1. Introduction

The study of the early universe, especially the inflationary models, provides a framework to understand the origin of the universe. The inflationary models predict a universe that is smooth and homogeneous on large scales, with small fluctuations that can be traced back to the quantum fluctuations during the inflationary period. These fluctuations are later amplified by gravitational effects and are observed as the cosmic microwave background radiation. The inflationary models also provide a natural explanation for the large-scale structure of the universe, including the distribution of galaxies and the large-scale anisotropies in the cosmic microwave background. The inflationary models have also led to the development of the cosmological constant term in the Einstein field equations, which is responsible for the observed acceleration of the universe.

The inflationary models are not without their challenges, however. One of the main challenges is the initial conditions of the universe, which are required to fine-tune the parameters of the inflationary model to match the observed data. Another challenge is the problem of the hierarchy between the energy scales of the inflationary model and the Standard Model of particle physics.

Despite these challenges, the inflationary models remain a cornerstone of modern cosmology and continue to be a subject of active research. The study of the inflationary models provides insights into the fundamental nature of the universe and the early stages of its development.
background field. The basic conclusion will be that the Fierz picture constructed in [4] is naturally present in the teleparallel construction. In this sense, we can say that the teleparallel equivalent of general relativity appears to be a natural framework to deal with the spin-2 theory. In fact, the small perturbations of the tetrad field are shown to reproduce correctly the behavior of the spin-2 field on the flat Minkowski spacetime. The antisymmetric piece of the tetrad turns out to be redundant, although taking into account its explicit contribution makes the underlying gauge symmetry more transparent. The generalization for the presence of gravitation is straightforward, and it represents an alternative way to describe the spin-2 particle interacting with an external gravitational field.

II. TELEPARALLEL GRAVITY: BASIC FACTS

The general structure of teleparallel gravity is presented in detail in [5, 7, 8, 9, 10]. In this section we summarize the fundamentals of this theory. In short, teleparallel approach can be understood as a gauge theory of the spacetime translation group. Using the Greek alphabet to denote spacetime indices, and the Latin alphabet to denote the local frame components, the corresponding gauge potential is represented by the nontrivial part of the tetrad field \( h^\alpha{}_{\mu} \). This tetrad gives rise to the so called Weitzenböck connection,

\[
\Gamma^\nu{}_{\mu\nu} = h^\alpha{}_{\mu} \partial^\nu h^\mu{}_{\nu},
\]

which introduces the distant parallelism on a four-dimensional spacetime manifold. It is a connection that presents torsion, but not curvature. Its torsion, \( T^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu} \), plays the role of the translational gauge field strength. The Weitzenböck connection can be conveniently decomposed into the Riemannian and the post-Riemannian pieces,

\[
\Gamma^\nu{}_{\mu\nu} = \tilde{\Gamma}^\nu{}_{\mu\nu} + K^\nu{}_{\mu\nu},
\]

where

\[
\tilde{\Gamma}^\nu{}_{\mu\nu} = \frac{1}{2} \partial^\nu (\partial^\mu g_{\sigma\nu} + \partial^\sigma g_{\nu\mu} - \partial^\sigma g_{\mu\nu})
\]

is the Christoffel symbol constructed from the spacetime metric \( g_{\mu\nu} = h^\mu{}_{\nu} h^\nu{}_{\mu} \), and

\[
K^\nu{}_{\mu\nu} = \frac{1}{2} (T^\nu{}_{\mu\nu} + T^\nu{}_{\nu\mu} - T^\nu{}_{\mu\nu})
\]

is the contortion tensor. Correspondingly, all other geometrical and physical objects and operations constructed with the help of the Riemannian connection \( \tilde{\Gamma}^\nu{}_{\mu\nu} \) will be denoted with a tilde.

The gauge gravitational field Lagrangian reads

\[
\mathcal{L}_G = \frac{\kappa}{16 \pi G} h S^\mu{}_{\nu\mu} T^\nu{}_{\mu\nu},
\]

where \( h = \det(h^\mu{}_{\nu}) \), and

\[
S^\mu{}_{\nu\rho} = -S^\nu{}_{\rho\mu} := \frac{1}{2} \left[ K^{\mu\rho\nu} - g^{\mu\nu} T^\rho{}_{\sigma} + g^{\rho\nu} T^\sigma{}_{\mu} \right]
\]

is a tensor written in terms of the Weitzenböck connection only. Inverting this equation, we obtain torsion in terms of the above tensor:

\[
T^\alpha{}_{\mu
u} = 2 \left( S^\mu{}_{\nu\alpha} - S^\nu{}_{\mu\alpha} + \delta^\alpha{}_{\mu} S^\rho{}_{\nu\rho} - \delta^\alpha{}_{\nu} S^\rho{}_{\mu\rho} \right).
\]

The Lagrangian (5) describes what is commonly known as the teleparallel equivalent of Einstein’s general relativity theory. Performing a variation with respect to the tetrad, we find the teleparallel version of the gravitational field equation,

\[
\partial^\sigma (h S^\lambda{}_{\sigma\nu}) - \frac{4 \pi G}{c^4} (h t^\lambda{}_{\nu}) = 0,
\]

where

\[
h t^\lambda{}_{\nu} = \frac{c^4}{4 \pi G} S^\nu{}_{\mu\lambda} \Gamma^\rho{}_{\nu\lambda} - \delta^\lambda{}_{\nu} \mathcal{L}_G
\]

is the energy-momentum (pseudo) tensor of the gravitational field. This equation is known to be equivalent to the Einstein’s equation of general relativity. It is important to notice that the left-hand side of the field equation (8) can be rewritten as the usual left-hand side of Einstein equations

\[
\partial^\sigma (h S^\lambda{}_{\sigma\nu}) - \frac{4 \pi G}{c^4} (h t^\lambda{}_{\nu}) \equiv \frac{h}{2} \left( \tilde{R}^\lambda{}_{\nu} - \frac{1}{2} \delta^\lambda{}_{\nu} \tilde{R} \right),
\]

which then provides an easy proof of the Lemma 2 of [4]. As the source of both field equations is the symmetric energy-momentum tensor, the equivalence alluded to above holds also in the presence of matter [11]. It is worth noticing that the teleparallel field equation has the same structure of the Yang-Mills equation, which is consistent with the fact that teleparallel gravity corresponds to a gauge theory. We see in this way that the teleparallel approach to gravitation is more closely related to field theory than the general relativity approach.

III. LINEARIZED THEORY

The trivial tetrad \( h^\alpha{}_{\mu} = \delta^\alpha{}_{\mu} \) describes the flat geometry, for which the metric has the diagonal Minkowski form, \( g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1) \). Let us then expand the tetrad field around the flat background as follows,

\[
h^\alpha{}_{\mu} = \delta^\alpha{}_{\mu} + u^\alpha{}_{\mu}.
\]

The Weitzenböck connection reads, correspondingly,

\[
\Gamma^\nu{}_{\mu\nu} = \partial^\nu u^\lambda{}_{\mu}.
\]

where \( u^\nu{}_{\mu} = \partial^\nu u^\alpha{}_{\mu} \). As a result, the torsion and its trace are, respectively,

\[
T^\nu{}_{\mu\nu} = \partial^\nu u^\rho{}_{\mu} - \partial^\rho u^\nu{}_{\mu}, \quad T^\mu{}_{\nu\mu} = \partial^\mu u^\nu - \partial^\nu u^\mu.
\]

where

\[
S^\mu{}_{\nu\rho} = -S^\nu{}_{\rho\mu} := \frac{1}{2} \left[ K^{\mu\rho\nu} - g^{\mu\nu} T^\rho{}_{\sigma} + g^{\rho\nu} T^\sigma{}_{\mu} \right]
\]

and

\[
\partial^\sigma (h S^\lambda{}_{\sigma\nu}) - \frac{4 \pi G}{c^4} (h t^\lambda{}_{\nu}) \equiv \frac{h}{2} \left( \tilde{R}^\lambda{}_{\nu} - \frac{1}{2} \delta^\lambda{}_{\nu} \tilde{R} \right),
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\]
with \( u = u^\nu \). Decomposing the perturbation tensor \( u_{\mu\nu} \) into the symmetric and antisymmetric pieces,

\[
u_{\mu\nu} = \phi_{\mu\nu} + a_{\mu\nu}, \tag{14}\]

with

\[
\phi_{\mu\nu} := u_{(\mu\nu)} \text{ and } a_{\mu\nu} := u_{[\mu\nu]} \tag{15}
\]
we compute immediately the contortion tensor:

\[
K^\nu_{\mu\rho} = \partial^\nu \phi_{\mu\rho} - \partial^\nu \phi_{\rho\mu} + \partial^\rho \partial_\mu \phi^\nu - \partial^\mu \partial_\nu \phi^\rho \tag{16}
\]

Substituting now the above expressions into (6), we obtain

\[
S^\nu_{\mu\rho} = \frac{1}{2} \left[ \partial^\mu \phi^\rho_{\nu} - \partial^\rho \phi^\mu_{\nu} - g^\mu\rho \left( \partial^\nu \phi - \partial_\nu \phi^\nu \right) + g_{\rho\nu} \left( \partial^\nu \phi^\rho - \partial^\rho \phi^\nu \right) + g^\nu\rho \partial_\mu \phi_{\nu\rho} - g^\mu\rho \partial_\nu \phi_{\nu\rho} \right]. \tag{17}
\]

Comparison with [4] shows that the first line is nothing but the Fierz tensor \( F^\nu_{\mu\rho} \) introduced by Noll and Neves. More specifically:

\[
S^\nu_{\mu\rho} = - F^\nu_{\mu\rho} + \frac{1}{2} \left( \partial^\mu \phi^\rho_{\nu} + g_{\rho\nu} \partial_\mu \phi^\nu - g^\mu\rho \partial_\nu \phi^\nu \right). \tag{18}
\]

Since we have identically

\[
\partial_\mu \left( \partial^\nu \phi^\rho_{\nu} + g_{\rho\nu} \partial_\mu \phi^\nu - g^\mu\rho \partial_\nu \phi^\nu \right) \equiv 0,
\]
the last term drops out completely from the linearized gravitational field equations (8), which then reads:

\[
\partial_\mu S^\mu_{\nu\lambda\tau} = 0. \tag{19}
\]

This equation yields the correct dynamics of the spin 2 particle in flat space time, as it is easily seen from the identity (10). Indeed, substituting the expansion (11) into it, we find that

\[
\partial_\mu S^\mu_{\nu\lambda\tau} = \frac{1}{2} G^L_{\nu\lambda\tau}, \tag{20}
\]
where the left-hand side represents the linearized Einstein tensor:

\[
G^L_{\nu\lambda\tau} = \square \left( \eta_{\nu\mu} \phi - \phi_{\nu\mu} \right) - \partial_\nu \partial_\mu \phi - \phi^\gamma_{\mu \nu} \partial_\lambda \phi^\gamma_{\rho \tau} + \phi^\gamma_{\mu \lambda} \partial_\rho \phi_{\nu \tau} + \phi^\gamma_{\nu \rho} \partial_\lambda \phi_{\mu \tau}.
\]

The identity (10) plays the fundamental role in the teleparallel theory, since it underlies the proof of the equivalence of Einstein’s gravity and the teleparallel gravity. Now, as a by-product of this identity we have straightforwardly derived the Lemma 2 of [4].

It is interesting to notice that the teleparallel Lagrangian (5) can be rewritten as

\[
\mathcal{L}_G = \frac{c^4}{8\pi G} \left( S^\mu_{\nu\rho} S^\nu_{\mu\rho} - S^\mu_{\nu\rho} S^\rho_{\mu\nu} - 3 S^\mu_{\nu\rho} S^\nu_{\mu\rho} \right).
\]

Inserting here (17), we can verify that the antisymmetric field \( a_{\mu\nu} \) drops out completely, in accordance with the analysis of the linearized field equations. This observation shows that, as a matter of fact, the antisymmetric field does not have physical importance. Indeed, one can show quite generally that, by means of a certain Lorentz transformation, it is always possible to choose a frame in which the tetrad matrix is symmetric [12, 13, 14]. In such a frame, the field \( a_{\mu\nu} \) vanishes, and as a result the tensor \( S^\mu_{\nu\rho} \) coincides (up to a sign) with the Fierz tensor introduced in [4].

However, there is a certain reason to keep \( a_{\mu\nu} \) non-trivial. In particular, we can verify the covariance of the linearized formalism with respect to “general coordinate” spacetime transformations. The demonstration of this property in [4] is rather long and not very transparent. In contrast, here it is sufficient to notice that the tensor (17) is explicitly invariant under the gauge transformations:

\[
\phi_{\mu\nu} \rightarrow \phi_{\mu\nu} + \partial_\nu \Lambda_\mu + \partial_\mu \Lambda_\nu, \tag{23}
\]

\[
a_{\mu\nu} \rightarrow a_{\mu\nu} - \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu. \tag{24}
\]

The proof is straightforward: One just needs to substitute these formulas into Eq. (17). The original perturbation field (14) transforms as \( u^\nu_{\mu} \rightarrow u^\nu_{\mu} + 2 \partial_\nu \Lambda^\mu \), in complete agreement with the geometrical meaning of the tetrad.

IV. SPIN-2 FIELD IN THE PRESENCE OF GRAVITATION

In order to construct the theory for a spin-2 particle on a curved spacetime, instead of the expansion (11), we consider the tetrad expansion

\[
h^\alpha_{\mu} \equiv \eta^\alpha_{\mu} + u^\alpha_{\mu}, \tag{25}
\]
around the nontrivial classical background \( \eta^\alpha_{\mu} \). Such an approach is analogous to the treatment of a consistent spin-2 model as a first order perturbation of the general relativity theory [2]. We will denote with an overline every other background objects and operations. In order to simplify the computations, in contrast to the above described flat-space discusion, we will choose the symmetric gauge [12, 13, 14] from the very beginning. Then, we have the symmetric tensor field

\[
\phi^\mu_{\nu} := \delta^\mu_{\nu} u^\nu, \quad \phi_{\mu\nu} = \phi^\nu_{\mu}, \tag{26}
\]
where \( \delta^\mu_{\nu} \) is the inverse background tetrad: \( \eta^\alpha_{\mu} \delta^\alpha_{\nu} = \delta^\mu_{\nu} \). From now on, the Greek indices will be raised and lowered with the help of the background metric \( \eta_{\mu\nu} = \delta^\mu_{\nu} \eta_{\mu\nu} \).

Substituting the expansion (25) into the Weitzenböck connection (1), up to first order in \( \phi^\mu_{\nu} \), we find

\[
\Gamma^\beta_{\mu\nu} = \overline{\Gamma}^\beta_{\mu\nu} + \nabla_\nu \phi^\beta_{\mu}, \tag{27}
\]
where $\Gamma_{\mu
u} = \tilde{\Gamma}_{\mu
u} \partial_\nu \tilde{\Gamma}_{\mu\rho} \partial_\rho$ is the background teleparallel connection, and $\nabla_\sigma$ is the covariant derivative in the connection $\Gamma_{\mu\nu}$. As a result, we obtain for the torsion

$$T^\rho_{\mu\nu} = \tilde{T}^\rho_{\mu\nu} + \nabla_\rho \tilde{\phi}_{\mu\nu} - \nabla_\mu \tilde{\phi}_{\rho\nu}. \quad (28)$$

Using this expression in Eqs. (6) and (22), we straightforwardly obtain the kinetic term for the spin 2 field Lagrangian in the presence of gravitation,

$$\mathcal{L}_G = \frac{\epsilon^4}{8 \pi G} \tilde{T} \left( F^\rho_{\mu\nu} F_{\mu\nu} - F^\sigma_{\mu\rho} F_{\mu\rho} \right) + \frac{\epsilon^4}{8 \pi G} \tilde{T} \nabla_\rho \phi_{\mu\nu}. \quad (29)$$

Here, $F^\rho_{\mu\nu}$ is the covariant generalization of the Fierz tensor:

$$F^\rho_{\mu\nu} = \frac{1}{2} \left( \tilde{\nabla}^\rho \phi_{\mu\nu} - \tilde{\nabla}^\rho \phi_{\mu\nu} + \Delta^\rho \left( \tilde{\nabla}^\rho \phi - \tilde{\nabla}_\sigma \phi_{\sigma\mu} \right) - g^{\rho\sigma} \left( \tilde{\nabla}_\sigma \phi - \tilde{\nabla}_\sigma \phi_{\rho\nu} \right) \right). \quad (30)$$

The main difference of the spin-2 model (29) from the theory studied in [4] is that the covariant derivatives in (30) are defined with the help of the teleparallel connection, and not with the help of the Riemannian (Christoffel) connection. We thus obtain an alternative way to describe a spin 2 particle in the presence of gravitation. The resulting theory is obviously generally covariant.

As it is well known [1, 2, 3, 4], the higher spin theories are generally inconsistent in the presence of the electromagnetic and/or gravitational field. Technically, this amounts to the non-vanishing covariant divergence of the corresponding field operator which, in turn, is related to the fact that the covariant derivative has a nontrivial commutator proportional to the curvature. The consistency conditions, derived for the higher spin fields, place strong restrictions on the spacetime curvature which are not fulfilled, in general.

In the teleparallel gravity, the spacetime curvature is zero, whereas the commutator of the covariant derivatives reads

$$\left[ \nabla_\mu, \nabla_\nu \right] = - T^\lambda_{\mu\nu} \nabla_\lambda. \quad (31)$$

Accordingly, the consistency condition for the theory (29) will be non-algebraic, as in the usual formulation, but differential instead: $\tilde{T}^\rho_{\mu\nu} \nabla_\rho \lambda \phi_{\mu\nu} = 0$.

If we demand that the consistency conditions should be satisfied for all field configurations, we then have to conclude that the torsion should vanish. This observation agrees with the general analysis of the higher spin theories on the Riemann-Cartan spacetime [14]. A possible way to avoid the consistency problem is to include non-minimal coupling terms in the Lagrangian [2, 3]. Since the curvature is zero, the non-minimal terms may involve only the torsion tensor. The latter, being of the 3rd rank, necessarily involves the derivative for the construction of the invariant contractions. It is straightforward to see that, adding the non-minimal interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \frac{\epsilon^4}{8 \pi G} \tilde{T} \left( K^\rho_{\lambda\mu\nu} - T^\rho_{\lambda\mu\nu} \right) \tilde{\nabla}_\rho \phi_{\lambda\mu} - \Delta^\rho \left( \tilde{\nabla}^\rho \phi - \tilde{\nabla}_\sigma \phi_{\sigma\mu} \right) + \Delta^\rho \left( \tilde{\nabla}^\rho \phi - \tilde{\nabla}_\sigma \phi_{\rho\nu} \right) \right] \int \tilde{T} \left( K^\rho_{\lambda\mu\nu} - T^\rho_{\lambda\mu\nu} \right) \tilde{\nabla}_\rho \phi_{\lambda\mu} \quad (32)$$

to the Lagrangian (29) yields the Riemannian Fierz Lagrangian of [4]. This allows then to solve the consistency problem.

The theory of massive spin-2 particle [2, 3] was satisfactorily formulated within the framework of the Fierz approach [4]. Our results also admit a direct generalization to the nontrivial mass which we though do not discuss here since it follows along the same lines as in [4].

V. DISCUSSION AND CONCLUSIONS

Novello and Neves [4] have demonstrated that the Fierz representation for the spin 2 theory has a number of advantages as compared to the alternative approaches (such as the non-ambiguity of the order of derivatives and the equivalence to the non-minimal curvature Einstein representation with the fixed coefficients of additional terms [2]). Here, we have shown that the Fierz representation can be naturally understood on the basis of the teleparallel gravity. In particular, we find that (i) the structure of the Fierz tensor defined in [4] through an ad hoc procedure is unambiguously fixed by the teleparallel theory; (ii) the gauge symmetry (23)-(24) underlying the corresponding spin 2 freedom, is manifested straightforwardly; (iii) the linearized Einstein operator arises immediately as a consequence of the fundamental identity (10) of the teleparallel theory. It is worthwhile to note that the antisymmetric piece of the tetrad field, while being dynamically redundant, plays a significant role in the formal derivations. A certain disadvantage of our approach is the need of the non-minimal coupling of the type (32) to solve the consistency problem on the curved spacetime. This is similar to the observations made within the Riemannian approach [2].

Summarizing, in this paper we have developed a new approach to the description of spin 2 in flat and curved spacetime on the basis of the teleparallel gravity theory. This approach appears to be a true origin for the Fierz representation proposed recently in [4].

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