Non-Abelian Giant Gravitons

Bert Janssen\textsuperscript{a} and Yolanda Lozano\textsuperscript{b}

\textsuperscript{a} Instituut voor Theoretische Fysica, K.U. Leuven
Celestijnenlaan 200 D, 3001 Leuven, Belgium
bert.janssen@fys.kuleuven.ac.be

\textsuperscript{b} Departamento de F\'isica, Universidad de Oviedo,
Avda. Calvo Sotelo 18, 33007 Oviedo, Spain
yolanda@string1.ciencias.uniovi.es

ABSTRACT

We argue that the giant graviton configurations known from the literature have a complementary, microscopical description in terms of multiple gravitational waves undergoing a dielectric (or magnetic moment) effect. We present a non-Abelian effective action for these gravitational waves with dielectric couplings and show that stable dielectric solutions exist. These solutions agree in the large $N$ limit with the giant graviton configurations in the literature.\textsuperscript{1}

1 Introduction

It is well-known that a collection of $p$-branes under the influence of a background field strength can undergo an “expansion” into a single, spherical, higher-dimensional $(p+2)$-brane. This is the so-called dielectric effect, a first analysis of which was performed in [1], at the level of the Abelian theory relevant to the description of the single expanded $(p+2)$-brane. It was some years later that the complementary description from the point of view of the lower-dimensional multiple branes was provided [2]. From this perspective, the expansion takes place because the embedding coordinates of the multiple branes are matrix-valued, and give rise to new non-Abelian couplings in the combined Born-Infeld-Chern-Simons action [3, 2]. The macroscopic (Abelian) and the microscopic (non-Abelian) descriptions have their own range of validity [2, 4], which however coincide in the limit where the number $N$ of $p$-branes becomes very large. All physical quantities, such as the energy of the configuration and the radius of the spherical $(p+2)$-brane take in this limit the same values in both descriptions.

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The dielectric effect was first derived (under that name) for the case of D-branes, but it has become clear since that other types of p-branes can undergo the same type of effect: the Abelian description given in [1] was actually done for fundamental strings and a non-Abelian analysis of this effect was provided in [5], in terms of Matrix string theory in weakly coupled (linear) background fields.

The dielectric effect can also occur for multiple coinciding gravitational waves. In [4] the uplifting of the non-Abelian couplings of D0-branes were interpreted as dielectric and magnetic couplings for gravitons in eleven dimensions and similar non-Abelian dielectric couplings have been introduced in the Matrix model action of the DLCQ description of the pp-wave in eleven dimensions [6]. A full derivation of the dielectric couplings in the effective actions for ten-dimensional gravitational waves, at least up to linear order in the background fields, has been given in [7]. For the case of the non-Abelian action for eleven-dimensional waves, the description could be easily extended beyond the level of linear approximation. In both cases solutions were found of multiple gravitational waves (or gravitons) expanding into D2-, D3- and M2-branes.

The question arising then is what is the corresponding, Abelian description of this effect. It has been suggested in the literature [4, 6, 7] that this might be the configurations of p-branes, known as giant gravitons. These giant gravitons [8] are spherical M2- and D3-branes with non-zero (angular) momentum in Minkowski space or AdS\textsubscript{m} \times S\textsuperscript{n} space-times that couple to external flux fields.\footnote{With external flux field we mean fields that form part of the background in which the brane lies and for which the brane is not the source.} They are stable due to a dynamical equilibrium between the tension of the spherical branes that makes them contract and the momentum in an external field that makes them expand.

The aim of this letter is to show that indeed the dielectric effect for gravitational waves of [7] and the giant gravitons of [8] are two complementary descriptions of the same effect. We will compute the radius and the energy of the spherical brane configuration and show that the values obtained in the microscopic, non-Abelian picture coincide in the large N limit with the values in the Abelian, macroscopic picture.

The paper is organised as follows: in section 2 we will review the comparison between the microscopical and the macroscopical pictures of the dielectric effect for the case of D-branes, since this turns out to be fully analogous to the case of the microscopic and macroscopic description of giant gravitons. In section 3 we give the construction of (Abelian) giant gravitons as given in [8]. In section 4 we construct a dielectric M-brane solution from a non-Abelian action for gravitational waves and compare the large N limit of this solution with the Abelian picture and find the the results agree up to order 1/N.

## 2 Macroscopic versus microscopic dielectric branes

Consider a spherical D2-brane probe laying, say, in the \( \theta \) and \( \phi \) directions of flat ten-dimensional spacetime

\[
    ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dy^2_{(6)}.
\]  

(2.1)
This is, as such, obviously not a stable configuration, since the tension of the D2-brane makes the brane contract, while there is no topological obstruction to keep the brane at finite radius. However it is possible to construct a stable configuration, if we switch on an external flux field and make an appropriate choice for the Born-Infeld vector on the D2-brane \[1, 2\]. Taking for the RR 3-form gauge field \(C_{\mu\nu\rho}\) and the Born-Infeld field strength \(F_{ab}\) the following values\(^3\)

\[
\begin{align*}
C_{t\theta\phi} &= -\frac{1}{3}fr^3\sin\theta \quad \implies \quad F_{t\theta\phi} = fr^2\sin\theta, \\
F_{\theta\phi} &= \frac{1}{2}N\sin\theta,
\end{align*}
\]

we notice that the D2-brane is embedded in a constant 4-form flux, while the Born-Infeld vector describes the fact that we have \(N\) D0-branes dissolved in the D2-brane world volume, as can be seen from the Chern-Simons action:

\[
S_{CS} \sim T_2 \int_{S^2 \times \mathbb{R}} P[C_1] \wedge F = NT_0 \int dt \, C_t,
\]

where \(P[C_1]\) denotes the pullback of the RR-vector \(C_{\mu}\) and we used that the tension of the D2 and the D0 are related via \(T_0 = 2\pi\lambda T_2\).

The full Born-Infeld Chern-Simons action

\[
S = -T_2 \int d^3\sigma \, e^{-\phi} \sqrt{\det |g_{ab} + \lambda F_{ab}|} + T_2 \int d^3\sigma \, P[C_3]
\]

gives then rise to the following potential as a function of the radius \(r\) of the spherical brane:

\[
V(r) = 2\lambda^{-1}T_0 \left[ \sqrt{r^4 + \frac{1}{4}\lambda^2N^2} - \frac{1}{3}fr^3 \right],
\]

which for \(N \gg 1\) can be expanded as

\[
V(r) = NT_0 + 2\lambda^{-2}T_0N^{-2}r^4 - \frac{2}{3}\lambda^{-1}T_0fr^3 + \ldots
\]

This potential has two extrema, a maximum at \(r_1 = 0\) and a minimum at \(r_2 = \frac{1}{4}\lambda Nf\), taking the values

\[
V(r_1) = NT_0, \quad \quad V(r_2) = NT_0 - \frac{1}{3}\lambda^2T_0N^3f^4.
\]

Clearly, the spherical D2-brane will stabilise at a finite radius \(r_2\) and has then an even lower energy than if it shrank to zero-size. At this radius, the contraction due to the D2-brane tension is compensated by the tendency of the dissolved D0-branes to expand in the presence of the 4-form flux. The spherical D2-brane has no net global charge under the 4-form field strength (2.2), but does have a local charge, which gives rise to a non-zero

\(^3\)Note that this 4-form RR field in flat space is not a consistent supergravity background. A proper solution of the dielectric effect satisfying the supergravity equations of motion, together with an argument that the probe approximation done here is actually valid, has been given in [9].
dipole moment $\int P[C_3] \neq 0$. Due to this non-zero dipole moment the above configuration is called a dielectric brane.

The same effect can also be described from the point of view of the dissolved D0-branes. It is well known that a set of $N$ coinciding D-branes exhibits an enhanced $U(N)$ symmetry [10] and should therefore be described by a non-Abelian $U(N)$ symmetric effective action and it has been realised that this non-Abelian action contains a set of non-Abelian couplings to the RR field of all ranks [3, 2]. It is precisely these couplings that give rise to the dielectric effect. In particular, a set of $N$ coinciding D0-branes in the presence of an external RR 3-form gauge field is described by the Born-Infeld and Chern-Simons actions

$$S = T_0 \int dt \text{Str} \left\{ \sqrt{[P_{tt} + E_{ii}(Q^{-1} - \delta^{ij}E_{ji})] \det Q} + P[(iX^iX_3)] \right\} \quad (2.8)$$

where

$$E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}, \quad Q^{ij} = \delta^{ij} + i[X^i, X^k]E_{kj}, \quad ((iX^iX_3))_\mu = X^jX^iC_{ij\mu} \quad (2.9)$$

and the $X^i$ are the $U(N)$ matrix valued coordinates transversal to the D0-branes. For the static case in a flat background with an external RR 4-form flux, and after the expansion of the Born-Infeld term and partial integrating the Chern-Simons term, this action gives rise to the potential [2]

$$V(X) = \lambda^{-2}T_0\lambda^2 \text{Str} \left\{ -\frac{1}{4}[X, X]^2 + \frac{1}{6}i\lambda[X^k, X^j]X^iF_{tijk} \right\}, \quad (2.10)$$

which has a solution to the equations of the form [2]

$$F_{tijk} = f\varepsilon_{ijk}, \quad X^i = -\frac{1}{4}\lambda fJ^i. \quad (2.11)$$

The $N \times N$ matrices $J^i$ are the generators of the $N$-dimensional representation of $SU(2)$, satisfying $[J^i, J^j] = 2i\varepsilon^{ijk}J^k$. We thus see that the three transverse coordinates $X^i$, whose eigenvalues indicate the position of the D0-branes in these directions, span a fuzzy (non-commutative) two-sphere with radius

$$r = \sqrt{\frac{1}{N} \text{Tr}(X^iX^i)} = \frac{1}{4}\lambda f \sqrt{N^2 - 1}. \quad (2.12)$$

This can be interpreted as the fact that the coincident D0-branes have, under the influence of the RR 4-form, expanded into a fuzzy, spherical D2-brane. The energy of this configuration can easily be computed and is given by

$$E = -\frac{1}{384}\lambda^2T_0f^4N(N^2 - 1), \quad (2.13)$$

which is lower then the zero-energy solution where all $X^i$ commute. Also here the 4-form field strength gives rise to a dipole moment, but not to a non-zero global D2-brane charge.

Each of these two descriptions, the Abelian macroscopic and the non-Abelian microscopic one, have their own range of validity [2, 4]. The first one is valid when the radius of
the spherical D2 is much bigger then the string scale, when the higher derivative terms in (2.4) are negligible. On the other hand, the non-Abelian calculation is trustable if the commutator corrections in higher orders in $F_{ab}$ are small and the expansion of the Born-Infeld action (2.8) converges rapidly, i.e. for the radius much smaller than $\lambda \sqrt{N}$. Clearly both conditions are simultaneously met for $N \gg 1$. Indeed we see that the expressions for the radii and the energy in both descriptions agree up to terms of the order $1/N$.

Another way to understand the agreement of both descriptions is by looking at the commutator of the $X^i$:

$$[X^i, X^j] = \frac{2iR}{\sqrt{N^2 - 1}} \varepsilon^{ijk} X^k.$$  \hspace{1cm} (2.14)

In the large $N$ limit the commutators will become very small and the non-Abelian character of the microscopic description will diminish rapidly.

3 Abelian giant gravitons

In this section we will review the construction of giant gravitons as presented in [8] and draw the attention to the analogies with the Abelian description of the dielectric effect.

There are several types of giant gravitons, the most relevant ones being the giant graviton in flat spacetime, the genuine giant graviton that lives in the spherical part of $AdS_m \times S^n$ and the dual giant graviton, living in the $AdS$ part of $AdS_m \times S^n$. In this letter we will mainly restrict ourselves to the giant graviton living in the spherical part of $AdS_7 \times S^4$. We will comment briefly on the other cases later on.

Consider in eleven dimensions an M2-brane probe wound around a two-cycle $\Omega_2$ of radius $L \sin \theta$ in the spherical part of $AdS_7 \times S^4$:

$$ds^2 = ds^2_{AdS} + L^2(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega_2^2).$$

$$d\Omega_2^2 = d\chi_1^2 + \sin^2 \chi_1 d\chi_2^2.$$  \hspace{1cm} (3.1)

Given that there are no non-trivial two-cycles in $S^4$, there is no topological obstruction that keeps the M2-brane from contracting to zero radius and slipping off the $S^4$. However when the M2-brane carries momentum in the transversal direction $\phi$, it will couple to the RR 3-form $C_{\phi \chi_1 \chi_2} = L^3 \sin^3 \theta \sin \chi_1$ of $AdS_7 \times S^4$ and will start to expand. A giant graviton is then the equilibrium configuration where the contraction by the brane tension $T_2$ is canceled by the expansion due to the momentum.

The Hamiltonian of the M2-brane with radius $L \sin \theta$ has been computed in [8]:

$$H = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta (1 - \frac{\tilde{N}}{T_2} \sin \theta)^2},$$  \hspace{1cm} (3.2)

where $\tilde{N} = 4\pi T_2 L^3$ is an integer that emerges through the quantisation condition of the 3-form flux on $S^4$. This Hamiltonian (3.2) has two stable minima, one for $\tan \theta = 0$, corresponding to a point-like object, and another for $\sin \theta = P_\phi/\tilde{N}$, corresponding to a finite sized configuration. The value of the Hamiltonian corresponds in both cases to $E = P_\phi/L$, i.e. to a massless particle (hence the name giant graviton). Note that the
finite radius of the $S^4$ implies an upper bound for the radius $L \sin \theta$ of the M2 and hence for the momentum: $P_\phi \leq \tilde{N}$.

We can express the total momentum of the giant graviton as $N$ units of momentum (gravitons): $P_\phi = NT_0$. In terms of the momentum $T_0$ of the gravitons, we find that $\tilde{N} = 2T_0L^3$. Therefore we can rewrite the Hamiltonian and its non-trivial solution as

$$H = \frac{NT_0}{L^3} \sqrt{1 + \tan^2 \theta \left( 1 - \frac{2L^3}{N} \sin \theta \right)^2}, \quad \sin \theta = \frac{P_\phi}{\tilde{N}} = \frac{N}{2L^3}.$$  

(3.3)

and the upper bound for the number of gravitons we can put in as $N \leq 2L^3$.

There is a remarkable analogy between the construction of giant gravitons and the construction of the spherical D2-brane in the macroscopic picture of the dielectric effect. In both cases we are dealing with spherical two-branes that are kept stable by a dynamical equilibrium between the tensions and the interaction with an external field. For the D2-brane this is done via dissolved D0-branes on the world volume, while for the M2 the interaction takes place via the angular momentum of the brane. The picture that arises then is of giant gravitons being M2-branes with dissolved momentum on the world volume. In both cases the radius of the D2 and the M2 are proportional to the number of dissolved D0’s or gravitons.\textsuperscript{4}

This analogy raises an obvious question: does there exist also a complementary, microscopic description of giant gravitons, from the point of view of the dissolved gravitons, of which the Abelian description is the large $N$ limit? For this to be true, we need a non-Abelian action for multiple gravitons (or gravitational waves) with dielectric couplings similar to the ones known for D-branes. In the next section we will show that such a description does indeed exist.

\section{Non-Abelian giant gravitons}

For the case of giant gravitons in flat space, the agreement between the microscopic and macroscopic picture has already been pointed out in [4], as an uplifting from the dielectric D0-brane case in ten dimensions. An analysis for the more complicated cases of giant gravitons in $AdS\times S^n$ is more difficult, since for this a non-Abelian action for dielectric gravitational waves in arbitrary backgrounds is needed.

Such an action was presented in [7], making use of Matrix and Matrix string theory techniques. In the presence of a 3-form potential $C$, this action is given by

$$S = T_0 \int d\tau \, \text{STr} \left\{ k^{-1} \mathcal{P} \left[ E_{tt} + E_{ti} (Q^{-1} - i \delta^{ij} E_{ji}) \right] \det Q \right\} + i P \left[ (ix_i x_i) C \right] \},$$  

(4.1)

where

$$E_{\mu\nu} = G_{\mu\nu} + k^{-1} (i k C)_{\mu\nu}, \quad Q^j_i = \delta^j_i + ik [x^i , x^k] E_{kj}.$$  

(4.2)

\textsuperscript{4}In the case of the giant gravitons in flat space, the analogy is even more striking [4, 7]. This is due to the fact that the Abelian D2-brane description is in fact the dimensionally reduced version of the flat giant graviton.
Note that this action is a gauged sigma model, where the propagation direction of the gravitational waves appears as a Killing direction which is projected out through the effective metric $G_{\mu \nu} = g_{\mu \nu} - k^{-2}k_\mu k_\nu$ and the contraction of the 3-form with the Killing vector $(i_k C)$ [7].

We will not review the construction of this action, but restrict ourselves to two arguments justifying its validity. First of all, the action (4.2) reduces to the well-known action (2.8) for coincident D0-branes derived in [3, 2], when reduced along the propagation direction of the waves. And secondly, in the Abelian limit, one recovers the known effective action for a single graviton. We believe these are non-trivial checks that confirm the interpretation of this action as describing non-Abelian M-theory gravitons.

In order to describe giant gravitons in the spherical part of $AdS_7 \times S^4$, we make the following Ansatz for the coordinates on the fuzzy 2-sphere and the 3-form:

\[ X^i = \frac{L \sin \theta}{\sqrt{N^2 - 1}} J^i, \quad C_{\phi ij} = -\epsilon_{ijk} X^k, \quad (4.3) \]

where $J^i, (i = 1, 2, 3)$ form an $N \times N$ representation of $SU(2)$ and $\phi$ is the propagation direction of the waves. With this Ansatz $(X^i)^2 = L^2 \sin^2 \theta \mathbf{1}$. In this particular background there is no contribution of the Chern-Simons action. The 3-form couples however in the Born-Infeld part of the action, through $E_{ij} = G_{ij} + k^{-1}C_{\phi ij}$, which implies that the fuzzy 2-sphere carries magnetic moment with respect to $C$.

Substituting the Ansatz above into the world volume action (4.2) we obtain the following potential:

\[ V(X) = \frac{T_0}{L \cos \theta} \text{STr}\left\{ \mathbf{1} - \frac{4L \sin \theta}{\sqrt{N^2 - 1}} X^2 + \frac{4L^4 \sin^2 \theta \cos^2 \theta}{N^2 - 1} X^2 X^2 + \frac{4L^2 \sin^2 \theta}{N^2 - 1} X^2 X^2 \right\}, \quad (4.4) \]

where we have taken into account that some contributions to $\text{det} Q$ do in fact vanish after taking the symmetrised average involved in the symmetrised trace prescription in (4.2). Since we are interested in the comparison with the Abelian calculation of section 3, it is convenient to look at the large $N$ limit. When $N$ is large

\[ \text{STr}\{(X^2)^n\} \approx \text{Tr}\{(X^2)^n\} + \mathcal{O}(\frac{1}{N^2 - 1}) = L^{2n} \sin^{2n} \theta N + \mathcal{O}(\frac{1}{N^2 - 1}), \quad (4.5) \]

since the commutators involved in the rewriting of (4.5) can be neglected. Therefore, the potential (4.4) can be written as:

\[ V(\theta) = \frac{N T_0}{L \cos \theta} \sqrt{1 - \frac{4L^3}{\sqrt{N^2 - 1}} \sin^3 \theta + \frac{4L^6}{N^2 - 1} \sin^4 \theta} = \frac{N T_0}{L} \sqrt{1 + \tan^2 \theta (1 - \frac{2L^3}{\sqrt{N^2 - 1}} \sin \theta)^2}, \quad (4.6) \]

where we are neglecting terms of order $(N^2 - 1)^{-\frac{3}{2}}$ in the expansion of (4.4). This potential admits two minima: the point-like graviton at $\sin \theta = 0$ and the giant graviton at

\[ \sin \theta = \frac{1}{2} L^{-3} \sqrt{N^2 - 1}, \quad (4.7) \]

\[ ^{5}\text{Recall that in our description of the gravitons, the propagation direction occurs as an isometry direction, and therefore we are dealing with a static configuration, for which we can compute the potential as minus the Lagrangian.} \]
both with an energy $E = N T_0 L^{-1} = P_\phi L^{-1}$, associated to a massless particle with angular momentum $P_\phi$. As in the Abelian case, due to the finiteness of the radius of the $S^4$ there is again an upper bound on the number of gravitons we can add: $\sqrt{N^2 - 1} \leq 2L^3$.

If we compare this non-Abelian computation with the Abelian one of section 3, it is clear that the radii, the energy, the upper bound for the angular momentum and even the the form of the action agree in the large $N$ limit, up to $1/N^2$ corrections. This justifies the statement that the giant gravitons of [8] are in fact the macroscopical description of a non-Abelian magnetic moment effect for gravitons.

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