To a first order, the evolution of the gravitational field is described by the Einstein equations. The evolution of the gravitational field is given by the equations of motion, which are derived from the Einstein field equations. The equations of motion are

\[ \Box g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

where \( g_{\mu\nu} \) is the metric tensor, \( T_{\mu\nu} \) is the stress-energy tensor, and \( G \) is the gravitational constant. The Einstein field equations are

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

where \( R_{\mu\nu} \) is the Ricci curvature tensor and \( R \) is the scalar curvature. The evolution of the gravitational field is governed by the equations of motion and the Einstein field equations.

The equations of motion and the Einstein field equations are the fundamental equations of general relativity. They describe the behavior of the gravitational field and are used to study the dynamics of spacetime. The equations of motion and the Einstein field equations are the basis of the general theory of relativity.

In this section, we will review the basic concepts of general relativity and the equations of motion and the Einstein field equations. We will also discuss the relationship between the equations of motion and the Einstein field equations and how they are used to study the dynamics of spacetime.
and the constraints
\[ \partial_{\pm} \partial_{\rho} r = (\partial_{\pm} g)^2 - (\partial_{\pm} f)^2. \] (6)

The field \( r \) plays a similar role to the areal radius in spherical symmetry \([8, 12]\).

In vacuum, \( f = g = 0 \), the general solution to the field equations is \([9]\)
\[ r = 2m - 4\lambda^2 x^+ x^- \] (7)
where the origin has been fixed. The constant \( m \) may be interpreted as the mass of the space-time, whose global structure has been described previously \([12]\). For \( m > 0 \) this describes the CGHS static black hole, analogous to the Schwarzschild black hole. The Penrose diagram is shown in Fig. 1(i).

Recently the solution \([10]\)
\[ r = a + 2\lambda^2 (x^+ - x^-)^2, \quad g = 2\lambda (x^+ - x^-), \quad f = 0 \] (8)
has been found, where the origin has again been fixed. If \( a > 0 \), this represents a traversable wormhole, hereafter called the HKL wormhole, with analogous global structure to a Morris-Thorne wormhole: a throat \( r = a \) at \( x^+ = x^- \), joining two regions with \( r > a \), an \( x^+ > x^- \) universe and a reflected \( x^+ < x^- \) universe, as depicted in Fig. 1(ii).

Thus the model naturally contains both static black holes and static traversable wormholes. A characteristic feature of both cases is the trapping horizons, defined by \( \nabla r \cdot \nabla r = 0 \), or equivalently \( \partial_+ r = 0 \) or \( \partial_- r = 0 \) \([5, 6, 7, 8]\). In the CGHS black hole, they coincide with the event horizons \( r = 2m \) at \( x^- \) = 0 and \( x^+ \) = 0 respectively. In the HKL wormhole, there is a double trapping horizon, \( \partial_+ r = \partial_- r = 0 \), at the throat \( r = a \). This illustrates how trapping horizons of different type may be used to locally define both black holes and wormholes \([5]\). Also relevant are the locally trapped regions where \( \nabla r \cdot \nabla r < 0 \), consisting of future trapped regions if \( \partial_+ r < 0 \) or past trapped regions if \( \partial_+ r > 0 \), as occur in black holes or white holes respectively. Locating the trapping horizons and the locally trapped regions is a key feature of the analysis of dynamic situations.

In the following, we wish to take delta-function profiles for the radiation energy densities
\[ \rho_{\pm} = (\partial_{\pm} f)^2 - (\partial_{\pm} g)^2 \] (9)
where units have been fixed. To avoid ill-defined square roots of delta functions, the model may be formally generalized to
\[ \partial_{\pm} \partial_{\pm} r = -\rho_{\pm} \] (10)
\[ \partial_{\pm} \partial_{\pm} r = -4\lambda^2 \] (11)
\[ \partial_{\pm} \partial_{\pm} r = 0 \] (12)
where the energy densities \( \rho_{\pm} \) are now regarded as basic and need not be derived from Klein-Gordon fields. The evolution equations \((11-12)\) have the general solutions
\[ r(x^+, x^-) = r_+(x^+) + r_-(x^-) - 4\lambda^2 x^+ x^- \] (13)
\[ \rho_{\pm}(x^+, x^-) = \rho_\pm(x^\pm) \] (14)
The constraints \((10)\) are preserved by the evolution equations in the \( \partial_\pm \) directions, and so may be reduced to
\[ \partial_\pm \partial_\pm \rho_{\pm} = -\rho_{\pm} \] (15)
The constraints are manifestly integrable for \( r_{\pm} \) given initial data
\[ \rho_{\pm} \quad \text{on } x^\pm = x_0^\pm \] (16)
\[ (r, \partial_+ r, \partial_- r) \quad \text{at } x^+ = x_0^+, \quad x^- = x_0^- \] (17)
for constants \( x_0^\pm \). The data consist of the energy-density profiles of the left-moving and right-moving radiation, plus lower-dimensional data for the metric. Then the general procedure is to specify this initial data according to the desired physical situation, integrate the constraints \((15)\) for \( r_{\pm} \), then the solution follows as \((13-14)\). Consequently, the effect of the radiation is much easier to see than in Einstein gravity, though the model shares various physically important features including gravitational collapse to black holes satisfying cosmic censorship \([12]\).

III. CONSTRUCTION OF A WORMHOLE FROM A BLACK HOLE

We study how to construct a traversable wormhole from a black hole by irradiating it with negative energy. Although a similar idea has been studied previously \([10]\), now we present a simpler solution involving impulsive radiation, which is a preliminary to construct an analytic solution in four-dimensional Einstein gravity. We consider a CGHS black hole subjected to impulsive negative-energy radiation at some positive value \( x_0 \) of the Kruskal-like coordinates \( x^\pm \), with energy density chosen in order to close up the future trapped region by merging its trapping horizons, followed by the constant irradiation needed to maintain the static wormhole
\[ \rho_{\pm} = 4\lambda^2 x_0 \delta(x^\pm - x_0) - 4\lambda^2 \Theta(x^\pm - x_0) \] (18)
where \( \Theta \) is the unit Heaviside step function and \( \delta \) the Dirac (delta-function) distribution. When differentiating to check solutions, it may be useful to remember that the
derivative $\delta'$ of the delta function, as a distribution acting on test functions $f$, satisfies $\delta'f = -\delta f$ or $(\delta f)' = 0$. Of course $\Theta' = \delta$.

Assuming a black hole of mass $m$ in the initial region, we obtain the solution

$$
r = 2m - 4\lambda^2 x^+ x^- + 2\lambda^2 (x^+ - x_0^+ - x^- + x_0^-) \Theta(x^+ - x_0^+) + 2\lambda^2 (x^- - x_0^-) \Theta(x^- - x_0^-),$$

(19)

The solution in the final region $x^\pm > x_0$ can be recognized as an HKL wormhole (8). In more detail, the trapping horizons

$$0 = \partial_x r = 4\lambda^2 (x^+ \Theta(x^+ - x_0^+) - x^-),$$

(20)

are located at

$$x^+ = x^- = 0, \quad x^+ = x^- < x_0^+, \quad x^+ = x^- > x_0^+,$$

(21)

and their radii are

$$r_0 = \begin{cases} 2m, & x^+ < x_0^+ > x^- = x_0^- \leq x_0 \leq 0, \\ 2m - 4\lambda^2 x_0^2, & x^+ > x^- > x_0. \end{cases}$$

(22)

as depicted in Fig. 2. In this solution, the throat radius $a = 2m - 4\lambda^2 x_0^2$ of the wormhole is smaller than the horizon radius $2m$ of the initial black hole. Thus we require $2\lambda^2 x_0^2 < m$ in order that a wormhole is constructed.

The results are similar to the previous solution [10], except that here the trapping horizons move discontinuously rather than continuously, a well-known property under infinitesimally thin mass shells [13]. This is a general feature of the solutions presented in this article, stemming from the fact that the field equations or Einstein equations relate $\partial_r \partial_\tau r$ to the energy densities $\rho_\pm$, so that delta-function $\rho_\pm$ causes discontinuous $\partial_r r$. This is, of course, an idealization of a situation where a

![Fig. 3: An HKL wormhole is subjected to a double burst of impulsive radiation with equal positive and negative energy. The operation shifts one wormhole mouth away from, then back to, its original position. The shaded regions are (i) past or (ii) future trapped, respectively expanding or contracting, so the wormhole becomes respectively larger or smaller.](image)

concentrated packet of radiation causes swift movement of the trapping horizon.

A recently discovered four-dimensional wormhole solution [14] can be similarly constructed from a Schwarzschild black hole in full Einstein gravity [15]. By the time reverse, we can also obtain a picture where a wormhole collapses into a black hole by beaming in impulsive radiation at the moment of switching off the supporting ghost radiation. In this case, the horizon radius of the black hole is larger than the throat radius of the initial wormhole. This reduces to the sudden collapse case [10] without the impulsive radiation and $x_0 = 0$.

IV. WORMHOLE ENLARGEMENT AND REDUCTION

We are interested in how to create wormholes with throat large enough for human beings to pass from one universe to another. It is practically useful if it can be achieved by processes from our universe only. In this section, we study wormhole operation by energy balance from one universe only. We irradiate the wormhole from our universe with impulsive radiation of equal positive and negative energy at different times:

$$
\rho_+ = -4\lambda^2 - \beta^2 \delta(x^+ - x_0) + \beta^2 \delta(x^+ - x_1),
\rho_- = -4\lambda^2
$$

(23)

where $x_1 > x_0$, so that the positive-energy impulse follows the negative-energy impulse. Assuming a wormhole of throat radius $a$ initially, we obtain the solution

$$
r = a + 2\lambda^2 (x^+ - x^-)^2 + \beta^2 (x^+ - x_0) \Theta(x^+ - x_0) - \beta^2 (x^+ - x_1) \Theta(x^+ - x_1),$$

(24)
Assuming a wormhole with initial throat radius $a$, we obtain the solution

$$r = a + 2\Lambda^{2} x^{2} + \frac{\beta^{2}(x, x_{0}) e^{x_{0}}}{x(x_{0})^{2}}$$

(28)

This also describes an HKL [h, k, l] in the final region $x > x_{1}$. The initial and final region have throat radius $a = \beta = 0, 4\Lambda^{2} x^{2}$, with radii

$$r_{0} = \begin{cases} a, & x = 0, x_{0} > x_{1} \hfill \text{(29)} \\
\beta^{2}(x, x_{0}) e^{x_{0}}, & x = 0, x_{0} < x_{1} \end{cases}$$

So that the throat of the final state becomes larger than that of the initial state.

On the other hand, if $x_{1} < x < 0$, where the negative energy impulsive, the energy impulse decreases the final state becomes smaller than that of the initial state. (Fig. 3ii). Then we need $\beta^{2}(x, x_{0}) e^{x_{0}}$, when the throat radius $a = \beta = 0, 4\Lambda^{2} x^{2}$.

This suggests how to operate the back-action of the trajectory to the initial and final state.

V. SYMMETRIC WORMHOLE ENLARGEMENT

Now we construct solutions to enlarge the throat by beaming in impulsive radiation symmetrically from both universes. We give two solutions, a simple case which can be so generalized, and a more delicate one which can be so generalized, as expected from the first law of wormhole dynamics [6].

Then the middle region is vacuum and the same part of a CGHS white hole. Assuming the initial region $a < x_{1}$ to be an HKL, the solution

$$\rho_{+} = 4\Lambda^{2} \beta^{2}(x, x_{0}) e^{x_{0}} + \frac{\beta^{2}(x, x_{0}) e^{x_{0}}}{x(x_{0})^{2}}$$

(30)
\[
\begin{align*}
    r &= a - 4\lambda^2(x^+ x^- + x_0^2) + 4\lambda^2 x_0(x^+ + x^-) + 2\lambda^2(x^+ - x_0)^2 \Theta(x_0 - x^+) + 2\lambda^2(x^- - x_0)^2 \Theta(x_0 - x^-) \\
    &\quad + \beta^2(x^+ - x_0) \Theta(x^+ - x_0) + \beta^2(x^- - x_0) \Theta(x^- - x_0) - \alpha^2(x^+ - x_1) \Theta(x^+ - x_1) \\
    &\quad - \alpha^2(x^- - x_1) \Theta(x^- - x_1) + 2\lambda^2(x^+ - x_1)^2 \Theta(x^+ - x_1) + 2\lambda^2(x^- - x_1)^2 \Theta(x^- - x_1). \\
\end{align*}
\] (31)

Now we want the final region \( x^+ > x_1 \) to be an HKL wormhole in the usual coordinates, Fig. 5. Then we find the relations

\[
x_0 = -\frac{\beta^2}{4\lambda^2}, \quad x_1 = -\frac{\alpha^2}{4\lambda^2}
\] (32)

between the energy and timing of the impulses. This simplifies the solution in the initial, middle and final regions:

\[
r = \begin{cases} 
  a + 2\lambda^2(x^+ - x^-)^2, & x_0 < x^+ < x_1 \\
  a + 4\lambda^2 x_0^2 - 4\lambda^2 x^+ x_0, & x_0 < x^+ < x_1 \\
  a + 4\lambda^2(x_0^2 - x_1^2) + 2\lambda^2(x^+ - x_1)^2, & x_1 < x^+, 
\end{cases}
\] (33)

which are recognizable as CGHS (7) white-hole or HKL wormhole (8) regions. The wormhole throats are at \( x^+ = x^- \) with radii

\[
r_0 = \begin{cases} 
  a, & x_0 < x^+ < x_1 \\
  a + 4\lambda^2(x_0^2 - x_1^2), & x_1 < x^+. 
\end{cases}
\] (34)

Thus the throat radius of the final wormhole is larger than that of the initial one, again due to the negative-positive energy ordering of the burst.

Note that \( x_0 < x_1 < 0 \), so that \( \alpha^2 < \beta^2 \), meaning that the positive-energy impulsive radiation does not completely balance the negative-energy impulsive radiation, since some supporting negative energy has already been removed. The difference \( \alpha^2 - \beta^2 = -4\lambda^2(x_1 - x_0) \) equals the energy \( \int x_0^2 \rho_4 \, dx^+ \) missing as compared with the static wormhole.

Combining with the results from the previous section, we have confirmed that the radius of the wormhole throat is enlarged when an expanding region of past trapped surfaces is opened and closed between the initial and final static wormholes, by bifurcating and merging the wormhole mouths, defined as trapping horizons. In addition, the solution just presented is one of the simplest where the wormhole is enlarged, in the sense that each relevant region is part of either a static white hole, a static wormhole or a pure-radiation region, joined at null boundaries, which is also possible in full Einstein gravity [15].

\section{VI. SUMMARY}

In this paper, we have used an exactly soluble dilaton gravity model to study wormhole dynamics under impulsive radiation, finding solutions where a traversable wormhole is created from a black hole or the throat radius of a wormhole is enlarged or reduced, the size being controlled by the energy and timing of the impulses. Where the solutions consist of black-hole, static-wormhole and pure-radiation regions matched along null boundaries, we can succeed to construct similar analytic solutions in full Einstein gravity, though the analytical details are much more complex [15].

The recipe to enlarge the wormhole is to cause the wormhole mouths to bifurcate, opening up an expanding region of past trapped surfaces, then merge again, by adding additional negative energy followed by compensating positive energy. The general proof involves the first and second laws of wormhole dynamics and is a future important work in the unified framework for black-hole and wormhole dynamics [5]. The second law determines whether the area increases or decreases, and the first law quantifies it in terms of the energy supplied and work done.

The results in this paper show how to create a traversable wormhole of human essence in principle, if negative-energy matter can be controlled. Self-inflating wormholes were discovered recently [11], but the present solutions are the first to describe stable wormhole enlargement. Clarifying the dynamical behavior of wormholes is a quite attractive subject, since the cosmic short-
cuts and time travel usually considered as science fiction are thereby closer to being realized.

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