Fermionic Subspaces of the Bosonic String

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Abstract. A universal symmetric truncation of the bosonic string Hilbert space yields all known closed fermionic string theories in ten dimensions, their D-branes and their open descendants. We highlight the crucial role played by group theory and two-dimensional conformal field theory in the construction and emphasize the predictive power of the truncation. Such circumstantial evidence points towards the existence of a mechanism which generates space-time fermions out of bosons dynamically within the framework of bosonic string theory. ††

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1. Introduction and conclusion

It is well-known that ten-dimensional fermionic strings can be analyzed in terms of bosonic operators, a consequence of the boson-fermion equivalence in two dimensions. The approach taken here is different. We show that the Hilbert space of the bosonic string compactified on suitable sixteen dimensional tori contain subspaces with fermionic degrees of freedom. This programme was initiated in 1986 in the framework of closed strings [1]. We revisited the approach in the last two years [2, 3] and extended the construction to the open string sectors. The recent developments in the Conformal Field Theory description of open strings [4] are instrumental to our results.

We determine the fermionic subspaces by performing a truncation of the bosonic Hilbert space. To ensure consistency of the truncation for open string sectors (and hence for D-branes) we impose a symmetric truncation in both closed string sectors. This consistency condition lifts all ambiguities about the fermionic subspaces found by truncation and, most importantly, allows the approach to be predictive. We can classify all the fermionic subspaces and exhibit how this classification is related to global properties of the group $SO(16)$. All fermionic strings live in subspaces of the bosonic string compactified on sublattices of the $E_8 \times SO(16)$ weight lattice and the sublattice $E_8 \times E_8$ of $E_8 \times SO(16)$ contains the supersymmetric theories IIA and IIB as well as the heterotic superstrings. All non-heterotic strings and all their (stable and unstable) D-branes are classified by the discrete subgroups of the centre of $SO(16)$. Significantly, the characteristic properties of fermionic D-branes (tension, charge conjugation, chirality changing T-dualities) are predicted from purely bosonic considerations. Furthermore, the Chan-Paton groups of tadpole-free open fermionic strings are also correctly obtained via truncation, and in particular, the anomalies in type I do cancel.

Truncation provides a dictionary translating all fermionic string properties to bosonic string ones. If a non-perturbative mechanism exists which isolates the fermionic subspaces, the scope of the M-theory quest would be considerably enlarged: there would be no elementary fermions at a fundamental level and supersymmetry would have a dynamical origin.

2. Symmetric truncation

The truncation of the bosonic string Hilbert space which yields all its ten-dimensional fermionic subspaces is highly constrained. A first constraint originates in the closed string sector, where coherence of the theory imposes that modular invariance be preserved by truncation, while a second constraint emerges from the open string sector, where one must require the truncation to be consistent with boundary conditions relating the left and right moving closed strings, i.e. with the introduction of D-branes. The resulting truncation must be symmetric and we now review how it works.

We perform a toroidal compactification of the 26-dimensional closed bosonic string theory at an enhanced symmetry point with gauge group $G_L \times G_R$, with $G_L, G_R$ two
semi-simple, simply laced Lie groups. With both groups of rank \( d = 24 - s \), \( 0 \leq s \leq 24 \), the compactified bosonic theory lives in \( s + 2 \) dimensions, and the original transverse Lorentz group \( SO(24)_{tr} \) becomes the Lorentz group \( SO(s)_{tr} \), which does not possess the spinorial representations needed to accommodate space-time fermions, and which cannot therefore play the role of the transverse Lorentz group of a fermionic theory in \( s + 2 \) dimensions. In order to manufacture an appropriate Lorentz group, one uses a stringy analog of the field theoretical construction which turns isospin into spin in four dimensional gauge theory \([5]\). Namely, one requires that \( \mathcal{G}_L \) and \( \mathcal{G}_R \) (in the heterotic case only \( \mathcal{G}_R \)) admit an \( SO(s)_{int} \) subgroup, and one takes as new transverse Lorentz group the diagonal \( SO(s)_{diag} \) group with algebra

\[
so(s)_{diag} = \text{diag} \left[ so(s)_{tr} \times so(s)_{int} \right]
\]

(2.1)
generated by \( J^{ij} = L^{ij} + K^{ij} \), \( i < j, i, j = 1, \ldots, s \). The algebraic set-up Eq.\((2.1)\) is a first step in creating spin from isospin. The second step is to ensure the closure of the full Lorentz algebra in \( s + 2 \) dimensions. This can be done only if all states corresponding to 12 compact dimensions are removed, except for some zero-modes, and the maximum value of \( s \) accommodating fermions turns out to be 8. Although these facts follow from the highly non-trivial closure condition of the Lorentz algebra, they can be understood in simpler qualitative terms. Indeed, the existence of space-time fermions in a covariant formalism is rooted in the existence of worldsheet supersymmetry. The central charge of the superghost is 11 and the timelike and longitudinal fermions contribute 1 to the central charge. The hidden superconformal invariance in the light-cone gauge thus requires the removal of 12 bosonic fields. The zero-modes kept in the 12 dimensions account for the oscillator zero-point energy which is equal to \((-1/24) \times 12\) (in units \( \alpha' = 1/2 \)) and which is to be removed. Therefore zero-modes kept in 12 dimensions must contribute an energy 1/2. In this way, space-time fermions can be obtained provided a truncation of the Hilbert space is performed. At this stage truncation is done by hand, but the group theory classification of fermionic D-branes and the host of correct predictions resulting from truncation strongly suggest the existence of an underlying dynamics.

We now specify the truncation and restrict hereafter to \( s = 8 \), i.e. to 10-dimensional fermionic strings. The toroidal compactification should therefore be performed on the lattice of a Lie group of rank \( d = 24 - 8 = 16 \) with subgroup \( SO(8)_{int} \). The compactification lattice in both sectors (or in the right sector only for the heterotic strings) is taken to be a sublattice of the \( E_8 \times SO(16) \) weight lattice which preserves the modular invariance of the partition function, whose \( \mathcal{G}_L \times \mathcal{G}_R \) lattice contribution \( P(\tau, \bar{\tau}) \) is separately modular invariant and given by,

\[
P(\tau, \bar{\tau}) = \sum_{\alpha, \beta} N_{\alpha \beta} \tilde{\alpha}_L(\bar{\tau}) \beta_R(\tau),
\]

(2.2)

where

\[
\beta_R(\tau) = \sum_{\sqrt{2\alpha'p_{\beta R}} \in (o)} \exp \left\{ 2\pi i \tau \left[ \frac{(p_{\beta R} + p_{\beta' R})^2}{2} + N_R^{(c)} - \frac{\delta}{24} \right] \right\}.
\]

(2.3)
Here $\beta$ is a partition function for a sublattice ($\beta$) of the $G_R = E_8 \times SO(16)$ weight lattice (i.e. $(\beta) = (o)_{E_8} \oplus (i)_{16}$, $i = o, v, s, c$) and $p_{\beta R}$ is a fixed vector, arbitrarily chosen, of the sublattice ($\beta$). $N_{R}^{(c)}$ is the oscillator number in the $\delta = 16$ compact dimensions. A similar expression holds for $\bar{\alpha}_{L}(\bar{\tau})$, $\bar{\alpha}$ labeling a partition function for a sublattice of the weight lattice of $G_L$. The coefficients $N_{\alpha \beta}$ are 0 or 1 and are chosen in such a way that $P(\tau, \bar{\tau})$ is modular invariant.

In order to proceed with the truncation (exemplified here in the right sector of the theory), we decompose the $SO(16)$ factor of $G_R$ in $SO'(8) \times SO(8)$ and first truncate all states created by oscillators in the 12 dimensions defined by the $E_8 \times SO'(8)$ root lattice. The group $SO(8)$ is identified with the internal symmetry group $SO(8)_{int}$. As discussed above, the closure of the new Lorentz algebra dictates we keep zero-modes in the 16 compact dimensions in such a way that

$$\frac{1}{2} p_{R}^{2}[E_8 \times SO(16)] = \frac{1}{2} p_{R}^{2}[SO(8)] + \frac{1}{2}, \quad (2.4)$$

with $p_{R}(G)$ a vector of the weight lattice of the group $G$. The zero-mode contribution 1/2 in Eq.(2.4) comes from $SO'(8)$ as there are no vectors of norm squared one in $E_8$. The only zero-mode contributions from $E_8 \times SO(8)'$ we keep are two fixed $SO(8)'$ 4-vectors $p'_o$ and $p'_s$, so that we truncate the lattice partition functions according to,

$$o_{16} \rightarrow v_8, \quad v_{16} \rightarrow o_8,$$

$$s_{16} \rightarrow -s_8, \quad c_{16} \rightarrow -c_8. \quad (2.5)$$

It follows from the closure of the Lorentz algebra that states belonging to $v_8$ or $o_8$ are bosons while those belonging to the spinor partition functions $s_8$ and $c_8$ are space-time fermions. In accordance with the spin-statistic theorem we have flipped the sign in the partition function of the space-time spinor partition functions.

All heterotic strings were obtained, using Eq.(2.5), in reference [6]. To obtain all fermionic D-branes in the non-heterotic theories, we must truncate both sectors of the modular invariant partition functions Eq.(2.2) according to Eq.(2.5)† [3]. As the $E_8$ lattice is Euclidean even self-dual, we concentrate on the $SO(16)$ weight lattice. Their are four even self-dual Lorentzian $SO(16)$ lattices. The corresponding modular invariant partition functions are (modulo the contribution from the $E_8$ lattice and from the non-compact dimensions),

$$OB_b = \bar{o}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} s_{16} + \bar{c}_{16} c_{16}, \quad (2.6)$$

$$OA_b = \bar{o}_{16} o_{16} + \bar{v}_{16} v_{16} + \bar{s}_{16} c_{16} + \bar{c}_{16} s_{16}, \quad (2.7)$$

$$IIb_b = \bar{o}_{16} o_{16} + \bar{s}_{16} o_{16} + \bar{o}_{16} s_{16} + \bar{s}_{16} s_{16}, \quad (2.8)$$

$$IIA_b = \bar{o}_{16} o_{16} + \bar{c}_{16} o_{16} + \bar{o}_{16} c_{16} + \bar{s}_{16} s_{16}. \quad (2.9)$$

They yield, after symmetric truncation, the four consistent non-heterotic ten-dimensional fermionic string partition functions, namely,

† Hence the terminology ‘symmetric truncation’.
Table 1. Bosonic D9-brane amplitudes.

<table>
<thead>
<tr>
<th>Theory</th>
<th>( A_{\text{tree}} )</th>
<th>( A_{\text{loop}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( OB_b )</td>
<td>( (1/2)(o_{16} + v_{16} + s_{16} + c_{16}) )</td>
<td>( o_{16} )</td>
</tr>
<tr>
<td>( OA_b )</td>
<td>( o_{16} + v_{16} )</td>
<td>( o_{16} + v_{16} )</td>
</tr>
<tr>
<td>( IIB_b )</td>
<td>( o_{16} + s_{16} )</td>
<td>( o_{16} + s_{16} )</td>
</tr>
<tr>
<td>( IIA_b )</td>
<td>( 2o_{16} )</td>
<td>( o_{16} + v_{16} + s_{16} + c_{16} )</td>
</tr>
</tbody>
</table>

\[
OB_b \rightarrow \bar{v}_8 v_8 + \bar{o}_8 o_8 + \bar{s}_8 s_8 + \bar{c}_8 c_8 \equiv OB, \tag{2.10}
\]
\[
OA_b \rightarrow \bar{v}_8 v_8 + \bar{o}_8 o_8 + \bar{s}_8 c_8 + \bar{c}_8 s_8 \equiv OA, \tag{2.11}
\]
\[
IIB_b \rightarrow \bar{v}_8 v_8 - \bar{s}_8 v_8 - \bar{v}_8 s_8 + \bar{s}_8 s_8 \equiv IIB, \tag{2.12}
\]
\[
IIA_b \rightarrow \bar{v}_8 v_8 - \bar{c}_8 v_8 - \bar{v}_8 s_8 + \bar{c}_8 s_8 \equiv IIA. \tag{2.13}
\]

3. Fermionic D-branes and torus geometry

The properties of the bosonic D9-branes pertaining to the four different theories compactified on \( E_8 \times SO(16) \) lattices can be related to the geometry of the configuration space torus characterizing each compactification. These tori are linked to each other through global properties of the universal covering group \( \tilde{SO}(16) \) as we shall now show.

The amplitudes \( A_{\text{tree}} \) describing the D9-branes in the tree channel are obtained from the torus partition functions Eqs. (2.6)-(2.9) by imposing Dirichlet boundary conditions on the compact space. In the tree channel, the latter consists in the following relation between compactified momenta,

\[
p_L - p_R = 0,
\]

as well as in a match between left and right oscillators. The amplitudes of elementary bosonic D9-branes are given in Table 1, both in the tree channel and in its S-dual loop channel.

In order to identify the configuration space torus on which each theory is defined, recall that in the conformal \( \sigma \)-model description of these theories in presence of torsion \( b_{ab} \), the left and right momenta are given by

\[
p_R = \frac{1}{2} m_b + n^a (b_{ab} + g_{ab}) e^b,
\]
\[
p_L = \frac{1}{2} m_b + n^a (b_{ab} - g_{ab}) e^b,
\]

where \( \{e^a\} \) is the lattice-dual basis of the basis \( \{e_a\} \) defining the configuration space torus

\[
x \equiv x + 2\pi n^a e_a \quad n^a \in \mathbb{Z}, \tag{3.3}
\]

and the lattice metric is given by \( g_{ab} = e_a.e_b \). The weight vectors \( 2e_a \) generate four sublattices of the weight lattice of \( SO(16) \). They can be read off from the second column in Table 1, as \( A_{\text{loop}} \) yields the winding lattice \( n^a e_a \). The classification of bosonic