The equations will contain the basic elements of the theory. In the case of the theory, the equations are used to describe the behavior of quantum systems. The equations are based on the principles of quantum mechanics and are used to predict the outcomes of quantum experiments.

**Introduction**

Potential parameters and constant parameters with multiple dimensions. The equations describe the behavior of the system and how it interacts with its environment. The equations are used to predict the outcomes of experiments and to test the validity of the theory. The equations are based on the principles of quantum mechanics and are used to describe the behavior of quantum systems. The equations are used to predict the outcomes of experiments and to test the validity of the theory. The equations are based on the principles of quantum mechanics and are used to describe the behavior of quantum systems.
since for the other two levels the same description applies (they are independent).

In order to understand the mechanism that we propose, it is convenient first to revise the method used in previous experiments [13, 14, 15] to keep an atom in a cavity. The interaction between the atom and cavity mode is characterized by a coupling constant $g(x)$, which depends on the atomic position $x$. In Fig. 2 we show the set-up, as well as the instantaneous energy levels of an atom as a function of its position (in one dimension). The ground state of the composed atom–cavity system is $|g, 0\rangle$, where $|n\rangle$ is the cavity state with $n$ photons (in this case $n = 0$). The corresponding energy, $E_n \propto |n\rangle$, is position independent. The first two excited levels are the dressed states of the Jaynes–Cummings Hamiltonian [24], $|\pm\rangle = \frac{1}{\sqrt{2}}(|g, 1\rangle \pm |g, 0\rangle)$, with corresponding energies $E_{\pm}(x) = \pm g(x)$ in the interaction picture, where we have taken $h = 1$. As Fig. 2 shows, the position-dependence of $E_{\pm}(x)$ provides the atom with a confining potential at the center of the cavity. Thus, if the atom can be prepared in the state $|\pm\rangle$ with a kinetic energy smaller than $g_0 = g(0)$, it will remain trapped [11, 12, 13, 14, 15]. As the state $|\pm\rangle$ contains a linear combination involving one photon, one can state that the atom is trapped by a single photon. On the other hand, the state $|\pm\rangle$ can be efficiently prepared by starting in the state $|g, 0\rangle$ and tuning the external laser field to be resonant with the transition $|g, 0\rangle \rightarrow |\pm\rangle$ near $x = 0$, as indicated in Fig. 2 [25].

The above discussion has omitted an important element which is present in all experiments, namely the dissipation mechanism. On the one hand, excited atoms may decay very fast (as long as the state $|e\rangle$ does not correspond to some Zeeman level, which is coupled to the cavity mode by some Raman transition [26]). More importantly, cavities have usually losses, so that the photons will leak the cavity after some time $t \approx 1/\kappa$, where $\kappa$ is the cavity damping rate. Any of these mechanisms will induce the spontaneous transition $|\pm\rangle \rightarrow |g, 0\rangle$, and therefore the atom will no longer experience the trapping force. The typical time scale of these processes is of the order of $\Gamma^{-1}$ and $\kappa^{-1}$, where $\Gamma$ and $\kappa$ are the spontaneous emission and the cavity damping rate, respectively. In practice [13, 14, 15] the atom can be promoted several times to the state $|\pm\rangle$ by the external laser, so that the trapping time inside the cavity can be several hundreds of $\kappa^{-1}$. Note, however, that these spontaneous transitions will break the atomic coherence if we are using more internal levels to store, for example, some quantum information in the atom (see Fig. 1).

Our idea is to detune the external laser slightly below the transition $|g, 0\rangle \rightarrow |\pm\rangle$ at $x = 0$. If the laser intensity is low enough, its only effect will be to produce an AC-Stark shift for the level $|g, 0\rangle$, whose energy $E_0(x)$ will now depend on the position, as shown in Fig. 3. Thus, if the atom is in the level $|g, 0\rangle$, it will experience a trapping force towards $x = 0$, and therefore, it can be trapped (as long as the corresponding potential supports bound states). Note also, that since the atom is basically in the ground state and no photon is present, all the dissipative mechanisms may be drastically reduced.

In the following sections we will compute the performance of our scheme. In the rest of this section we will use very simple estimates to characterize the trapping potential and the corresponding time scales.

Denoting by $\Omega$ the Rabi frequency of the external laser, and by $\Delta$ its corresponding detuning with respect to the
\([g] \rightarrow \ket{e}\) transition \((\Delta < 0)\), we have that the regime of validity of our analysis will be
\[
\Omega \ll |\Delta + g| \ll g_0.
\] (1)

In this case, the depth of the trapping potential \(V_0\) will be approximately equal to the AC-Stark shift of the level \([g, 0]\) due to its coupling to \([-\) at \(x = 0\), i.e.
\[
V_0 \simeq \frac{\Omega^2}{8|\Delta + g|}.
\] (2)

On the other hand, losses will be due to the small contamination of level \([g, 0]\) with level \([-\) given by the off-resonant coupling. The population of this level will be of the order of \(\Omega^2 / 4|\Delta + g|^2\), and therefore the lifetime of the state will be
\[
\tau \simeq \frac{4|\Delta + g|^2}{\Omega^2} \min(\Gamma^{-1}, \kappa^{-1}).
\] (3)

Equations (2) and (3) indicate that the lifetime can be made arbitrarily long at the expense of reducing the potential depth.

In three dimensions, one can easily estimate the condition for a potential to possess a bound state. It is given by \([27]\) \(2m\Omega L^3 / h^2 \gtrsim 1\), where \(L\) is the cavity length, \(m\) is the atomic mass, and we have included \(h\) to make the dimensions more explicit. We can rewrite this as \(V_0(L / \lambda)^2 \gtrsim g_0\), where \(\lambda\) is an optical transition wavelength and \(g_0\) the corresponding energy of one photon recoil. Since \(L \gtrsim \lambda\) in all cases we see that by taking \(V_0 > g_0\) we will always have a bound atomic state.

Note that in a one dimensional set-up there is always a bound state for any value of \(V_0\) \([27]\).

So far, we have shown that it is possible to have atoms trapped in the cavity with basically zero photons and in the atomic ground state. However, the trapping potential may become very weak. Thus, in order to trap atoms it will be required that they move very slowly in the cavity in the state \([g, 0]\) and then, when they are close to \(x = 0\), the external field is turned on. Let us estimate what will be, in this case, the lifetime of the trapped state. We will assume that we have all atoms and the kinetic energy is of the order of one optical recoil \((E_R = h^2 k^2 / 2m\), where \(k\) is the optical wavevector). Thus, we can take \(E_R = 4kHz \ll V_0 = 10kHz\). Let us analyze separately the optical and microwave regimes.

For the optical regime we take the parameters from \([15]\). There the \(5^2 S_{1/2} F = 3 \rightarrow 5^2 P_{3/2} F = 4\) transition of \(^{85}\)Rb with a frequency of \(3.8 \times 10^{12}\text{Hz}\) was used. The maximal coupling between cavity and atom is \(g_0 \approx 16 \times 10^{12}\text{MHz}\), the cavity loss rate is \(\kappa \approx 1.4 \times 2\pi\text{MHz}\) and the spontaneous decay rate is \(\Gamma \approx 3 \times 2\pi\text{MHz}\). We can estimate (3) a decay time of \(2.1 \times 10^{-3}\text{s}\). For the microwave regime we consider circular Rydberg states, so we have \([11, 28]\) \(g_0 \approx 67 \times 2\pi\text{kHz}, \kappa \approx 1.6 \times 2\pi\text{Hz}\) and \(\Gamma \approx 1.6 \times 2\pi\text{Hz}\), where \(\Gamma\) is the spontaneous decay rate of the Rydberg transition. We reach a life time of \(40\text{s}\) \((3)\).

These estimates look very promising. They will be optimized and compared with numerical calculations in the following sections. On the other hand, let us stress that we have calculated here the lifetime for a single loss event, since it will destroy the coherence present in the atomic state. For the reference \([13, 14, 15]\), in the optical experiments in which the atom is trapped in a cavity this time is of the order of \(1\mu s\), i.e. one order of magnitude smaller than the one we obtain. In the following sections we will also analyze the trapping time if several loss events are allowed.

**MODEL**

In this section we will introduce in detail the model that describes the situation we have in mind. In the first subsection we will start with the full Hamiltonian characterizing the atom–cavity interaction and perform some approximations in order to derive the estimates given in the previous section. Then we will introduce the decay mechanisms in this picture.

**Hamiltonian dynamics**

The Hamiltonian describing the dynamics of the atom and the cavity mode can be written as follows
\[
H = \frac{p^2}{2m} + \omega_0 (\sigma^+ a + \frac{1}{2} \sigma_z) + g(x) (\sigma^+ a + a^\dagger \sigma^-) + \frac{\Omega}{2} (\sigma^+ e^{-i \omega_L t} + \sigma^- e^{i \omega_L t}).
\] (4)

Here, \(\omega_L\) and \(\omega_0\) are the laser and atomic transition frequency, respectively, \(\Omega\) is the Rabi frequency and \(g(x)\) the position dependent coupling constant between the cavity mode and the atom. Note that we have not included the position dependence of the laser plane wave to make more explicit the fact that the laser exerts no force on the atom (in any case, since this laser only gives rise to AC-Stark shift, its position dependence will cancel out).

In order to make the analysis simpler, we will project our system in the subspace spanned by the states \([[g, 0], [e, 0], [g, 1]\}). In any case, the reader can easily verify that the population of all other levels will be much smaller than the last two, which will be scarcely populated. The Hamiltonian (4) in this subspace can be rewritten as \(H = \frac{p^2}{2m} + H'(x)\), and this last can be diagonalized exactly in the rotating frame. Instead of doing that, we calculate the eigenstates and eigenvalues of \(H'(x)\) in lowest order perturbation theory with respect to \(\Omega\), which is assumed to be small with respect to \(|\Delta + g(x)|\) for all values of \(x\) (see Fig. 3), where \(\Delta = \omega_L - \omega_0\). We
obtain
\[
\begin{align*}
|\Psi_0\rangle &= |g, 0\rangle + \frac{\Omega}{\Delta + g(x)} (g(x)|g, 1\rangle + \Delta|e, 0\rangle) \\
|\Psi_1\rangle &= \frac{1}{\sqrt{2}} \left(|g, 1\rangle - |e, 0\rangle - \frac{\Omega}{\Delta + g(x)} |g, 0\rangle\right) \\
|\Psi_2\rangle &= \frac{1}{\sqrt{2}} \left(|g, 1\rangle + |e, 0\rangle - \frac{\Omega}{\Delta - g(x)} |g, 0\rangle\right)
\end{align*}
\]
and the corresponding eigenvalues
\[
\begin{align*}
\lambda_3(x) &= \frac{\Delta}{2} + \frac{\Omega^2}{8} \left(\frac{1}{\Delta + g(x)} + \frac{1}{\Delta - g(x)}\right) \\
\lambda_1(x) &= -\frac{\Delta}{2} - g(x) - \frac{\Omega^2}{8(\Delta + g(x))} \\
\lambda_2(x) &= -\frac{\Delta}{2} + g(x) - \frac{\Omega^2}{8(\Delta + g(x))}.
\end{align*}
\]
As we see, the ground state is basically $|g, 0\rangle$ with a vanishing contribution of levels $|g, 1\rangle$ and $|e, 0\rangle$ in the limit $\Omega \ll |\Delta + g(x)|$. However, it acquires a position-dependent shift in its energy. The two terms in the shift come from the AC-Stark shifts due to $[-\lambda]$ and $[+\lambda]$, respectively, which, with the chosen detuning, do not compensate each other. The shift is maximal at $x = 0$. To obtain the potential depth $V_0$ we have to subtract the shift at $g(x) = 0$. This gives
\[
V_0 = -\frac{\Omega^2g_0^2}{4\Delta(\Delta^2 - g_0^2)}.
\]
In the limit (1) we have $\Delta \approx -g_0$. If we plug this into (7) we obtain (2).

Starting out with $\Omega = 0$, if the atom is initially in $|g, 0\rangle$ and has a small velocity near $x = 0$, and we turn the laser on, it will basically remain in the eigenstate $|\Psi_0\rangle$, and therefore will experience the potential $\lambda_3(x)$. Note that for this picture to be valid, we need the kinetic energy of the atom to be smaller than $|\Delta + g_0|$ since in this case we can adiabatically eliminate the levels $|g, 1\rangle$ and $|e, 0\rangle$ and obtain the effective Hamiltonian
\[
H_{ad} = \frac{p^2}{2m} + \lambda_3(x)|g, 0\rangle\langle g, 0|.
\]

Dissipation

We introduce a cavity decay rate $\kappa$ and a spontaneous decay rate $\Gamma$ for the atom. To take both into account we use the master equation that describes the time evolution of this open quantum systems. The state of the system, which is now in general a mixed one, is given by a density matrix $\rho$. For our system we obtain
\[
\dot{\rho} = -i[H, \rho] + (\mathcal{L}_{\text{cav}} + \mathcal{L}_{\text{sp}})\rho.
\]
Here,
\[
\mathcal{L}_{\text{cav}}\rho = \kappa (2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)
\]
describes cavity damping, whereas
\[
\mathcal{L}_{\text{sp}}\rho = \frac{\Gamma}{2} \left(2 \int_{-\infty}^{\infty} N(u) \sigma^- e^{-it\sigma^x} \rho e^{it\sigma^x} \sigma^+ du - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-\right)
\]
describes spontaneous emission. The first term in this expression accounts for the photon recoiled experienced by the atom after photon emission. We have included here a one dimensional version, since in our numerical calculations we have investigated this case.

To simulate a single trajectory we use the Quantum Jump Approach [29, 30, 31, 32]. Therefore one defines an effective non-Hermitian Hamiltonian $H_{\text{eff}}$ which describes the time evolution of the system under the condition that no emission takes place. The master equation can then be written in the form
\[
\dot{\rho} = -i \left[H_{\text{eff}} + \frac{p^2}{2m}, \rho\right] \\
+ 2\kappa a^\dagger a + \Gamma \int_{-\infty}^{\infty} N(u) \sigma^- e^{-it\sigma^x} \rho e^{it\sigma^x} \sigma^+ du.
\]
The decay rates contribute to the effective time evolution as damping terms. Therefore the norm of the state decreases. This means that the probability to find no photon in the time interval $(0, t)$ decreases with time $t$.

Dissipation occurs in our model due to the small contamination of level $|\Psi_0\rangle$ with the states $|g, 1\rangle$ and $|e, 0\rangle$, which in turn decay due to cavity damping and spontaneous emission, respectively. In order to determine the effective decay rate (or jump time) we take the sum over the probabilities for the excited states $|g, 1\rangle$ and $|e, 0\rangle$ in $|\Psi_0\rangle$ weighted with the cavity decay rate $\kappa$ and the spontaneous emission rate $\Gamma$. For the coupling constant we assume that the atom is in the center of the cavity $g(x) = g_0$. We obtain
\[
\Gamma_{\text{eff}} = \kappa \frac{g_0\Omega/2}{\Delta^2 - g_0^2} + \Gamma \left|\Delta\Omega/2\right|^2 \\
= \frac{\Omega^2(g_0^2 + \Gamma \Delta^2)}{4(\Delta^2 - g_0^2)^2}.
\]
This gives an effective decay time of
\[
\tau_{\text{eff}} = \frac{1}{\Gamma_{\text{eff}}} = \frac{4(\Delta^2 - g_0^2)^2}{\Omega^2(g_0^2 + \Gamma \Delta^2)}.\]
For the estimation of the life time in Eq. (3) we neglected the contribution of the upper dressed level $|+\rangle$. If we consider the is and the approximation $\Delta \approx -g_0$ and plug it with $\kappa = \Gamma = \max(\kappa, \Gamma)$ into (14) we end up with the expression (3).
FIG. 4: Effective decay time $\tau_{\text{eff}}$ (A) and potential depth $V_0$ (B) versus laser detuning $\Delta = \omega_c - \omega_0$. For the Rabi frequency of the laser we took $\Omega = 0.70 \times 2\pi$ MHz. The coupling strength between cavity and atom is $g_0 = 16 \times 2\pi$ MHz, the cavity loss rate $\kappa = 1.4 \times 2\pi$ MHz and the spontaneous decay rate $\Gamma = 3 \times 2\pi$ MHz.

Discussion

In Fig. 4 and Fig. 5 we have plotted the potential depths $V_0$ and the effective life time $\tau_{\text{eff}}$ versus $\Delta$ and $\Omega$. From Fig. 4(A) we see that in order to get a long decay time it would be desirable to have $|\Delta| \gg |g_0|$. In Fig. 4(B) the region $-g_0 < \Delta < 0$ is not of interest since there one obtains no attractive potential ($V_0 < 0$). One has to find a compromise between $\Delta$ close to $-g_0$ in order to get a deep potential and $|\Delta| \gg |g_0|$ in order to obtain a long decay time. This behavior is not surprising because if the detuning is close to $-g_0$ the population in $|-\sum\rangle$ increases. This leads to a short decay time and a deep potential. The same reasoning explains the plots in Fig. 5 since the Rabi frequency $\Omega$ is a measure for the coupling strength between the atomic transition and the laser.

For the parameters from [15] and a potential depth of $V_0 = 10$ kHz the longest effective life time we can achieve in the optical regime is $\tau_{\text{eff}} = 0.18$ ms. The corresponding values for the laser parameters are

$$\Omega = 0.70 \times 2\pi \text{ MHz},$$
$$|\Delta| = 1.909 g_0^2 = 30 \times 2\pi \text{ MHz}.$$  \hspace{1cm} (15)

In the microwave regime [11, 28] we get $\tau_{\text{eff}} = 1.26 \text{ sec}$ for $g_0 = 2.06 \text{ MHz}$. The other parameters are the same as in Fig. 4.

\begin{equation}
\Omega = 54 \times 2\pi \text{ kHz},
\end{equation}

\begin{equation}
|\Delta| = 2.06 \times 2\pi \text{ MHz}.
\end{equation}

It is important to mention that since we used the expressions from (7) and (14) we are not in the limit $\Delta \approx -g_0$ (1). This leads to a significantly longer life time in the optical regime.

NUMERICAL RESULTS

Here we investigate the behavior of the system numerically. For the analytic results we made certain approximations. The comparison with the numerical results will show if these assumptions are justified for realistic parameters. Furthermore we will include spontaneous emission and photon recoil.

We denote the state of the system by $|\Phi\rangle$. For the simulation we write it as $|\Phi\rangle = |\phi_{g0}\rangle + |\phi_{g1}\rangle + |\phi_{e0}\rangle$, where $|\phi_i\rangle = |i\rangle|\Phi\rangle$. We consider only the contributions of $|g, 0\rangle, |g, 1\rangle$ and $|e, 0\rangle$ since the population of the levels with two and more excitations is negligible. As for the analytic estimations we restrict the investigations to one dimension. The probability amplitudes for the system being in the states $|g, 0\rangle, |g, 1\rangle$ and $|e, 0\rangle$ at position $x$ are given by $\phi_{g0}(x) = \langle x|\phi_{g0}\rangle$, $\phi_{g1}(x) = \langle x|\phi_{g1}\rangle$ and $\phi_{e0}(x) = \langle x|\phi_{e0}\rangle$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{Effective decay time $\tau_{\text{eff}}$ (A) and potential depth $V_0$ (B) versus Rabi frequency $\Omega$ of the laser. For the laser detuning we took $|\Delta| = 1.909 g_0 = 30 \times 2\pi$ MHz. The other parameters are the same as in Fig. 4.} \label{fig:fig5}
\end{figure}
In the first subsection we calculate the ground state of the system with and without the assumptions made above. In the following we include dissipation and compare the decay time with the effective decay time \( \tau_{\text{eff}} \) we estimated in the last section. Finally we consider spontaneous emissions and recoil and investigate how long the atom remains in the cavity for different parameters. Apart from one simulation with the parameters from the analytic estimation we will only consider the optical regime in this section.

The ground state

To obtain the ground state we apply the \textit{imaginary time evolution} \cite{33} to an arbitrary initial state until it remains unchanged. Instead of the time evolution operator \( e^{-iH\Delta \tau} \) one uses a modified operator \( e^{-iH\Delta \tau} \). After a sufficient number of iterations this damps away all states orthogonal to the one with the lowest eigenvalue, which is the ground state of the system.

The Hamiltonian of the system is given in Eq. (4). We denote its ground state by \( |\Phi_0\rangle \). For \( \Omega \) and \( \Delta \) we took the values from (15). According to the analytic approximation they should give the maximal decay time which is achievable for a potential depth of 10 kHz. The numerical simulation of the ground state leads to the probability distribution shown in Fig. 6. The three plots show the population distribution of the three internal states separately. The excited states are only very weakly populated. The probability to find an atom in the center of the cavity with the system being in state \( |g, 1\rangle \) or \( |e, 0\rangle \) is three to four orders of magnitude smaller than to find it there with the state \( |g, 0\rangle \). We also found that the atom is well localized in the center of the cavity. At 0.1 \( \sigma \), where \( \sigma \) is the width of the cavity, the probability to find the atom is already reduced by more than 1/2.

In order to validate the approximations made for the analytic estimation we calculated the ground state also using the Hamiltonian from Eq. (8). We denote its ground state solution by \( |\kappa_0\rangle \). We find

\[
|\Phi_0\rangle \approx |g, 0\rangle \otimes |\kappa_0\rangle.
\]

This means that nearly all the population is in \( |g, 0\rangle \). So the approximations in the analytical approach are justified and one can trap an atom in a basically empty cavity.

Dissipation and photon emissions

In this subsection we include the coupling of the system to the environment and as a consequence the spontaneous emission of photons. First we will only consider the time evolution of the system under the condition that no photon is emitted and compare the decay time with the analytic estimation. Then we include also spontaneous emissions and the recoil kick the atom experiences.

In the Quantum Jump Approach one describes the time evolution of the system with an effective Hamiltonian as long as no photon is emitted. The emissions which cause the system to jump in a different state are destroyed by reset operators. We obtain an expression for the effective Hamiltonian by comparing Eqs. (9),(10) and (11) with Eq. (12). This gives

\[
H_{\text{eff}} = \frac{P^2}{2m} + \frac{\Delta}{2} \left( |g, 1\rangle\langle g, 1| + |e, 0\rangle\langle e, 0| - |g, 0\rangle\langle g, 0| \right) + g(x) \left( |e, 0\rangle\langle g, 1| + |g, 1\rangle\langle e, 0| \right) + \frac{\Omega}{2} \left( |g, 0\rangle\langle e, 0| + |e, 0\rangle\langle g, 0| \right) - i\kappa |g, 1\rangle\langle g, 1| - i\Gamma |e, 0\rangle\langle e, 0|,
\]

where we used an interaction picture rotating with the laser frequency \( \omega_L \) and assumed that there is at most one excitation in our system.

After preparing the system in the ground state \( |\Phi_0\rangle \) using the imaginary time evolution we simulated the time evolution with the effective Hamiltonian \( H_{\text{eff}} \). This leads to a damping of the state of the system. So the probability \( |\Phi_0|^2 \) that no photon has been emitted also decreases. We compare the time after which this probability has

\[
\text{FIG. 6: Numerical simulation of the ground state. The plots show } |\varphi_0(x)|^2, |\varphi_{1}\rangle |^2 \text{ and } |\varphi_{2}(x)|^2 \text{ versus } x. \text{ They satisfy the normalization condition } \int |\varphi_0(x)|^2 + |\varphi_{1}\rangle |^2 + |\varphi_{2}(x)|^2 \text{ dx } = 1. \text{ The plots are in units of } \sigma, \text{ which is the cavity width. For the laser detuning we took } \Delta = 1.90 \times 30 = 3 \times 2 \pi \text{ MHz and for the Rabi frequency of the laser } \Omega = 0.70 \times 2 \pi \text{ MHz. The coupling strength between cavity and atom is } g_0 = 36 \times 2 \pi \text{ MHz, the cavity loss rate } \kappa = 1.4 \times 2 \pi \text{ MHz and the spontaneous decay rate } \Gamma = 3 \times 2 \pi \text{ MHz.}
\]

reached $1/e$ with the effective life time $\tau_{\text{eff}}$ we estimated analytically. For the parameters from (15) we obtained $\tau_{\text{eff}} = 0.18\text{ ms}$, which agrees with the decay time from the numerical simulation of $0.14\text{ ms}$.

We assumed that either the atom or the cavity emit a photon when $|\Phi|^2 = 0.5$. In the Quantum Jump Approach the jumps are described by reset operators. We obtain them from the master equation (12). For the spontaneous emission of the atom we get

$$e^{-ix\sigma}\sqrt{\Gamma}|g, 0\rangle\langle e, 0|,$$  \hspace{1cm} (19)

where $e^{-ix\sigma}$ describes the momentum shift "$u$" due to the photon recoil. If the cavity emits a photon one has to apply

$$\sqrt{2\kappa}|g, 0\rangle\langle g, 1|.$$  \hspace{1cm} (20)

In both cases the population of the excited level gets shifted to the ground state. After that one has to normalize the wave function.

In the following we will discuss the trapping time $\tau_{\text{trap}}$ of the atom. We define it as the time when the probability to find the atom in the cavity ($|x| < \sigma$) is reduced to $0.5$. The atom has an initial kinetic energy and gains a momentum kick if it spontaneously emits a photon. When the kinetic energy is bigger than the trapping potential the atom leaves the cavity. So it is desirable to achieve a long decay time in order to get a low photon emission rate. On the other hand a deeper potential provides the possibility of a bound state for an atom which experienced more recoil kicks. From our analytic estimations we know that these demands contradict each other.

For the simulation we took the parameters from (15) for a potential depth of $V_0 = 100\text{ kHz}$ and the longest corresponding effective decay time $\tau_{\text{eff}} = 0.18\text{ ms}$. As the initial kinetic energy we assumed the energy of one photon recoil $E_{\text{RF}} = 4\text{ kHz}$. This is also the energy the atom gains if it emits a photon. Due to the damping and the jumps $|\Phi\rangle$ evolves into a state which is no longer an eigenstate of (4) but as mentioned above the atom leaves the cavity only if its kinetic energy is bigger than the potential depth. In our simulation this happens after a time $\tau_{\text{trap}} = 0.73\text{ ms}$.

In order to achieve a longer trapping time we first varied the detuning $\Delta$ and left the Rabi frequency $\Omega = 0.70 \times 2\pi\text{ MHz}$ unchanged. The result is shown in Fig. 7. A larger detuning of the laser leads to a smaller potential depth $V_0$ and a longer effective life time $\tau_{\text{eff}}$. From Fig. 7 we see that the longer life time has a bigger influence on the trapping time since $\tau_{\text{trap}}$ increases with growing detuning. It is not surprising that the effective life time has a crucial influence on the trapping time since it determines how fast the kinetic energy of the atom grows.

A larger Rabi frequency causes a smaller effective life time and a deeper potential. So consistently we expect a decreasing trapping time when we enlarge the Rabi frequency. This is confirmed by Fig. 8, where we plotted $\tau_{\text{trap}}$ versus $\Omega$ for a fixed detuning $|\Delta| = 1.90\text{ g}_0 = 30 \times 2\pi\text{ MHz}$. If we compare this plot with the plot in Fig. 5(A) we ascertain a qualitative agreement. This is again what we expect if we assume that the decisive variable for the trapping time $\tau_{\text{trap}}$ is the effective decay time $\tau_{\text{eff}}$.

The longest trapping times we can achieve in the optical regime are of the order of 1 ms. For the microwave regime we took $g_0 = \Omega/2 \times 2\pi\text{ kHz}$, $\kappa = 1.6 \times 2\pi\text{ Hz}$ and $\Gamma = 1.6 \times 2\pi\text{ Hz}$. The trapping time we obtained for the laser parameters from (16) was $\tau_{\text{trap}} \approx 10\text{ s}$. The reason for the good result are the very small decay rates for the Rydberg state and the micro-cavity. Another important advantage over the optical regime is that the recoil due to spontaneous emissions is practically zero.

CONCLUSIONS

We showed that it is possible to trap an atom in the vacuum field of a high Q cavity. To do this we need a weak laser which couples directly to the atom in the
cavity. It induces a position dependent AC Stark shift to the ground state of the cavity-atom system. We use this energy shift as a trapping potential and as we showed by an analytic estimation and a numerical simulation it is deep enough to trap an atom with a realistic initial momentum.

The advantage of this approach is the low effective decay rate due to the little amount of excitation in the system. This requires to cool the atom to a lower kinetic energy than the potential depth. In order to obtain a long life time it would be good to have an initial kinetic energy of the order of one photon recoil. This is still difficult to achieve, even though it is possible to cool an atom below one photon recoil with the method of velocity-selective coherent population trapping [34] or Raman-cooling [35]. Another possibility would be a cavity assisted cooling method [36].

The trapping time we can achieve in the optical regime with our approach is of the same order or even lower as observed already in experiments [13, 14, 15]. The benefit of this method is that the time after which the first jump occurs is longer because there is only very little excitation in the system. As mentioned before this decay time is very important for any kind of quantum information application since the jump destroys the coherence in the atomic state. In the microwave regime the trapping times can be much longer.

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