Effective theory for close limit of two branes

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We discuss the effective theory for the close limit of two branes in a covariant way. To do so we solve the five dimensional Einstein equation along the direction of the extra dimension. Using the Taylor expansion we solve the bulk spacetimes and derive the effective theory describing the close limit. We also discuss the radion dynamics and braneworld black holes for the close limit in our formulation.

I. INTRODUCTION

The purpose of this paper is to write down the effective theory for two branes system at the closed limit. This kind of system is a situation just before the collision of two branes. Since the brane is a key object in superstring theory and braneworld scenarios \([1–4]\), the collision process of branes definitely could be an important and fundamental process. In a braneworld the brane-brane collision could provide us a new picture of the creation of the big-bang universe \([5–8]\). A brane-bubble collision also will be interesting because it might give us a new scenario of inflation and reheating process \([9]\). Thus it is worth deriving an effective theory on the brane for close limit of two branes.

To derive an effective theory on the brane, we must solve the bulk spacetime. In the ordinary Kaluza-Klein theory it was a trivial task. In the braneworld, it is quite non-trivial because the gravity on the brane is non-trivially coupled to the gravitational fields in the bulk. Recently it has been reported that we can do so for a two branes system if we employ the derivative expansion scheme \([10–12]\), that is, long wave approximations. The two junction conditions on the branes gives us sufficient boundary conditions to determine the bulk geometry. Thus, the bulk geometry could be uniquely determined. Finally, a low energy effective theory can be derived (See also Refs. \([13,14]\) for earlier works and Refs. \([15–18]\) for the other issues.). In this paper, on the other hand, we point out that we can apply the Taylor expansion to the closed limit of two branes. In this method, the high energy effects can be easily seen, which are difficult to deal with in low energy approximations. The price of it is that the separation between two branes should be small.

In this paper we will use a toy model to concentrate on the dealing of the gravitational effect and dynamics for two branes system of the Randall-Sundrum model \([2]\) where the bulk stress energy is a cosmological constant. The brane will be treated as the thin domain wall. In this approximation, we will be able to use the Israel junction formalism \([19]\). However a problem comes out here. From the derivation it is obvious that the Israel junction condition cannot be applied to just a moment of the collision of two branes with \(Z_2\)-symmetry. That is, under \(Z_2\)-symmetry, the size of the extra dimension shrinks to zero at the collision. So it is fair to say that our present analysis cannot treat the collision in Randall-Sundrum type models. Our formalism can be applied just before the collision of two branes. In order to treat the collision, we need the interaction between branes, that is, the brane field action which will be reduced to something such as the Born-Infeld action \([20]\) at the thin wall limit. This is, however, beyond the scope of present study.

In this paper we will derive the effective theory describing the close limit of two branes and discuss the radion dynamics, which expresses the evolution of the brane separation, and its relation to the bulk geometry. We also give an insight into the braneworld black holes. Our present study can be regarded as the generalization of that by Khoury and Zhang \([21]\) who considered the Friedman brane cosmology in an empty bulk and discussed the meaning of dark radiation \([22]\). We will work in the covariant curvature formalism \([23,24]\) which is most useful to keep the four dimensional covariance and obtain the full view.

The rest of this paper is organised as follows. In Sec. II, we review the covariant curvature formalism. In Sec. III, we solve the bulk geometry by Taylor expansion and then derive the four dimensional effective theory at the close limit. In Sec. III the radion dynamics and the braneworld black holes will be addressed. Finally we will give the summary and discussion in Sec. IV.

II. COVARIANT CURVATURE FORMALISM

In this section we review the covariant curvature formalism developed in Refs. \([23,24]\). Note that the central issue is how to solve the bulk seriously. Without this step we cannot obtain a correct effective theory on the brane.

We employ the metric form

\[ ds^2 = e^{2\phi(x)}dy^2 + q_{\mu\nu}(y,x)dx^\mu dx^\nu. \] (1)

In the above it is supposed that the positive and negative tension branes are located at \(y = 0\) and \(y = y_0\),
respectively. The proper distance between two branes is given by
\[ d_0(x) = \int_0^{y_0} e^{\phi(y)} dy = y_0 e^{\phi(x)}. \] 
where \( q_{\mu\nu}(y, x) \) is the induced metric of \( y = \text{const} \) hypersurfaces. At the close limit of two branes we expect that the coordinate used in the metric of Eq. (1) does not break down for a wide class of the braneworld spacetimes.

From the Gauss-Codacci equations, we have two key equations
\[
(4) G_{\nu}^{\mu} = \frac{3}{2} \delta_{\nu}^{\mu} + K K_{\nu}^{\mu} - K_{\alpha}^{\mu} K_{\nu}^{\alpha} - \frac{1}{2} \delta_{\nu}^{\mu} (K^2 - K_{\beta}^{\beta} K_{\alpha}^{\alpha}) = E_{\nu}^{\mu} \quad (2)
\]
and
\[
D_{\mu} K_{\nu}^{\mu} - D_{\nu} K = 0, \quad (3)
\]
where \( D_{\mu} \) is the covariant derivative with respect to \( q_{\mu\nu} \), \( \ell \) is the bulk curvature radius. \( (4) G_{\mu\nu} \) is the four-dimensional Einstein tensor with respect to \( q_{\mu\nu} \) and \( K_{\mu\nu} \) is the extrinsic curvature of \( y = \text{constant} \) hypersurfaces defined by
\[
K_{\mu\nu} = \frac{1}{2} \xi \partial_{\gamma} q_{\mu\nu} = \nabla_{\mu} n_{\nu} + n_{\mu} D_{\nu} \phi, \quad (4)
\]
where \( n = e^{-\phi} \partial_{y} \). Here note that \( a^{\mu} = n^{\mu} \nabla_{\nu} n^{\nu} = -D^{\mu} \phi(x) \). \( E_{\mu\nu} \) is a part of the projected Weyl tensor defined by
\[
E_{\mu}^{\nu} = (5) C_{\mu\nu\beta\sigma} n^{\beta} n^{\sigma} = -D^{\mu} D_{\nu} \phi - D^{\mu} \phi D_{\nu} \phi - \xi \nabla_{\mu} K_{\nu}^{\alpha} + 2 D_{\mu} \phi D_{\nu} (\phi) \xi E_{\mu}^{\nu} + \frac{1}{2} \delta_{\nu}^{\mu}, \quad (5)
\]
where \( (5) C_{\mu\nu\beta\sigma} \) is the five-dimensional Weyl tensor.

The junction conditions on the branes give us the relation between the extrinsic curvatures and the energy-momentum tensor on the branes:
\[
[K_{\mu}^{\nu} - \delta_{\nu}^{\mu} K]_{y=0} = -\frac{\kappa^2}{2} \left( -\sigma_1 \delta_{\nu}^{\mu} + T_1^{\mu \nu} \right) \quad (6)
\]
and
\[
[K_{\nu}^{\mu} - \delta_{\mu}^{\nu} K]_{y=0} = -\frac{\kappa^2}{2} \left( -\sigma_2 \delta_{\mu}^{\nu} + T_2^{\mu \nu} \right). \quad (7)
\]

\( T_1^{\mu \nu} \) and \( T_2^{\mu \nu} \) are the energy-momentum tensor localised on the positive and negative branes. \( \sigma_1 \) and \( \sigma_2 \) are the brane tensions. If one substitutes the above conditions to Eq. (2), we might be able to derive the Einstein equation on the brane [23]. However, it turns out that \( E_{\mu\nu} \) is not still known, because it is a five-dimensional quantity. For the single brane \( E_{\mu\nu} \) is identical to the contribution from the Kaluza-Klein modes in the linearised theory [25] and it will be higher order correction at large distance, that is, \( E_{\mu\nu} \) is negligible at low energy in the linearised theory. For the two branes system, on the other hand, \( E_{\mu\nu} \) is not negligible as seen later. Anycase, we need the evolution equation for \( E_{\mu\nu} \) to the bulk.

To evaluate \( E_{\mu\nu} \) in the bulk, we derive its evolution equation. The result is given by
\[
\xi E_{\mu\nu}^{\alpha\beta} = \xi D^{\mu} B_{\mu(\alpha\beta)} + K^{\mu(4)} C_{\mu\nu\alpha\beta} + 4 K_{(\alpha} E_{\beta)\mu}^{\mu} - \frac{3}{2} K E_{\alpha\beta} - \frac{1}{2} q_{\alpha\beta} K_{\nu}^{\nu} E_{\mu\nu} + 2 D^{\beta} \phi B_{\beta(\mu\nu)}
\]
\[
+ 2 \tilde{K}_{\mu}^{\alpha} \tilde{K}_{\nu}^{\beta} - \frac{7}{6} \tilde{K}_{\mu}^{\alpha} K_{\nu}^{\mu} \tilde{K}_{\alpha}^{\beta} - \frac{1}{2} q_{\mu\nu} \tilde{K}_{\alpha}^{\beta} \tilde{K}_{\alpha}^{\nu} \tilde{K}_{\nu}^{\rho}, \quad (8)
\]
where \( B_{\mu\nu} = q_{\mu\nu} (5) C_{\mu\nu\alpha\beta} n^{\alpha} n^{\beta} \) and \( \tilde{K}_{\mu}^{\alpha} = K_{\mu}^{\alpha} - \frac{1}{4} q_{\mu\nu} K \). Since the right-hand side contains \( B_{\mu\nu} \), \( K_{\mu\nu} \), and \( (4) C_{\mu\nu\alpha\beta} \), we need their evolution equations too. After a long calculation we obtain
\[
\xi E_{\mu\nu} = \xi D^{\mu} B_{\mu(\alpha\beta)} + 2 D^{\beta} \phi B_{\beta(\mu\nu)}
\]
\[
+ 2 \tilde{K}_{\mu}^{\alpha} \tilde{K}_{\nu}^{\beta} + 2 B_{\alpha\beta} K_{\mu\nu}^{\mu} + 2 D_{\mu} E_{\nu(\alpha\beta)} + 2 D_{\nu} E_{\mu(\alpha\beta)} + 2 D_{\mu\nu} E_{\alpha\beta}
\]
\[
+ (4) R_{\mu\rho\nu\alpha} - K_{\mu\alpha} K_{\nu\beta} + K_{\mu\beta} K_{\nu\alpha}) D_{\gamma} \phi = 0, \quad (9)
\]
and
\[
e^{-\phi} \xi \partial_{\gamma} K_{\mu}^{\nu} = -D^{\mu} D_{\gamma} \phi - D^{\mu} \phi D_{\gamma} \phi - K_{\mu}^{\nu} K_{\alpha}^{\alpha} + \frac{1}{2} \delta_{\nu}^{\mu}, \quad (11)
\]

Equation (11) is a rearrangement of Eq. (5).

The junction condition directly implies the boundary condition on the branes for \( K_{\mu\nu} \) and \( B_{\mu\nu} \) because of
\[
B_{\mu\nu} = 2 D_{\mu} E_{\nu(\alpha\beta)} \quad (12)
\]

For the discussion in the next section, we write down the second derivative of \( K_{\mu}^{\nu} \):
\[
\partial_{\gamma} K^{\mu}_{\nu} = e^{2\phi} \left[ K_{\alpha}^{\mu} D_{\nu(\alpha)} \phi + K_{\nu}^{\mu} D_{\nu(\alpha)} \phi + 3 K_{\alpha}^{\mu}(D_{(\beta} D_{\nu(\gamma)} \phi + D_{\gamma} \phi D_{\nu(\beta)} \phi) \right]
\]
\[
+ K_{\nu}^{\mu}(D_{(\beta} D_{\nu(\gamma)} \phi + D_{\gamma} \phi D_{\nu(\beta)} \phi) - K_{\gamma}^{\mu} E_{\nu}^{\gamma} + K_{\nu}^{\mu} E_{\nu}^{\gamma} + \frac{3}{2} K E_{\nu}^{\nu} + \frac{1}{2} \delta_{\nu}^{\mu} K_{\nu}^{\nu} E_{\nu}^{\gamma} + D_{\alpha} \phi D_{\mu} K_{\nu}^{\nu} C_{\nu}^{\gamma} - K_{\nu}^{\mu}(D_{\phi} \phi) E_{\nu}^{\gamma}
\]
\[
+ B_{\nu(\alpha\beta)} D_{\gamma} \phi - D_{\nu(\gamma)} \phi (-B_{\alpha\beta}^{\nu} + B_{\alpha\beta}^{\mu}) + \chi_{\nu}^{\mu} \quad (13)
\]
where \( \chi_{\nu}^{\mu} \) is defined by
\[
\chi_{\nu}^{\mu} = -\frac{1}{2} (D_{\beta} B_{\nu(\beta}^{\alpha}) D_{\nu(\alpha)}^{\mu} - K_{\gamma(\beta}^{\mu}(4) C_{\gamma(\alpha)}^{\nu})
\]
\[
+ 2 K_{\nu}^{\mu} K_{\gamma}^{\beta} K_{\beta}^{\gamma} - \frac{7}{6} K_{\nu}^{\mu} K_{\gamma}^{\beta} K_{\beta}^{\gamma} - \frac{1}{2} \delta_{\nu}^{\mu} K_{\gamma}^{\beta} K_{\beta}^{\gamma} K_{\gamma}^{\mu}. \quad (14)
\]
Using Eqs (11) and (13), Eq. (15) becomes
\[
E^\nu_\nu = \delta_\nu^\nu T d_\nu + \frac{1}{\ell}E^{(0)}_\nu + O(d_\nu),
\]
where \(K^\mu_\mu\) and \(E^\mu_\nu\) in the right-hand side is evaluated at \(y = 0\).

Let us regard Eq. (16) as the equation for \(E^\mu_\nu\). To solve the equation, we assume that two branes are close enough to satisfy the conditions
\[
d_0/\ell, \quad \kappa^2 T_1 d_0, \quad \kappa^2 T_2 d_0 < 1.
\]

Then we expand \(E^\mu_\nu\) as
\[
E^\mu_\nu = E^{(1)}_\nu + \frac{1}{d_0} E^{(0)}_\nu + O(d_0).
\]

Substituting the above into Eq. (16) and solving iteratively we see that \(E^{(1)}_\nu\) and \(E^{(0)}_\nu\) has the following form:
\[
E^{(1)}_\nu = K^\mu_\nu (y_0, x) - K^\mu_\nu + D^\mu D_\nu d_0
\]
and
\[
E^{(0)}_\nu = K^\mu_\nu (y_0, x) - \frac{1}{\ell^2} \delta_\nu^\mu - \frac{1}{2} \left( - K^\mu_\nu + K^\nu_\mu \right) + \frac{3}{2} K^\mu_\nu + \frac{1}{2} \delta_\nu^\mu \kappa^\alpha K^\alpha_\nu
\]
\[
+ \frac{1}{2} K^\mu_\nu \frac{(Dd_\nu)^2}{d_0} - \frac{1}{2} D_\nu d_0 D^\alpha_\nu K^\alpha_\nu
\]
\[
+ \frac{1}{2} D^\alpha_\nu (2B^\alpha_\mu - B^\alpha_\mu).
\]
where we defined
\[ \frac{1}{\ell_1} = \frac{1}{6} \kappa_1^2 \sigma_1 \quad \text{and} \quad \frac{1}{\ell_2} = -\frac{1}{6} \kappa_2^2 \sigma_2. \tag{22} \]

Since we write down \( E_\nu^\mu \) in terms of four dimensional quantities, we can derive the effective gravitational equation on the brane
\[
(4) G_\nu^\mu = 6 \left( \frac{1}{\ell_1} - \frac{1}{\ell_2} \right) \delta_\nu^\mu - 9 \left( \frac{1}{\ell_1} - \frac{1}{\ell_2} \right) \delta_d^\mu \delta_\nu^d + \frac{3}{d_0} \left( \frac{1}{\ell_2} - \frac{1}{\ell_1} \right) \delta_\nu^\mu + \frac{\kappa_1^2}{2d_0} (T_1^\mu_\nu + T_2^\nu_\mu) \\
+ \frac{2\kappa_1^2}{2} \left( \frac{1}{\ell_1} - \frac{1}{\ell_2} \right) (T_1^\mu_\nu + \frac{\kappa_1^2}{2\ell_1} (T_1^\mu_\nu - 3T_2^\mu_\nu) \\
+ \left( 1 - \frac{d_0}{\ell_1} \right) \frac{D^\mu D_\nu d_0 - \delta_\nu^d D^2 d_0}{d_0} \\
+ \frac{2D^\mu d_0 D_\nu d_0 + \delta_\nu^d (Dd_0)^2}{2d_0}. \tag{24} \]

Here reminded that the first term is much bigger than the second term in the first line of the right-hand side. It is easy to check that the above is identical to the close limit of the low energy effective theory obtained in Refs. [10,11].

IV. APPLICATIONS

In this section we consider the applications of our formalism: the radion dynamics and the braneworld black holes.

A. Radion dynamics and bulk geometry

First, let us remember the work of Khoury and Zhang [21]. They investigated the modified Friedman equation on the brane in empty bulk and observed that \( E_{00} \) is proportional to the energy density on the brane. It corresponds to Eq (19) in our general analysis of the close limit because we can neglect the effect of bulk curvature for \( d_0 \ll \ell \). At the leading order of \( d_0/\ell, d_0\kappa_1^2 T \), the effective equation on the positive tension brane becomes
\[
(4) G_\nu^\mu = -E_\nu^\mu, \\
-E_\nu^\mu = \frac{\kappa_1^2}{2d_0} (T_1^\mu_\nu + T_2^\nu_\mu) + \frac{D^\mu D_\nu d_0 - \delta_\nu^d D^2 d_0}{d_0}, \\
D^2 d_0 = \frac{\kappa_1^2}{6} (T_1 + T_2). \tag{25} \]

From these effective equations, it is easy to derive the modified Friedman equation on the brane
\[
H^2 = -H \frac{d_0}{d_0} + \frac{\kappa_1^2 (\rho_1 + \rho_2)}{6d_0}, \tag{26} \]

where \( H \) is the Hubble parameter on the positive tension brane. \( \rho_1 \) and \( \rho_2 \) are energy densities on positive tension brane and negative tension brane, respectively. This agrees with Khoury and Zhang’s equation. From the evolution equation for radion and the conservation law for the energy-momentum tensor, the right-hand side of Eq. (26) can be rewritten as
\[
H^2 = C a^{-4}, \tag{27} \]

where \( a \) is the scale factor on the positive tension brane and \( C \) is the integration constant of the radion \( d_0 \). Thus we can understand the behavior of dark radiation in terms of the dynamics of the radion and that the bulk geometry is related to the initial condition for the radion. Hence the present formalism gives us a straightforward way to see what they observed.
Next we discuss the radion dynamics for the deSitter-like branes. For simplicity we consider the vacuum energy cases
\[ \ell = \ell_1 = \ell_2 \] (28)
and
\[ T^\mu_1_\nu = -\lambda_1 \delta^\mu_\nu, \quad T^\mu_2_\nu = -\lambda_2 \delta^\mu_\nu. \] (29)
Then the branes will be deSitter spacetimes if the maximum symmetry is imposed. But, we do not restrict ourselves to the consideration on the exact deSitter branes. In this case the equation for the radion is extremely simplified as
\[ \frac{2\kappa^2}{3d_0} \left[ (1 + \frac{d_0}{\ell}) \lambda_1 + (1 - \frac{d_0}{\ell}) \lambda_2 \right] = 0, \] (30)
where \( 1/L = 1/\ell + \kappa^2 \lambda_1/6 \). Defining \( \varphi = d_0/\ell (\ll 1) \) and the conformally transformed metric \( g_{\mu\nu} = (1 + \varphi)^{-1} q_{\mu\nu} \), the equation becomes
\[ D^2 \varphi - V'(\varphi) = 0, \] (31)
where
\[ V(\varphi) \simeq \alpha \varphi + \beta \varphi^2 + \cdots. \] (32)
The constants \( \alpha \) and \( \beta \) are defined by
\[ \alpha = \frac{2\kappa^2}{3\ell} (\lambda_1 + \lambda_2) \] (33)
and
\[ \beta = -\frac{\kappa^2}{18} (\lambda_1 - \lambda_2) + \frac{\kappa^4}{18} \lambda_1 \lambda_2. \] (34)
The close limit corresponds to \( \varphi \to 0 \) limit. At the very close limit, the potential is approximated by
\[ V(\varphi) \simeq \alpha \varphi. \] (35)
So the brane could collide if \( \alpha < 0 \) due to the attractive force. If \( \alpha > 0 \) the repulsive force acts between two branes. This feature is same as the result in Ref. [8]. The term proportional to \( \beta \) describes the correction in the next order. We should emphasize that high-energy terms such as \( \kappa^2 \lambda_1 \lambda_2 \) can be included in our method. In the case of \( \alpha = 0 \), the next order should be taken into account and the potential becomes
\[ V(\varphi) \simeq -\frac{2\kappa^2}{3\ell} \lambda_1 \left( 1 + \frac{12}{\ell} \lambda_1 \ell \right) \varphi^2. \] (36)
From the above we can see that the repulsive force will be induced between two branes when \( \lambda_1 > 0 \).

In general the potential at the close limit is given by
\[ V(\varphi) \simeq \frac{\kappa^2}{6L} (T_1 + T_2) \varphi + \cdots. \] (37)
Thus the attractive force acts if \( T_1 + T_2 > 0 \) and the repulsive force if \( T_1 + T_2 < 0 \). If one thinks of the dust universe, the repulsive force does.

C. An implication into black holes

Let us finally consider brane world black holes in the case of \( \ell = \ell_1 = \ell_2 \) and \( T^\mu_1_\nu = T^\mu_2_\nu = 0 \). In this case the equation for the radion and the gravitational equation become
\[ \frac{1}{d_0} \left( 1 + \frac{d_0}{\ell} \right) D^2 d_0 - \frac{(Dd_0)^2}{\ell d_0} = 0 \] (38)
and
\[ (4)G^\mu_\nu = \left( 1 - \frac{d_0}{\ell} \right) D^\mu D^\nu d_0 - \frac{\delta^\mu_\nu D^2 d_0}{d_0} + \frac{2D^\mu d_0 D^\nu d_0 + \delta^\mu_\nu (Dd_0)^2}{2\ell d_0}. \] (39)
Up to this order the higher order correction disappears because they are written by the energy-momentum tensor on the branes. The above system is equivalent to the scalar-tensor theory. In the present case it is easy to see that the static and spherical solution on the brane is only Schwarzshild spacetimes due to the Bekenstein theorem [26]. At the close limit, thus, the black holes in the bulk are described by the adS black string solution [27]. However, if the brane distance increase, it is natural to expect that there should be black holes solution confined on the each branes. That is, there might be a phase transition from black holes to the adS black string [28]. Thus we hope we will be able to see this transition using our formalism. This expectation might be supported because the higher order corrections than that we considered here will contain the terms like \( D^\alpha (4)R_{\mu\nu\alpha\beta} D^\beta \varphi \) and \( (4)R_{\mu\nu\alpha\beta}(4)R^\alpha\beta \) and then the modified solution from the Schwarzshild could be emerged such as the situation in the superstring theory [29].

V. SUMMARY AND DISCUSSION

In this paper we derived the effective theory at the close limit of two branes and discussed the radion dynamics/braneworld black holes. Our effective theory is complimentary with the low energy effective theory [10,11] and can be regarded as the generalisation of the Khoury and Zhang’s recent work [21].

Finally we should stress on the limitation of our formulation. (i) To determine the bulk spacetime we used
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