n)}_{\mu\lambda,.\nu} - \Gamma^{(n)}_{\nu\lambda,.\mu} + \sum_{m+m' = n} \Gamma^{(m)}_{\mu\lambda} \Gamma^{(m')\beta}_{\nu\lambda} - \Gamma^{(m)}_{\lambda\beta} \Gamma^{(m')\lambda}_{\mu\nu}.$$  

The harmonicity condition reads

$$0 = \Gamma^{\lambda} = g^{\lambda\rho} g^{\mu\nu} \Gamma_{\rho\mu\nu} = \sum_n \Gamma^{(n)}_{\mu\nu}. \quad (16)$$

In the space-time regions where $|h_{\mu\nu}| << 1$ and $|h_{\mu\nu,\alpha}| << 1$, the linearised theory can be applied taking into account only terms which are of the first order in $h$. In particular only the term

$$\Gamma^{(1)}_{\mu\lambda,\nu} = \eta^{\mu\nu} [(\eta^{\lambda\rho} h_{\rho\nu})_{,\nu} + (\eta^{\lambda\rho} h_{\nu\rho})_{,\mu}] - \eta^{\lambda\rho} (\eta^{\mu\nu} h_{\rho\nu})_{,\rho} \quad (17)$$

contributes to the sum (16). In this approximation, the Einstein equations read

$$R^{(1)}_{\mu\nu} = 0 = \frac{1}{2} [\eta^{\rho\sigma} h_{\rho\sigma,\mu\nu} - \eta^{\rho\sigma} (h_{\rho\sigma,\mu\nu} + h_{\nu\rho,\mu\sigma})]$$

and, because of the harmonicity condition $\Gamma^{(1)}_{\mu\nu} = 0$, they reduce to the well known wave-equation

$$\eta^{\alpha\beta} \partial_{,\alpha} \partial_{,\beta} h_{\mu\nu} = 0. \quad (18)$$

Up to now, apart from the harmonicity condition, no special assumptions either on the form or on the analytic properties of the perturbation $h$ have been done.  

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§ In fact, $\Gamma_{\rho\mu\nu}$ is intrinsically of the first order in $h$.  

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Even if not explicitly declared, the standard textbooks analysis of the polarization is performed for the \textit{square integrable} solutions of the wave-equation (18). Indeed, they can be always Fourier developed in terms of plane-wave functions with a \textit{light-like} vector wave $k_{\mu}$. The harmonicity condition for the plane wave solutions of Eq. (18), $h_{\mu\nu} = e_{\mu\nu}e^{i\rho} + e^{*}_{\mu\nu}e^{-i\rho}$ with $\rho = k_{\mu}x^{\mu}$ and $k_{\mu}k^{\mu} = 0$, reduces to

$$\frac{1}{2}k_{\lambda}h_{\mu\nu}^{\lambda}e_{\mu\nu} = \eta^{\mu\nu}k_{\nu}e_{\mu\lambda}. \quad (19)$$

It is trivial to see that the symmetry group of this equation, which encodes the harmonic nature of the coordinate system, reduces to linear transformations and more precisely to Poincaré transformations [25] (these are nothing but the usual "gauge transformations" of the linearised gravity). It can be easily shown that, for \textit{square integrable} perturbations, one can always choose the transverse-traceless gauge. In other words, it is always possible to eliminate, with a suitable gauge transformation, the components of the perturbations with one index in the propagation direction [25], \textit{i.e.} \textit{square integrable} perturbations of spin-1 do not exist. For these reasons, the \textit{canonical} energy-momentum pseudo-tensor, that is gauge invariant in the sense of the linearised gravity, "cannot see" the energy and momentum of gravitational fields given by Eq.(15) which have one index in the propagation direction (but they are not \textit{square integrable} so they cannot be gauged away). As it will now be explained in more detail, the fact that \textit{square integrable} perturbations with one index in the propagation direction are always \textit{pure gauge} is equivalent to the fact that, for such perturbations, the canonical energy-momentum pseudo-tensor identically vanishes.

\subsection{4.1.1 A special perturbation}

The components of a perturbation of the form $h = d \ln \sqrt{x^2 + y^2} d \ln |z - t|$ satisfy the linearised field equations $\Box h_{\mu\nu} = 0$, but they are not \textit{square integrable} and cannot be Fourier developed, as before. Nevertheless, the metric

$$g = \eta + d \ln \sqrt{x^2 + y^2} d \ln |z - t|,$$

being asymptotically flat, still represent a physically interesting gravitational field\footnote{Of course, it is the exact solution (15) with $w = \ln \sqrt{x^2 + y^2}$}. A perturbation $h$ of the form given above has, in the harmonic coordinates $(x, y, z, t)$ an \textit{off-diagonal} form. This implies that, upon gauge fixing, the matrix of derivatives of $h$ ($u_{\alpha\beta} \equiv \partial h_{\alpha\beta}/\partial \rho$) has the following structure

$$\|u_{\mu\nu}\| = \begin{bmatrix} 0 & \mathbf{F} \\ \mathbf{F}^T & 0 \end{bmatrix}, \quad (20)$$

where $\mathbf{F}$ is a 22 matrix of the form

$$\mathbf{F} = \begin{bmatrix} A & B \\ -A & -B \end{bmatrix}.$$
It follows that $\eta^{\mu\nu}u_{\mu\nu}$ and $u^{\mu\nu}u_{\mu\nu} = 2(u^2_{11} + u^2_{22} - u^2_{14} - u^2_{24})$ vanish. Thus, the canonical energy-momentum pseudo-tensor, which to the first order in $h$ reads

$$16\pi t_\mu^\nu = \frac{1}{2} \left[ u_{\alpha\beta} u^{\alpha\beta} - \frac{1}{2} (\eta^{\alpha\beta} u_{\alpha\beta})^2 \right] k^\nu k_\mu,$$

vanishes, just because, as it has been said, the perturbation has one index in the propagation plane and one in the transverse plane. Moreover, the gauge transformations represented by the Poincaré group, preserve this feature. As it will be shown below, this is also the origin of the non-standard polarization of the gravitational fields of Eq. (5).

### 4.2 The Landau–Lifshitz energy-momentum pseudo-tensor

Fortunately, besides the canonical energy-momentum pseudo-tensor, a deep physical significance can be given to the Landau-Lifshitz energy-momentum pseudo-tensor $\tau^\rho{}^\kappa$ [10] defined by

$$\tau^\rho{}^\kappa = \frac{1}{16\pi k} \left\{ (2\Gamma_\lambda^\nu \Gamma_\mu^\sigma - \Gamma_\lambda^\sigma \Gamma_\mu^\nu) (g_\rho^\lambda g_\kappa^\mu - g_\rho^\kappa g_\lambda^\mu) ight. + g_\rho^\beta g_\mu^\alpha (\Gamma_\kappa^\nu \Gamma_\sigma^\mu + \Gamma_\kappa^\mu \Gamma_\sigma^\nu - \Gamma_\nu^\sigma \Gamma_\mu^\sigma - \Gamma_\mu^\nu \Gamma_\sigma^\sigma) \\
+ g_\kappa^\lambda g_\mu^\nu (\Gamma_\rho^\nu \Gamma_\sigma^\mu + \Gamma_\rho^\mu \Gamma_\sigma^\nu - \Gamma_\nu^\sigma \Gamma_\mu^\sigma - \Gamma_\mu^\nu \Gamma_\sigma^\sigma) \\
+ \left. g^{\rho\lambda} g^{\kappa\sigma} (\Gamma_\nu^\rho \Gamma_\sigma^\mu + \Gamma_\nu^\mu \Gamma_\sigma^\rho - \Gamma_\rho^\nu \Gamma_\sigma^\sigma) \right\}. \quad (21)$$

There are strong evidences that, in some cases, it gives the correct definition of energy [15]. In fact, the energy flux radiated at infinity for an asymptotically flat space-time, evaluated with the Landau-Lifshitz energy-momentum pseudo-tensor, has been seen to agree with the Bondi flux [2] that is with the energy flux evaluated in the exact theory. With respect to the canonical energy-momentum pseudo-tensor there are situations where they do coincide [10, 24], but in general this is not true. For gravitational fields in Eq. (5) it reduces to

$$\tau^\rho{}^\kappa = \frac{1}{16\pi k} \left\{ -\Gamma_\lambda^\nu \Gamma_\mu^\sigma (g_\rho^\lambda g_\kappa^\mu - g_\rho^\kappa g_\lambda^\mu) \\
+ g_\rho^\beta g_\nu^\mu \Gamma_\sigma^\kappa \Gamma_\lambda^\mu - g_\rho^\kappa g_\mu^\nu \Gamma_\sigma^\lambda \Gamma_\rho^\nu \\
+ g^{\rho\lambda} g^{\kappa\sigma} (\Gamma_\nu^\rho \Gamma_\sigma^\mu + \Gamma_\nu^\mu \Gamma_\sigma^\rho - \Gamma_\rho^\nu \Gamma_\sigma^\sigma) \right\}. \quad (22)$$

where the harmonicity condition $\Gamma_\mu^\nu g^{\mu\nu} = 0$ and the property $\eta^{\mu\nu} h_{\mu\nu} = 0$, which in turn implies $\Gamma_\nu^\sigma = 0$, have been used.

It is easy to check that the components $p^\mu \equiv \tau^\mu_0$ of the 4-momentum density are

$$\begin{cases} p^0 = \frac{4}{(t-z)^2} [C_1 (w_{xx})^2 + C_2 (w_{xy})^2] + \frac{4}{(t-z)^2} C_3 \nabla ||\nabla w||^2 \nabla w], \\ p^1 = 0, \\ p^2 = 0, \\ p^3 = p^0, \end{cases}$$

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where $C_i$ are some positive numerical constants, $\nabla = (\partial_x, \partial_y)$ and the harmonicity condition for $w$ has been used. The use of the Bel’s superenergy tensor $[1]$

$$T^{\alpha\beta\lambda\mu} = \frac{1}{2} \left( R^{\alpha\rho\lambda\sigma} R_{\rho\sigma}^{\beta\mu} + * R^{\alpha\rho\lambda\sigma} * R_{\rho\sigma}^{\beta\mu} \right),$$

where the symbol $*$ denotes the volume dual, leads to the same result. Indeed, in adapted coordinates the metric has the form

$$g = dy^1 dy^1 + dy^2 dy^2 + (w(y^1, y^2) - 2y^4) dy^3 dy^3 + 2 dy^3 dy^4$$

and the only non vanishing independent components of the covariant Riemann tensor $R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R_{\rho\beta\gamma\delta}^{\alpha\beta\gamma\delta}$ are

$$R_{1313} = -w_{11}; \quad R_{1323} = -w_{12}; \quad R_{2323} = -w_{22}.$$

It follows that the density energy represented by the Bel’s scalar

$$W = T_{\alpha\beta\lambda\mu} U^{\alpha} U^{\beta} U^{\lambda} U^{\mu},$$

the $U^{\alpha}$’s denoting the components of a time-like unit vector field, depends on the squares of $w_{ij}$. Thus, both the Landau-Lifshitz pseudo tensor and the Bel superenergy tensor single out the same physical degrees of freedom. In particular, we can take the components $h_{1x}$ and $h_{1y}$ as fundamental degrees of freedom for the gravitational wave (15). Concerning the definition of the polarization, the above form for $\tau_{0}^{\mu}$ is particularly appealing because, apart from a physically irrelevant total derivative that does not contribute to the total energy flux, the component $\tau_{0}^{0}$ representing the energy density is expressed as the sum of square amplitudes. The momentum $p^{i} = \tau_{0}^{i}$ is non vanishing only in the $z$-direction and it is proportional to the energy with proportionality constant $c = 1$; that is these waves move with fundamental velocity along the $z$-axis. Moreover, this result is perfectly consistent with the one obtained with the Pirani criterion. The study of the polarization of this wave, performed with the same procedure described for the linearised case, leads to a non standard result, that is $\partial_t h_{1x} = (z - t)^{-1} \partial_{xx} w$ and $\partial_t h_{1y} = (z - t)^{-1} \partial_{xy} w$ represent spin-1 fields. In the forthcoming section a detailed derivation of these results is presented.

5 Spin

Even more controversial than for the energy and momentum, the definition of spin or polarization for a theory, such as general relativity, which is non-linear and possesses a much bigger invariance than just the Poincaré one, deserves a careful analysis [16]. It is well known that the concept of particle together with its degrees of freedom like the spin may be only introduced for linear

\[\text{\textsuperscript{1}}\text{There is nothing to forbid the existence of two spin-1 fields, but one consequence is that particles with the same orientation repel and particles with opposite orientation attract.}\]
theories (for example for the Yang-Mills theories, which are non linear, it is necessary to perform a perturbative expansion around the linearised theory). In these theories, when Poincaré invariant, the particles are classified in terms of the eigenvalues of two Casimir operators of the Poincaré group, $P^2$ and $W^2$ where $P^\mu$ are the translation generators and $W^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma\tau} P^\rho M_{\sigma\tau}$ is the Pauli-Ljubanski polarization vector with $M^{\mu\nu}$ Lorentz generators. Then, the total angular momentum $J = L + S$ is defined in terms of the generators $M_{\mu\nu}$ as $J^i = \frac{1}{2} \epsilon^{ijk} M_{jk}$. The generators $P^\mu$ and $M_{\mu\nu}$ span the Poincaré algebra $ISO(3,1)$

$$\left\{ \begin{array}{c} [M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \\
[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \\
[P_\mu, P_\nu] = 0. \end{array} \right. \quad (23)$$

Let us briefly recall a few details about the representation theory of this algebra. The Pauli-Ljubanski operator is a translational invariant Lorentz vector, that is $[P^\mu, W_\nu] = 0$ and $[M_{\mu\nu}, W_\rho] = i(\eta_{\nu\rho}W_\mu - \eta_{\mu\rho}W_\nu)$. In addition, it satisfies the equation

$$W_\mu P^\mu = 0. \quad (24)$$

The unitary (infinite-dimensional) representations of the Poincaré group fall mainly into three different classes:

- $P^2 = m^2 > 0$, $W^2 = -m^2 s(s + 1)$, where $s = 0, \frac{1}{2}, 1, ...$ denotes the spin.
  From Eq. (24) we deduce that in the rest frame the zero component of the Pauli-Ljubanski vector vanishes and its space components are given by $W_i = \frac{1}{4} \epsilon_{ijk} P^0 S^{jk}$ so that $W^2 = -m^2 s^2$. This representation is labelled by the mass $m$ and the spin $s$.

- $P^2 = 0, W^2 = 0$. In this case $W$ and $P$ are linearly dependent
  $$W_\mu = \lambda P_\mu;$$
  the constant of proportionality is called helicity and it is equal to $\pm s$.
  The time component of $W$ is $W^0 = \vec{P} \cdot \vec{J}$, so that
  $$\lambda = \frac{\vec{P} \cdot \vec{J}}{P_0}$$
  which is the definition of helicity for massless particles like photons.

- $P^2 = 0, W^2 = -\rho^2$, where $\rho$ is a continuous parameter. This type of representation, which describes particles with zero rest mass and an infinite number of polarization states, labeled by $\rho$ does not seem to be realised in nature.

Let us turn now to the gravitational fields represented by Eq. (15). As it has been shown, they represent gravitational waves moving at the velocity of light,
that is, in the would be quantised theory, particles with zero rest mass. Thus, if a classification in terms of Poincaré group invariants could be performed, these waves would belong to the class of unitary (infinite-dimensional) representations of the Poincaré group characterized by $P^2 = 0$, $W^2 = 0$. But, in order for such a classification to be meaningful, $P^2$ and $W^2$ have to be invariants of the theory. This is not the case for general relativity, unless we restrict to a subset of transformations selected for example by some physical criterion or by experimental constraints. For the solutions of the linearised vacuum Einstein equations the choice of the harmonic gauge does the job \[25\]. There, the residual gauge freedom corresponds to the sole Lorentz transformations. For these reasons, only gravitational fields represented by Eq. (15) will be considered, which, besides being exact solutions, solve the linearised vacuum Einstein equations as well. There exist several equivalent procedures to evaluate their polarization. For instance, one can look at the $\tau_{00}^R$ component of the Landau-Lifshitz pseudo-tensor and see how the metric components that appear in $\tau_{00}^R$ transform under an infinitesimal rotation $R$ in the plane $(x, y)$ transverse to the propagation direction. The physical components of the metric are $h_{tx}$ and $h_{ty}$ and under the infinitesimal rotation $R$ in the plane $(x, y)$ transform as a vector. Applied to any vector $(v_1, v_2)$ the infinitesimal rotation $R$ has the effect

$$Rv_1 = v_2, \quad Rv_2 = -v_1,$$

from which

$$R^2 v_i = -v_i \quad i = 1, 2,$$

so that $iR$ has the eigenvalues $\pm 1$. Thus, the components of $h_{\mu\nu}$ that contribute to the energy correspond to spin-1 fields, provided that only Lorentz transformations are allowed. Spin-0 and spin-1 gravitons have been considered, in a different context, in \[12, 18\]. As it is usual, the observable effects of the gravitational wave \(15\) follow from the study of relative motion of test particles described by the \textit{geodesic deviation equation}. Since the only nonvanishing components of the Riemann tensor field $R^\lambda_{\mu\alpha\nu}$ are determined by Eq. (6), for small velocity $(ds \sim dt)$ and in the usual weak field approximation, the geodesic deviation equation reduces to

$$\frac{d^2 V^\lambda}{dt^2} + \frac{V^\alpha}{4(z-t)^2} \frac{\partial^2 w}{\partial x^\alpha^2} = 0,$$

the vector field $V$ representing the \textit{space-like} separation of the close geodesics. Thus, it is clear that the deviation depends explicitly on the choice of the harmonic function $w$ and then on the polarization. A complete analysis will be performed in a forthcoming paper by introducing a source generating the gravitational wave described by the Eq. (15).

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\*\*With respect to the Minkowskian background metric, this plane is orthogonal to the propagation direction. With respect to the full metric this plane is transversal to the propagation direction and orthogonal only in the limit $|z-t| \to \infty$.

\*\*\*It has been said before, that this transformation preserves the harmonicity condition.
6 Conclusions

In this paper, starting from the physical properties of already known exact solutions of the Einstein equations, the origin of spin-1 gravitational waves is discussed. The first step has been the analysis of the wave character of these solutions. By using either the Zel’manov or the Pirani criterion, it has been shown that these solutions represent gravitational waves. The second step has been the study of the energy content and of the propagation direction of the waves. The energy content of the waves, and, in particular, the physical degrees of freedom, i.e. the metric components that contribute to the total energy flux, have been analysed by using either Landau-Lifshitz energy-momentum pseudo tensor or either the Bel superenergy tensor, both tensors giving the same results. The propagation direction has been determined by using both the Landau-Lifshitz energy-momentum pseudo tensor and the Pirani criterion, and in this case also the two methods agreed. Eventually, the conclusion is that gravitational fields (11) represent spin-1 gravitational waves and that the reason why it is commonly believed that spin one gravitational waves do not exist is that, in dealing with the linearised Einstein theory, all authors implicitly assume a \textit{square integrable} perturbation. In other words, \textit{square integrable} spin-1 gravitational waves are always \textit{pure gauge}. However, it has been proven that there exist interesting non \textit{square integrable} wave-like solutions of the linearised Einstein equations that have spin-1. These solutions are very interesting for two reasons. Firstly, they are asymptotically flat in the plane orthogonal to the propagation direction. Secondly, they are solutions of the exact equations too, so that the spin-1 cannot be considered as an “artifact” of the linearised theory.

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REFERENCES


A On the harmonic coordinates when $\sigma = 1$.

In order for $\tilde{x}^\mu$ to be harmonic they have to solve

$$g^{\mu\nu} \partial_\nu J^\lambda_\mu = \Gamma^\mu J^\lambda_\mu,$$

where $\Gamma^\mu = \Gamma^\mu_{\sigma\nu}g^{\sigma\nu}$ and $J = ||\partial_\nu \tilde{x}||$ is the Jacobian of the transformation to the harmonic coordinates. After some calculation we find, in the adapted coordinates:

$$\Gamma^a = -\frac{\partial_a (\ln \mu)}{2f}, \quad a = 1, 2, \quad \Gamma^3 = 0, \quad \Gamma^4 = -\frac{2}{\mu}.$$

The above equations admit a solution of the following type

$$\begin{cases}
\tilde{x}^1 = x(y^1, y^2), \\
\tilde{x}^2 = y(y^1, y^2), \\
\tilde{x}^3 = (z + t)(y^1, y^2, y^3, y^4), \\
\tilde{x}^4 = (z - t)(y^3, y^4).
\end{cases}$$

Indeed, the equations for the harmonic coordinates become:

$$\begin{cases}
\Delta_\mu x = 0, \\
\Delta_\mu y = 0, \\
\frac{1}{2f} \Delta_\mu (z + t) - [w(y^1, y^2) - 2y^4](z + t)_{,44} + 2((z + t)_{,34} - (z + t)_{,4}) = 0,
\end{cases}$$

where $\Delta_\mu = \Delta + (\partial_1 \ln|\mu|)\partial_1 + (\partial_2 \ln|\mu|)\partial_2$ is the $\mu$-deformed Laplace operator.

This system of equations has many solutions. A solution in which an explicit time coordinate can be identified is the following

$$\begin{cases}
z - t = \exp(y^3) \\
z + t = -(w(y^1, y^2) - 2y^4) \exp(-y^3),
\end{cases}$$

where $x$ and $y$ are two independent solutions of the $\mu$-deformed Laplace equation.