populated orthogonal quantum states. Since $S_0 = 1 - 2\pi\hbar \int \tilde{P}(Q, P) d\Gamma_s$, where $d\Gamma_s \equiv dQdP$, this entropy has a natural classical analog (denoted $S_c$) obtained by replacing $\tilde{P}$ with $\tilde{\rho}$. That is, $S_c = 1 - 2\pi\hbar \int \tilde{\rho}(Q, P) d\Gamma_s$. A more detailed, but representation-dependent description of decoherence is the decay of off-diagonal density matrix elements such as $\langle Q_1|\tilde{\rho}(t)|Q_2\rangle$. Significantly, we discover that the classical analog of these matrix elements can also be constructed. Specifically, noting that $\langle Q_1|\tilde{\rho}(t)|Q_2\rangle = \int dP \tilde{\rho}(Q, P) \exp[\imath\Delta QP/\hbar]$, where $Q \equiv (Q_1 + Q_2)/2$ and $\Delta Q = Q_1 - Q_2$, we define the classical analog (denoted $\tilde{\rho}_c(Q_1, Q_2, t)$) of $\langle Q_1|\tilde{\rho}(t)|Q_2\rangle$ as the Fourier transformed classical distribution function, i.e., $\tilde{\rho}_c(Q_1, Q_2, t) = \int dP \tilde{\rho}(Q, P, t) \exp[\imath\Delta QP/\hbar]$. This approach can be readily extended to the momentum representation.

Perturbative treatments have proved to very useful in understanding decoherence dynamics [10, 11]. Here, to examine classical vs. quantum decoherence dynamics at short times, a regime of great interest in the control of decoherence, we consider a second-order expansion with respect to time variable $t$ for both $S_c$ and $S_q$, i.e., $S_c(t) = S_c(0) + t/\tau_{\gamma,1} + t^2/\tau_{\gamma,2}^2 + \cdots$ and $S_q(t) = S_q(0) + t/\tau_{\gamma,1} + t^2/\tau_{\gamma,2}^2 + \cdots$. Using the definitions of Poisson and Moyal brackets and assuming that the initial distribution function is decorrelated with initial bath statistics, we obtain

$$\frac{1}{\tau_{\gamma,1}} = \frac{1}{\tau_{\gamma,2}} = 0,$$

$$\frac{1}{\tau_{\gamma,2}^2} = \frac{C_b}{\hbar} \int dQ_1 dQ_2 |\tilde{\rho}_c(Q_1, Q_2, 0)|^2 \Delta Q^2 \left[ \frac{df(Q)}{dQ} \right]^2,$$

and

$$\frac{1}{\tau_{\gamma,2}^3} = \frac{C_b}{\hbar} \int dQ_1 dQ_2 |\langle Q_1|\tilde{\rho}(0)|Q_2\rangle|^2 \Delta Q^2 \left[ \frac{\Delta f(Q)}{\Delta Q} \right]^2,$$

where $C_b = \sum_{j=1}^N C_j^2 \coth(\beta\hbar\omega_j/2)/(2m\hbar\omega_j)$. $\Delta f(Q) = f(Q + \Delta Q/2) - f(Q - \Delta Q/2)$, and $\beta$ is the Boltzmann factor. Note that the factor $\hbar$ appearing in the classical result [Eq. 2] is just due to the definitions of $S_c$ and $\tilde{\rho}_c(Q_1, Q_2, 0)$, and that the initial variances of the bath variables $\gamma$ have been evaluated using quantum statistics to ensure the same initial quantum state for the ensuing classical and quantum dynamics. Note also that the decoherence time scale indicated in the derived and simple quantum result of Eq. 3 is consistent with, but is more transparent than, a previous perturbative result [Eq. (3) in Ref. [12]] obtained using a sophisticated influence functional approach.

Equation (1) shows that zero first-order decoherence rate i.e., $1/\tau_{\gamma,1} = 0$, has a strict classical analog. More interestingly, Eqs. 2 and 3 show that, for the same fixed initial distribution function, the ratio of $1/\tau_{\gamma,2}^3$ to $1/\tau_{\gamma,2}^2$ is $\hbar$-independent. As seen from Eqs. 2 and 3, $1/\tau_{\gamma,2}^2 - 1/\tau_{\gamma,2}^3$ arises from the difference between the derivative $df/dQ$ and the finite-difference function $\Delta f/\Delta Q$, weighted by $\Delta Q^2$ and the initial state. As a result: (1) For any given $f(Q)$, as long as $\langle Q_1|\tilde{\rho}(0)|Q_2\rangle$ decays fast enough with $\Delta Q$ such that $\Delta f/\Delta Q \approx df/dQ$, there would be excellent QCC in early-time decoherence dynamics. The smaller the $\hbar$, the more rigorous is this requirement. (2) If $f(Q)$ depends only linearly or quadratically upon the coupling coordinate $Q$, then $1/\tau_{\gamma,2}^2 - 1/\tau_{\gamma,2}^3 \approx 0$ for any initial state. Significantly, in all traditional decoherence models [13] where $f(Q) = Q$ is assumed, there exists perfect QCC in early decoherence dynamics, regardless of $\hbar$, and irrespective of the system potential $V(Q)$ [14]. Indeed, in the case of $f(Q) = Q$ Eq. 2 reduces to an important result, previously obtained quantum mechanically [10]:

$$\frac{1}{\tau_{\gamma,2}^2} = \frac{1}{\tau_{\gamma,2}^3} = \frac{\delta^2 Q}{\hbar} \sum_{j=1}^N C_j^2 \coth(\beta\hbar\omega_j/2),$$

where the initial state of the system is assumed to be pure, with the initial variance in $Q$ given by $\delta^2 Q$. (3) For nonlinear $f(Q)$ where $\Delta f/\Delta Q \neq df/dQ$ over the range of the initial state, QCC can be very poor.

The second-order perturbative treatment is most relevant to short times and for weak decoherence. The results are particularly significant for studies of decoherence control where early-time dynamics of weak decoherence is important. In these circumstances it is useful to understand the extent to which (quantum) decoherence is equivalent to classical entropy production, i.e., to increasing $S_c(t)$. In particular, if there exists good correspondence between classical and quantum decoherence dynamics, then the essence of decoherence control is equivalent to the suppression of classical entropy production, and various classical tools may be considered to achieve decoherence control. If not, then fully quantum tools are required.

As an example, consider decoherence for an initial superposition state of two well-separated and strongly localized Gaussian wavepackets located at $Q_a = Q_{ab} - \Delta Q_{ab}/2$ and $Q_b = Q_{ab} + \Delta Q_{ab}/2$ with $Q_{ab} = 0$. For this initial state,
Then in a cubic decoherence model, for example, where \( f(Q) = Q^2 \), one would obtain 1/\( \tau_{c2}^2 \) ~ 0 since \( df(Q)/dQ = 0 \). However, here 1/\( \tau_{c2}^2 \) >> 1/\( \tau_{c2}^2 \), i.e. there is appreciable decoherence without classical entropy production. By contrast, in another nonlinear decoherence model where \( f(Q) = \sin(2\pi Q/\Delta Q_{ab} + \pi/4) \), 1/\( \tau_{c2}^2 \) ~ 0 since \( f(Q) = f(Q_0) \). Here, however, 1/\( \tau_{c2}^2 \) > 1/\( \tau_{c2}^2 \), i.e., the system is decoherence-free but with substantial classical entropy production. Since we find that the ratio of \( \tau_{c2}^2 \) to \( \tau_{c2}^2 \) in early-time decoherence dynamics is independent of \( \pi \) for fixed initial state, these two examples lead to a rather counter-intuitive result: given a macroscopic object which is initially in a superposition state of two distinguishable states and is nonlinearly coupled with an environment, classical dynamics could totally fail to predict its initial entropy production or its decoherence rate. Indeed, Eqs. (2) and (3) suggest that, as long as \( df(Q)/dQ \neq 0 \) and \( |f(Q)| \) is bounded, then 1/\( \tau_{c2}^2 \) saturates with increasing \( \Delta Q_{ab} \), whereas 1/\( \tau_{c2}^2 \) does not. Thus, one can conclude that decoherence dynamics must be quantum and that the system-environment coupling must be nonlinear if the saturation behavior of early-time decoherence rates is observed experimentally[15]. Further, it is clear that in the limit of large \( \Delta Q_{ab} \), classical decoherence dynamics in the general case of nonlinear system-environment coupling predicts much faster decoherence than does quantum decoherence dynamics. This leads to the rather surprising inference that initial superposition states of well-separated wavepackets would be more susceptible to nonlinear system-environment coupling if they are propagated by classical dynamics than by quantum mechanics.

To go beyond the perturbation results we now consider a strong decoherence model in which decoherence is assumed to be much faster than the system dynamics, so that \( H^s \) can be set to zero [1]. We consider both the “off-diagonal elements” \( \rho_c(Q_1, Q_2, t) \) as well as the entropy \( S_c(t) \) and compare them to the quantum results.

In this case the classical Liouville dynamics gives

\[
\frac{\partial F_c[\bar{Q}, \Delta Q, \{q_{j}, p_{j}\}, t]}{\partial t} = \sum_{k=1}^{N} \frac{\partial H_{k}^c}{\partial q_{k}} \frac{\partial F_c[\bar{Q}, \Delta Q, \{q_{j}, p_{j}\}, t]}{\partial p_{k}} - \sum_{k=1}^{N} \frac{\partial H_{k}^c}{\partial p_{k}} \frac{\partial F_c[\bar{Q}, \Delta Q, \{q_{j}, p_{j}\}, t]}{\partial q_{k}} + \sum_{k=1}^{N} C_{k} f(\bar{Q}) \frac{\partial F_c[\bar{Q}, \Delta Q, \{q_{j}, p_{j}\}, t]}{\partial p_{k}} - \frac{i}{\hbar} \Delta Q \sum_{k=1}^{N} C_{k} \frac{df(\bar{Q})}{dQ} q_{k} F_c[\bar{Q}, \Delta Q, \{q_{j}, p_{j}\}, t],
\]

where \( F_c[\bar{Q}, \Delta Q, \{q_{j}, p_{j}\}, t] \equiv \int dP \exp[i\Delta Q P/\hbar] \rho_c(\bar{Q}, P, \{q_{j}, p_{j}\}, t) \). Since \( \bar{Q} = 0 \) due to \( H^s = 0 \), and \( \Delta Q \) is a time-independent parameter introduced in the Fourier transformation, Eq. (5) leads to

\[
\frac{dF_c[\bar{Q}, \Delta Q, \{q_{j}(t), p_{j}(t)\}, t]}{dt} = \frac{\partial F_c[\bar{Q}, \Delta Q, \{q_{j}(t), p_{j}(t)\}, t]}{\partial t} + \sum_{k=1}^{N} \frac{\partial F_c[\bar{Q}, \Delta Q, \{q_{k}(t), p_{k}(t)\}, t]}{\partial q_{k}(t)} q_{k}(t) + \sum_{k=1}^{N} \frac{\partial F_c[\bar{Q}, \Delta Q, \{q_{k}(t), p_{k}(t)\}, t]}{\partial p_{k}(t)} p_{k}(t).
\]

\[
= \frac{-i}{\hbar} \Delta Q \sum_{k=1}^{N} C_{k} q_{k}(t) \frac{df(\bar{Q})}{dQ} F_c[\bar{Q}, \Delta Q, \{q_{j}(t), p_{j}(t)\}, t],
\]

where \( \{q_{j}(t), p_{j}(t)\} \) satisfy \( \dot{q}_{j}(t) = -\partial H_{j}^c / \partial p_{j}(t) \) and \( \dot{p}_{j}(t) = -\partial H_{j}^c / \partial q_{j}(t) - C_{j} f(\bar{Q}) \), of which the solution is

\[
q_{j}(t) = \frac{C_{j} f(\bar{Q})}{m_{j} \omega_{j}^{2}} [\cos(\omega_{j} t) - 1] + q_{j}(0) \cos(\omega_{j} t) + \frac{p_{j}(0)}{m_{j} \omega_{j}} \sin(\omega_{j} t),
\]

and \( p_{j}(t) = m_{j} \dot{q}_{j}(t) \). Analytically integrating Eq. (6), and using \( d\Gamma_{0}^{N}(t) = d\Gamma_{0}^{N}(0) \) and \( \tilde{\rho}_{c}(Q_1, Q_2, t) = \int d\Gamma_{0}^{N}(t) F_c[\bar{Q}, \Delta Q, \{q_{j}(t), p_{j}(t)\}, t] \), we have

\[
\tilde{\rho}_{c}(Q_1, Q_2, t) = \int d\Gamma_{0}^{N}(0) F_c[\bar{Q}, \Delta Q, \{q_{j}(0), p_{j}(0)\}, 0] \times \exp[-\frac{i}{\hbar} \int_{0}^{t} dt \Delta Q \sum_{k=1}^{N} C_{k} \frac{df(\bar{Q})}{dQ} q_{k}(t)].
\]
system, and assuming that the equilibrium state of the bath is maintained, we obtain
\[
\frac{\rho_c(Q_1, Q_2, t)}{\rho_c(Q_1, Q_2, 0)} = \exp \left[ i\phi_c(t) - (\Delta Q)^2 \left( \frac{df(Q)}{dQ} \right)^2 B_2(t) \right],
\]
where \( \phi_c(t) \equiv (\Delta Q) f(Q)|df(Q)/dQ| B_1(t)/\hbar, \) with \( B_1(t) = \sum_{j=1}^{N} C_j^2[t - \sin(\omega_j t)/\omega_j][(m_j\omega_j)^2], \) and \( B_2(t) = \sum_{j=1}^{N} C_j^2 \coth(\beta\hbar\omega_j)/2[1 - \cos(\omega_j t)/(2m_j\hbar\omega_j^2)]. \) Interestingly, the classical result [Eq. (9)] displays two dynamical aspects of \( \rho_c(Q_1, Q_2, t), \) i.e., coherent dynamics of its phase \( \phi_c(t), \) and incoherent decay due to bath statistics. The classical linear entropy \( S_c(t) \) can then be obtained from Eq. (9) as
\[
S_c(t) = 1 - \int dQ_1 dQ_2 |\rho_c(Q_1, Q_2, 0)|^2 \exp \left[ -2(\Delta Q)^2 \left( \frac{df(Q)}{dQ} \right)^2 B_2(t) \right].
\]

With similar manipulations for quantum strong decoherence dynamics, we obtain the quantum result
\[
\frac{\langle Q_1|\hat{\rho}(t)|Q_2 \rangle}{\langle Q_1|\hat{\rho}(0)|Q_2 \rangle} = \exp \left[ i\phi_q(t) - \Delta Q^2 \left( \frac{\Delta f(Q)}{\Delta Q} \right)^2 B_2(t) \right],
\]
where \( \phi_q(t) \equiv \Delta Q f(Q)|\Delta f(Q)/Q| B_1(t)/\hbar, \) and
\[
S_q(t) = 1 - \int dQ_1 dQ_2|\langle Q_1|\hat{\rho}(0)|Q_2 \rangle|^2 \exp \left[ -2(\Delta Q)^2 \left( \frac{\Delta f(Q)}{\Delta Q} \right)^2 B_2(t) \right].
\]

These results extend those in Ref. [16] to nonlinear \( f(Q) \) using a simple approach and demonstrate a direct classical analog to quantum strong decoherence dynamics.

Since \( dB_2(t)/dt(t = 0) = 0 \) and \( \frac{d^2 B_2(t)}{dt^2}(t = 0) = C_0/\hbar, \) one finds that in the short time limit, Eqs. (10) and (12) reduce to previous perturbation results of \( 1/\tau_{1,1}, 1/\tau_{1,2}. \) Furthermore, the classical results [Eqs. (9) and (10)] are again much similar to the quantum results [Eqs. (11) and (12)], with the only difference being that \( \Delta f/\Delta Q \) in the quantum expression is replaced by \( df/dQ \) in the classical result.

This result makes clear that our previous QCC results based upon second-order perturbation theory are generalizable to all orders of time in the strong decoherence case. In particular, defining \( \gamma_c(t) \equiv d\ln[\rho_c(Q_1, Q_2, t)]/dt \) and \( \gamma_q(t) \equiv d\ln[\langle Q_1|\hat{\rho}(t)|Q_2 \rangle]/dt, \) we have \( \gamma_c(t) = -\Delta Q^2 \left( \frac{df(Q)}{dQ} \right)^2 (dB_2(t)/dt), \) and \( \gamma_q(t) = -\Delta Q^2 \left[ \frac{\Delta f(Q)}{\Delta Q} \right] (dB_2(t)/dt). \) Then, in the case of linear and/or quadratic coupling, e.g., \( f(Q) = aQ + bQ^2, \) one has \( \gamma_c(t) = \gamma_q(t) \) and \( S_c(t) = S_q(t), \) showing that there is perfect QCC in decoherence dynamics for all times.

By contrast, in the case of nonlinear coupling, \( \gamma_c(t) \) in general does not saturate with increasing \( \Delta Q \) whereas \( \gamma_q(t) \) does saturate for bounded \( [f(Q)]. \) As such, in the limit of large \( \Delta Q, \) one has \( |\gamma_c(t)| \gg |\gamma_q(t)| \) and thus \( [1 - S_c(t)] \ll [1 - S_q(t)] \) as \( t \) increases, with \( |\gamma_c(t)/\gamma_q(t)| \) independent of \( \hbar. \) This observation is of conceptual importance: it says that decoherence can dramatically improve QCC, but as far as some detailed characteristics of decoherence dynamics are concerned, decoherence itself does not necessarily suffice to ensure that the dynamics of quantum entropy production equals that of classical entropy production. That is, even in the presence of strong decoherence, subtle quantum classical differences may persist in some measures (e.g., \( 1/[1 - S_c(t)] \) vs. \( 1/[1 - S_q(t)] \)) for all finite times. Note, however, the entropy measures such as \( 1/[1 - S_c(t)] \) are not a quantum mechanical observables and hence do not allow one to directly measure the subtle difference between classical and quantum decoherence dynamics at later times.

Thus, from both the perturbation and strong decoherence results, we obtain that QCC depends critically upon the initial quantum state and the nature of the system-environment coupling. This result should have an impact on our current understanding of decoherence even when the role of the dynamics of the system is important. For example, it is worthwhile reexamining the relationship between classical Lyapunov exponents and decoherence rates in classically chaotic systems, since previous studies [17] only dealt with the case of linear system-environment coupling.

In conclusion, we have examined, using analogous measures, the decoherence dynamics of an initial quantum state coupled to a bath that is subjected to either classical or quantum dynamics. Within the framework of a second-order perturbative treatment and a strong decoherence theory, we have exposed the system-independent conditions under which the quantum decoherence dynamics is either well or, poorly, approximated by classical dynamics. Further studies are ongoing to assess QCC in cases beyond the short time and strong decoherence approximations. Preliminary computational results [8] support the conclusions drawn herein.

This work was supported by the U.S. Office of Naval Research and the Natural Sciences and Engineering Research Council of Canada.
[8] H. Han, J. Gong, and P. Brumer, to be published.
[14] It is not surprising, in retrospect, that the classical and quantum results are in perfect agreement in the cases where the bath is harmonic, the coupling is linear or quadratic, and the system can be neglected. In this case it is expected that the quantum and classical Liouville dynamics are identical. What is important, however, is that this equivalence can be extended to the process of decoherence, a phenomenon assumed to be solely quantum in nature.