Spin 1/2 Particle on a Cylinder with Radial Magnetic Field

Chryssomalis Chryssomalakos,
Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
Apdo. Postal 70-543, 04510 México, D.F., MEXICO
chryss@nuclecu.unam.mx

Alfredo Franco, and Alejandro Reyes-Coronado
Instituto de Física
Universidad Nacional Autónoma de México
Apdo. Postal 20-364, 01000 México, D.F., MEXICO
alfredof@fisica.unam.mx, coronado@fisica.unam.mx

ABSTRACT: We study the motion of a quantum charged particle, constrained on the surface of a cylinder, in the presence of a radial magnetic field. When the spin of the particle is neglected, the system essentially reduces to an infinite family of simple harmonic oscillators, equally spaced along the axis of the cylinder. Interestingly enough, it can be used as a quantum Fourier transformer, with convenient visual output. When the spin 1/2 of the particle is taken into account, a non-conventional perturbative analysis results in a recursive closed form for the corrections to the energy and the wavefunction, for all eigenstates, to all orders in the magnetic moment of the particle. A simple two-state system is also presented, the time evolution of which involves an approximate precession of the spin perpendicularly to the magnetic field. A number of plots highlight the findings while several three-dimensional animations have been made available on the web.
Contents

1 Introduction 2

2 The Spinless Case 3
  2.1 The spectrum .......................................................... 3

A First Order Corrections to the Ground State: the Standard Treatment 3

1 Introduction

The quantum mechanical description of the motion of charged particles in a magnetic field is a classic application of the theory, having given rise to some of its most striking results. Among them, the seminal analysis by Dirac [0], of the motion in the field of a magnetic monopole, continues to inspire decades after its inception, and motivates the study of similar quantum systems that share the characteristic of providing insights into the fundamentals without too much distraction by analytical complexity. Such systems are invaluable pedagogically, as they furnish a manageable, yet captivating testing ground of the fundamentals of the theory.

The problem of the motion of a non-relativistic quantum particle in a plane, in the presence of a perpendicular homogeneous magnetic field is presented in several textbooks (see, e.g., [0]) — nevertheless, it seems to be the only standard example of this type available. The main purpose of this paper is to draw attention to the fact that the analogous problem for the cylinder is also manageable, even when augmented to include a spin 1/2. In this latter case, we also show how the use of the creation and annihilation operator machinery greatly simplifies the perturbative analysis of the problem, in comparison to the standard textbook procedure.

Despite the simplicity of the problem and it being an obvious variation on the monopole theme, we have not been able to find a treatment in the literature. The motion of a spin-1/2 particle in the field of a magnetic monopole has been studied in detail, both in the non-relativistic [0, 0, 0, 0] and relativistic [0] cases. Symmetry aspects of the problem have also been considered extensively (see, e.g., [0]), with the discovery of an underlying supersymmetry among the most notable results [0, 0, 0]. On the other hand, quantum spinless particles moving on curves or surfaces have been extensively studied (see, e.g., [0, 0, 0] and references therein) with a general discussion of the effects of a vector potential given in [0]. It is our hope that the use of the above simple system will enhance the exposition of this fascinating part of the theory. It should also be of interest in practical applications, such as constrained quantum mechanics and carbon nanotube physics.

Consider a classical charged particle, constrained to move on the surface of an infinite cylinder, in the presence of a radial magnetic field,

We study, in this paper, the quantum mechanical version of the above problem, adding, at a later stage, a spin-1/2 to the particle. The treatment of the spinless case, contained in Sect. 2, is exact — the problem separates and reduces to an infinite collection of harmonic oscillators along $z$. We find, nevertheless, the resulting quantum system particularly rich and with surprising properties — it functions, for example, as a quantum Fourier transformer with convenient visual output (see Sec. ??). The addition of spin is treated perturbatively in Sect. ??, with a non-conventional method that greatly simplifies the calculations. We are able to give recursion relations for the corrections to the wavefunctions and the energy to all orders, for all unperturbed eigenstates, and apply the results to compute second-order corrections to the ground state. Several plots highlight the findings. We also make available on the web several three-dimensional color animations of the time evolution of the wavefunction, with or without spin, and corresponding to various initial conditions. An appendix shows how the standard perturbation theory treatment of the problem reproduces, albeit laboriously, our first order results.
2 The Spinless Case

2.1 The spectrum

The magnetic field of Eq. (??) can be obtained, in the vicinity of the surface of the cylinder, from the vector potential

The requirement that the perturbed eigenket be normalized implies that the corrections, order by order in $\epsilon$, have to be orthogonal to the unperturbed eigenket $|n\rangle_0$. This in turn implies that, for $k > 0$, the coefficient of $x^n$ in $f_n^{(k)}(x)$ must vanish. Then so does the coefficient of $x^n$ in $x\partial_x f_n^{(k)}(x)$. Using this information, we can extract the coefficient of $x^n$ on both sides of (??) — the resulting equation fixes recursively the energy corrections $E_n^{(k)}$.

We look in some detail now at the wavefunctions and resulting spin configurations, including up to quadratic corrections. For the first three $f_n^{(k)}$, Eqs. (??), (??) give

$$f_0^{(0)}(x) = 1 \quad (1)$$
$$f_0^{(1)}(x) = e^{-b^2/4} \int_0^1 \frac{ds_1}{s_1} (1 - e^{-b \sqrt{2} s_1}) \quad (2)$$
$$f_0^{(2)}(x) = e^{-b^2/2} \int_0^1 \int_0^1 \frac{ds_2 ds_1}{s_2 s_1} \left\{ 2 - e^{b^2 s_1/2} - e^{-b \sqrt{2} s_2} + e^{b^2 s_1} e^{-b \sqrt{2} s_2 (1-s_1) x} - e^{-b \sqrt{2} s_2 s_1 x} \right\}, \quad (3)$$

while for the corresponding energy corrections we get

We end with a comment on the form of the unperturbed hamiltonian used, Eq. (??). When dealing with the motion of a quantum particle on a surface, one can use a 3-D Laplacian in the hamiltonian and constrain the motion of the particle on the surface using a steep confining potential in the radial direction. It is well known that, in this approach, which seems to be the one appropriate for practical applications, there is an induced potential for the motion along the surface, proportional to the square of the difference between the two principal curvatures of the surface (see, e.g., [0, 0] and references therein). In our case, this is a constant which only shifts the energy eigenvalues. Nevertheless, an obvious extension of our problem here would be the study of the motion on the surface of a slightly curved cylinder, in which case the above mentioned induced potential would have to be taken into account.

A First Order Corrections to the Ground State: the Standard Treatment

We outline here the standard first order perturbative analysis of the problem, deriving the corrections to the ground state. Given that the zeroth order spectrum is degenerate, we need to first diagonalize the interaction hamiltonian in each degenerate subspace. One easily sees that the interaction only connects the pairs of eigenstates $|n, \ell, +\rangle$, $|n, \ell + 1, -\rangle$. The appropriate zeroth order basis is given by the symmetric and antisymmetric linear combinations

$$|n^{(a)}_\ell \rangle \equiv \frac{1}{\sqrt{2}} (|n, \ell, +\rangle + |n, \ell + 1, -\rangle)$$
$$|n^{(a)}_\ell \rangle \equiv \frac{1}{\sqrt{2}} (|n, \ell, +\rangle - |n, \ell + 1, -\rangle). \quad (4)$$

Notice that we use the label $\ell$ for states that are equally localized at $-\ell b$ and $-(\ell+1)b$. The reason for renaming the states with numerical superscripts, instead of letters, will become apparent below. The expectation value
of $H_{\text{int}}$ in these states reproduces our result (§7) for the first order correction to the energy. It is interesting to see how the first order correction to the wavefunction, conventionally given by an infinite sum, is brought into the closed form (§10). The matrix elements of $H_{\text{int}}$ in the above basis are

$$\langle n_0^\ell | H_{\text{int}} | m_0^\ell \rangle = \epsilon \langle n_0^\ell | m_{\ell+1} \rangle \delta_{\ell \ell'} \delta_{\tilde{n} \tilde{n}}$$
$$\langle n_1^\ell | H_{\text{int}} | m_1^\ell \rangle = -\epsilon \langle n_1^\ell | m_{\ell+1} \rangle \delta_{\ell \ell'} \delta_{\tilde{n}, \tilde{n}+1} \ ,$$

where $\langle n_\ell | m_{\ell+1} \rangle$ is the overlap between SHO eigenstates $|n\rangle$, $|m\rangle$, at a distance $b$ apart and $\tilde{n}$ is the parity of $n$. Specifying to the symmetric ground state and taking $\ell = 0$, we find the first order correction

---