Analytical Calculation of Current Distribution in Multistrand Superconducting Cables

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Abstract—In recent years the problem of current distribution in multistrand superconducting cables has received increasing attention for large scale superconductivity applications due to its effect on the stability of fusion magnets and the field quality of accelerator magnets. A modeling approach based on distributed parameters has revealed to be very effective in dealing with long cables made of some tens or hundreds of strands. In this paper we present a fully analytical solution equation for a distributed parameters model in cables made of an arbitrary number of strands, whose validity is subjected to symmetry conditions generally satisfied in practical cables. We give in particular analytical formulae of practical use for the estimation of the maximum strand currents, time constants and redistribution lengths as a function of the cable properties and the external voltage source.

Index Terms—Current distribution, superconducting cables.

I. INTRODUCTION

SUPERCONDUCTORS for large scale applications are often manufactured from a large number of strands or tapes. The strands and tapes are twisted or transposed in order to reduce the induced currents and the resulting AC loss generated by time dependent external magnetic fields. The transposition is however usually incomplete due to the following reasons:

- in a twisted cable the strands are still coupled with the self-field generated by the transport current;
- field gradients longitudinal and transverse to the cable axis induce imbalance of the flux linked by the single strands;
- manufacturing tolerances or cable compaction result in deviations from the ideal geometry.

Current distribution among the strands of a large, multistage superconducting cable is of great interest for magnets as it affects the stability and AC loss properties of the cable. Here we study current distribution by means of a distributed parameters model [1], [2] that gives rise to a system of parabolic (diffusion) differential equations. The model is a special case of a multi-conductor transmission line for which standard electrical engineering theory applies. We will show that in special but relevant cases it is possible to find an analytic solution of the model equations for a cable made of an arbitrary number N of strands.

An analytical treatment of current distribution and redistribution under much simplified conditions of a cable consisting of two strands has already been undertaken by several authors [3]–[6]. In particular Krempasky and Schmidt [5] have per-

Fig. 1. Distributed parameters circuit model of the elemental mesh of cable used to describe current distribution in multistrand superconducting cables.

formed an extensive study of the transient and steady state response of a two-strand cable subjected to a field transient with variation of the field derivative along the cable axis. For this particular case they have given the closed-form analytical solution of the equation and have analyzed in detail the transient current diffusion.

In this work we report the analytical solution of the equations of current diffusion for cables made of an arbitrary number of strands. Our general solution reduces to the solution given in [5] when a two-strand cable exposed to a localized external flux change is considered. We then give analytical formulae of practical use for the estimation of the maximum strand currents, time constants and redistribution lengths as a function of the cable properties and the strength of the external voltage sources inducing current imbalance.

II. MODEL FOR CURRENT DISTRIBUTION

The model for the current distribution that we have developed [1], [2] is shown schematically in Fig. 1 (for three strands). The cable has a total length L and is composed of N strands. The details of the internal structure of a single strand are neglected. The key postulate in the model is the hypothesis that the current can flow continuously from each strand to all the other strands through distributed contacts. Similarly the voltages induced by external sources, such as, e.g., a time dependent external magnetic field dB/dt are also distributed along the cable length. The N strands have initial currents i_i and voltages V_i at the coordinate x (with $i = 1, N$). Over an elemental length dx the currents change by di_i because of current transfer across the interstrand contact resistances $R_{i,j} = 1/(g_{i,j} dx)$, where $g_{i,j}$ is the interstrand conductance per unit length between the i th and j th strand. The voltages drop by dV_i due to inductive voltages, and the voltage source v_i^{ext} . We neglect the effect of parallel resistance to avoid complicating the discussion, although it can be included easily [1], [2] and [7].

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The current in each strand evolves as dictated by the following parabolic system of partial differential equations:

$$\mathbf{g} \mathbf{l} \frac{\partial \mathbf{i}}{\partial t} + \frac{\partial^2 \mathbf{i}}{\partial x^2} - \mathbf{g} \mathbf{v}^{ext} = 0 \quad (1)$$

where t is time, x is the distance along the cable axis, $\mathbf{i}$ is the vector that contains the N strand currents, while $\mathbf{g}$ and $\mathbf{l}$ are the cable conductance and inductance matrices respectively, that we define as follows:

$$\mathbf{l} = [l_{i,j}] \quad (2)$$

$$\mathbf{g} = \begin{bmatrix} -\sum_{\substack{k=2 \\ k \neq 1}}^N g_{1,k} & g_{1,2} & \cdots & g_{1,N} \\ g_{2,1} & -\sum_{\substack{k=1 \\ k \neq 2}}^N l_{2,k} & \cdots & g_{2,N} \\ \vdots & & & \\ g_{N,1} & g_{N,2} & \cdots & -\sum_{\substack{k=1 \\ k \neq N}}^N g_{N,k} \end{bmatrix} \cdot \quad (3)$$

The $l_{i,j}$ in (2) are the inductance coefficients among strands, defined for a unit of strand length. In this study we neglect the cable transport current, so that the boundary conditions can be written as:

$$i_k(x=0, t) = i_k(x=L, t) = 0 \quad \text{with} \quad k = 1, \dots, N. \quad (4)$$

This choice results in a drastic simplification of the formulae discussed here without loss of generality. For the most general case, including also the effect of a longitudinal resistance in the strand, we refer to [1], [2] and [7].

III. ANALYTIC SOLUTION

Equation (1) for a cable of length L with initial and boundary conditions given by (4) can be solved analytically when the matrices $\mathbf{l}$ and $\mathbf{g}$ are symmetric and circulant [7]. We discuss in [7] the meaning of these hypotheses. Here it is sufficient to note that both properties are verified either exactly or on average in typical superconducting cables for large scale applications and power transmission, so that the analytical solution discussed here is valid. In the scope of this paper we restrict further our analysis to the special case of uniform longitudinal conductance $g_{i,j} = g$, and identical self inductances $l_{i,i}$ and mutual inductances $l_{i,j}$ for all the strands and strand couples. The self and mutual inductances are indicated with $l = l_{i,i}$ and $m = l_{i,j}$ respectively. We finally consider the cable subjected to a longitudinal voltage excitation localized over a short length δ placed in the middle of the cable and acting for a time t_1 . This situation is representative of the voltage induced by a changing external magnetic flux in the presence of a localized transposition error in the cable, or in the initial phase of a normal transition. The resulting evolution of the current distribution will be used to establish the typical response characteristics for a more general

excitation. Two situations are considered: the case of a voltage source acting on all strands (but different from strand to strand) and the particular case of a voltage source localized in a single strand.

A. Voltage Source on All Strands

The voltage per unit length is v_k^{ext} on the k th strand. The analytic solution of (1) under these conditions can be written as follows [7]:

for $t \leq t_1$:

$$i_k(x, t) = \frac{4}{\pi \alpha} I_k \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left[\frac{1}{n^2} (1 - e^{-(t/\tau_n)}) \sin\left(\frac{n \alpha x}{w}\right) \sin(n \alpha) \right] \quad (5)$$

for $t > t_1$:

$$i_k(x, t) = \frac{4}{\pi \alpha} I_k \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left[\frac{1}{n^2} (1 - e^{-(t_1/\tau_n)}) e^{-((t-t_1)/\tau_n)} \cdot \sin\left(\frac{n \alpha x}{w}\right) \sin(n \alpha) \right] \quad (6)$$

where we have used the following definitions for the parameters appearing in (5) and (6):

$$\alpha = \pi \frac{L - \delta}{2L} \quad (7)$$

$$w = \frac{L - \delta}{2} \quad (8)$$

$$\tau_n = \frac{\tau}{n^2}. \quad (9)$$

The analytic solution expressed by (5) and (6) consists of two separate parts: an exponential time dependence with a sequence of time constants τ_n , where the first mode $\tau_1 = \tau$ has the longest time constant, and a space dependence expressed by the trigonometric series. The term I_k in the multiplying constant of (5) and (6) is the regime current reached in each strand under steady excitation, defined as follows:

$$I_k = \frac{wg\delta}{2} \sum_{\substack{j=1 \\ j \neq k}}^N (v_k^{ext} - v_j^{ext}). \quad (10)$$

The total current in a strand of a multi-strand cable only depends on the voltage differences among strands and it can be obtained as the linear combination of the currents flowing in all possible loops formed by strand-couples in a cable. This value of the regime current I_k is reached only after a time sufficiently long if compared to the longest time constants of the system. For the specific conditions of uniform cable self and mutual inductance chosen a single time constant τ defines the dynamic response. The definition of τ is:

$$\tau = N(l - m)g \left(\frac{L}{\pi} \right)^2 \quad (11)$$

B. Voltage Source on One Strand Only

A special case of the above equations is obtained considering that the external voltage source acts only on one strand, e.g., the first one in the cable. For convenience we define the total voltage V^{ext} acting on the strand as the product of the voltage per unit length v_1^{ext} and the length δ :

$$V^{ext} = \delta v_1^{ext}. \quad (12)$$

The solution for this case is identical to the one given in the previous section apart for the regime current that can be written explicitly for all strands using (10). In the first strand the regime current is:

$$I_1 = (N-1) \frac{wgV^{ext}}{2} \quad (13)$$

while in all the other strands the regime current is the same and is given by:

$$I_{k \neq 1} = -\frac{wgV^{ext}}{2}. \quad (14)$$

IV. DISCUSSION

The solution given by (5) and (6) is formally identical to the results found by Krempaski and Schmidt in [5]. Indeed in the case of a cable of two strands ($N = 2$) with voltage localized in the first strand it reduces exactly to the case analyzed by Krempaski and Schmidt. As the expressions given above are however valid in a more general case it is interesting to examine the scaling predicted as a function of the number of strands in the cable, i.e., as a function of the size of the cable.

A. Regime Current

The regime current scales for a cable of N strands as expected from the results on 2 strands [5]. It is proportional to the interstrand conductance and the cable length (through w). In addition, from (13), the current is proportional to the number of strands N in the cable. We can interpret this scaling as a result of the fact that the current flowing in one strand has multiple return paths provided by the other $N - 1$ strands in the cable.

B. Time Constant

As for the case of 2 strands the time constant is proportional to the stray inductance of the loop formed by two strands ($l-m$), the interstrand conductance and the square of the cable length. In addition the time constant scales with the number of strands in the cable. Similar to the regime current, this scaling can be interpreted as the result of the multiple return paths for the induced current in a multi-strand cable. The return currents from each strand in the cable can close on several parallel resistive paths, thus reducing the equivalent resistance while maintaining the loop inductance constant, and leading to an increase of the time constant.

C. Characteristic Length

In order to examine the scaling of the characteristic length for the current diffusion we define a normalized current:

$$\eta_k = \frac{i_k}{I_k} \quad (15)$$

and a normalized time θ :

$$\theta = \frac{t}{\tau}. \quad (16)$$

If we consider (5) the normalized current is given by:

$$\eta_k(x, t) = \frac{4}{\pi\alpha} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left[\frac{1}{n^2} (1 - e^{-n^2\theta}) \sin\left(\frac{n\alpha x}{w}\right) \sin(n\alpha) \right] \quad (17)$$

where we note that in normalized time the space dependence of η_k does not make reference to the number of strands in the cable. The same result applies to the current decay in (6). This means that the characteristic length as well as the scaled profile of the current diffusion at a given normalized time is the same for any cable size. The only parameters changing as a function of the number of strands in the cable are the absolute current and the speed at which the diffusion takes place, through the dependence on the regime current and the time constant. We recall, however, that a significant change in the cable size as implied by a change in the number of strands may affect substantially the cable inductance and conductance, thus invalidating the hypotheses made for this study.

V. EXAMPLES OF APPLICATION

A. Short Twisted Cable

To show explicitly the scalings discussed above we have computed the evolution of the current in the strands of a cable of length $L = 2.3$ m as caused by a voltage source of $10 \mu\text{V/m}$ in one strand. The voltage source is localized in the center of the cable with $\delta = 0.1$ m and acts for a time t_1 equal to 10 s. The cable self and mutual inductances are $l = 0.5 \mu\text{H/m}$ and $m = 0.25 \mu\text{H/m}$, while the conductance is $g = 7.463 \text{ MS/m}$, so that the time constant τ for a 2-strand cable ($N = 2$) is 2 s. These conditions resemble, in the case of 2 strands, the experimental conditions examined in [8].

In Fig. 2 we plot the current in the strands in the center of the cable (i.e., at $x = 1.15$ m) for the cases $N = 2$ and $N = 5$. The current rises in the first strand driven by the voltage source, approaching steady-state conditions for times much longer than the time constant. The current in the other strands is negative, so to maintain the total zero current condition in the cable, and equal in all strands. We see in Fig. 2 the clear increase in the time constant as a function of the number of strands. As soon as the voltage source is removed the current decays. The maximum current reached during the evolution increases going from $N = 2$ to $N = 5$. The increase is however less than implied by (13) because regime conditions are reached only in the case of $N = 2$, while for $N = 5$ the currents are still in transient.

Fig. 2. Comparison of the evolution of current in the strands of 2.3 m long cables made of 2 and 5 strands. A $10 \mu\text{V/m}$ voltage acts on the first strand (positive current), while the currents in all other strands are negative and equal to each other.

B. A LHC Rutherford Cable

We have considered next a Rutherford cable of the LHC type, made of $N = 28$ strands with 1 mm diameter, 15.1 mm width, an average thickness of 1.89 mm and a twist pitch of 115 mm. For this cable the inductance matrix is more complex than assumed previously, as the mutual inductances for all possible strand-couples differ depending on the relative distances of the strands. In this case, as discussed in [7], the solution of the system (1) has multiple time constants that can be calculated from the eigenvalues of $\mathbf{g}$ and $\mathbf{l}$. However, all self inductances l are equal (for a sufficiently long cable) and it is hence possible to define an effective mutual inductance m such that the magnetic energy stored in an arbitrary current distribution (e.g., uniform current) is the same. The equivalent inductances are $l = 1.1 \mu\text{H/m}$ and $m = 0.6 \mu\text{H/m}$. The interstrand conductance for a LHC cable is expected to be around 1 MS/m or smaller. Taking a short cable length, e.g., 2.3 m as in the case considered previously, the time constant τ is computed to be 7.2 s, compared in Fig. 3 to the spectrum obtained using the method described in [7]. The time constant obtained using homogenized parameters is well centered in the spectrum and thus provides a good estimate of the average response of the cable. Note finally that in a LHC magnet the piece length of a cable is of the order of 500 m, and the time constant for current diffusion resulting from the

Fig. 3. Spectrum of time constants computed for a 2.3 m long Rutherford cable of the LHC class (28 strands), as compared to the average time constant estimated using equivalent inductances $l = 1.1 \mu\text{H/m}$, $m = 0.6 \mu\text{H/m}$ and $g = 1 \text{ MS/m}$ in (11). Only one half of the spectrum (14 of 28 modes) is plotted because of symmetry.

scaling (11) gives then an exceedingly high value of $3.4 \cdot 10^5 \text{ s}$, or 4 days.

VI. CONCLUSION

We have reported a closed form analytical solution for the current distribution in a superconducting cable made of N strands. The solution is the natural extension of the two-strand model used by other authors, but has the advantage of generality and broader validity. The analytical solution predicts that the time constant of current redistribution in a cable with uniform properties should scale with the number of strands. A similar scaling applies to the regime current imbalance caused by a localized voltage source as, e.g., a strand transposition error in conjunction with an external field change or a localized resistive transition. In contrast, once time and current are normalized, the current distribution length seems to be a constant, dependent on the strand inductances and interstrand conductances but independent of the number of strands. These scaling predictions should be verified by a dedicated experiment to provide a validation of our theory.

REFERENCES

- [1] L. Bottura, "Modeling stability in superconducting cables," *Physica C*, vol. 310, pp. 310–316, 1998.
- [2] A. Akhmetov, L. Bottura, and M. Breschi, "A continuum model for current distribution in Rutherford cables," *IEEE Trans. Appl. Sup.*, vol. 11, pp. 2138–2141, 2001.
- [3] B. Turck, "Influence of a transverse conductance on current sharing in a two-layer superconducting cable," *Cryogenics*, vol. 24, pp. 448–454, August 1974.
- [4] G. Ries, "Stability in superconducting multistrand cables," *Cryogenics*, vol. 20, pp. 513–519, 1980.
- [5] L. Krempasky and C. Schmidt, "Theory of 'supercurrents' and their influence on field quality and stability of superconducting magnets," *J. Appl. Phys.*, vol. 78, no. 9, pp. 5800–5810, 1995.
- [6] N. Mitchell, "Analysis of nonuniform current distribution effects in multistage cable-in-conduit conductors," *Cryogenics*, vol. 39, pp. 539–556, 1999.
- [7] L. Bottura, M. Breschi, and M. Fabbri, "Analytical solution for the current distribution in multistrand superconducting cables," *J. Appl. Phys.*, to be published.
- [8] L. Krempasky and C. Schmidt, "Experimental verification of 'supercurrents' in superconducting cables exposed to AC fields," *Cryogenics*, vol. 39, pp. 23–33, 1999.