We define where $\beta f - \beta = e$ and $\frac{d}{e} \frac{d}{f}$, and $\frac{d}{e} \frac{d}{f}$ is their covariance. We have
\[(\gamma x^2)^{-\frac{1}{2}} (\gamma x^2)^{-\frac{1}{2}} = (\gamma x^2)^{-\frac{1}{2}} \frac{d}{e} \frac{d}{f} \gamma x^2 = \frac{d}{e} \frac{d}{f} \gamma x^2 \frac{d}{e} \frac{d}{f} \gamma x^2\]
\[(\gamma x^2)^{-\frac{1}{2}} (\gamma x^2)^{-\frac{1}{2}} = (\gamma x^2)^{-\frac{1}{2}} \frac{d}{e} \frac{d}{f} \gamma x^2 = \frac{d}{e} \frac{d}{f} \gamma x^2 \frac{d}{e} \frac{d}{f} \gamma x^2\]

\[(\gamma x^2)^{-\frac{1}{2}} (\gamma x^2)^{-\frac{1}{2}} = (\gamma x^2)^{-\frac{1}{2}} \frac{d}{e} \frac{d}{f} \gamma x^2 = \frac{d}{e} \frac{d}{f} \gamma x^2 \frac{d}{e} \frac{d}{f} \gamma x^2\]
confirming both (5) and our numerical accuracy.

The cut in \( \tilde{G} \) itself is

\[
\Delta \tilde{G}(x, y; -i \gamma) = -2 i \gamma \frac{q(x) f(x, -i \gamma) f(y, -i \gamma)}{J_+(-i \gamma) J_-(-i \gamma)}
\]

[cf. below (2) for the \( \pm \) subscripts]. Although \( f \) can be stably integrated when matching to a Born approximation instead of to \( f_{\rightarrow -\infty} \sim e^{-\gamma x} \) \cite{20}, we prefer Jaffe's series \cite{22, 23}. Figure 3 shows a typical result. For some refinements, see Ref. [1]. For \( \gamma > 0 \), (6) reproduces \( G(x, y; t \rightarrow \infty) \) \cite{15}.

The important limit is \( x, y \rightarrow \infty \). Since \( f(y, -i \gamma) \sim e^{\gamma y} \) in general also \( f(x, -i \gamma) \sim e^{\gamma x} \), (6) has a strong position dependence—simply a result of the long propagation time, and removed if \( t \) is measured from the first arrival at \( t_0(x, y) \equiv x - y \). Define \( G^L(t') \equiv \lim_{x, y \rightarrow \infty} G(x, y; t_0 + t') \), so \( \tilde{G}^L \equiv J^- - J^+ \); see Fig. 4 for results. Figure 5, line a shows \( f_{5, 4} g_{5, 4} ; e^{-\gamma x} \times \Delta \tilde{G}^L(-i \gamma) \approx \Delta \tilde{G}^L(t') \) (accurate except for very small \( t' \)).

The cut \( \Delta \tilde{G}^L(-i \gamma) \) vanishes at (a) the zeros of \( q(\gamma) \) \cite{6}, and (b) \( \gamma = \frac{1}{2}, 1, \frac{3}{2}, \ldots \). The former depend only on \( V(x \rightarrow \infty) \) \cite{22}, the latter only on \( V(x \rightarrow \infty) \) scaling with \( \lambda \). If the two tails are separately adjusted the sequences are independent, but for \( \lambda = 1 \) they share one member at \( \Gamma \). Some members of the two sequences are close, cf. the insets in Fig. 4a. The many zeros also keep \( \Delta \tilde{G}^L \) small generically, cf. the discussion.

**Unconventional QNM.** Although \( \Delta \tilde{G}^L(-i \Gamma) = 0 \), surprisingly \( \Delta \tilde{G}^L \) is largest (close to a dipole) near \( \gamma = \Gamma \) (Fig. 4b); the dominance is less pronounced for finite \( y \), cf. Fig. 3. This reveals a pair of nearby QNM poles \( \omega_{\pm} \): if \( \tilde{G}^L_{\pm}(\omega) \approx (a_{\pm} i \omega) / (i \omega - \gamma \pm \imath \omega) \), with \( a_{\pm} \), \( a_2 \) real and \( c > 0 \), so \( \omega_{\pm} \) are on the unphysical sheets, then

\[
\Delta \tilde{G}^L(-i \gamma) \approx \frac{2 i a_2 (\gamma - \Gamma)}{(\gamma - \Gamma - b)^2 + c^2}.
\]

**Fig. 2:** Plot of \( q(\gamma) \).

**Fig. 3:** Plot of \(-i \Delta G(x, y; -i \gamma) \) for \( x = 0.2 \), \( y = 0.1 \).
Here, $a_1 c + a_2 b = 0$ enforces the zero at $\gamma = \Gamma$. The broken line in Fig. 4b shows this fit, yielding

$$\omega_+ + i \Gamma = \mp c - ib \approx \mp 0.027 + 0.0033 i.$$  

(8)

The extrapolation into the unphysical sheet can also be carried out analytically, by assuming that $J_+(\infty \Omega)$ can be linearized up to the nearest zero,

$$\omega_+ + i \Gamma \approx \frac{J_1(\Omega)}{J'_1(\Omega)}.$$  

(9)

Since $\Delta J(\Omega) = 0$, the sheet of $J(\Omega)$ need not be indicated. Following the methods of [19], one readily obtains

$$J'_1(\Omega) = \frac{i}{2^{\frac{1}{2}} 2\pi 10^{-6}} \left[ -17122265640585(\gamma F + \ln 8 - i \pi) ight.$$ 

$$- 24581051235861775 Ei(8 + i\eta)e^{8} + 36326230655979688 \right].$$  

(10)

where $\gamma F$ is Euler's constant, $Ei(z) \equiv \int_{-\infty}^{z} \frac{dt}{e^t/t}$, and $\eta > 0$ is an infinitesimal. Insertion into (9) yields

$$\omega_+ + i \Gamma \approx -0.03248 + 0.003436i;$$  

(11)

especially the agreement of $\text{Im} \omega_+$ with (8) is remarkable. Since the latter value is also found by extrapolation, it need not be more accurate.

Discussion. Like the hydrogen atom in quantum mechanics, the Schwarzschild black hole is the simplest compact object in relativity, and its spectrum also contains discrete and continuum parts. While the former is a classic of physics, the latter is much more difficult, being spheroidal rather than hypergeometric [22]. We have characterized the continuum, recovering the behavior both for $\gamma \downarrow 0$ and near the miraculous point $\Gamma$. This leads to the identification of an essentially new type of damped excitation $\omega_\pm$, which clearly affects the dynamics more than QNMs on the physical sheet at larger $|\text{Im} \omega_+|$. Consider the Kerr hole. By comparing numerics for moderately small $a$ [24] with the QNM multiplet found analytically to branch off from $\Omega$ at $a = 0$, we conclude that one additional multiplet has to emerge (as $a$ increases) near $\Omega$. Rather than the possibilities considered in Ref. [19], this multiplet may well be due to $\omega_\pm$ splitting (since spherical symmetry is broken) and moving through the NIA as $a$ is tuned. As a first step, the continuum should be studied also for $a > 0$.

One is led to consider another Fourier contour going into the unphysical sheet and detouring around $\omega_\pm$ (Fig. 1, line b), including them as QNMs (Fig. 5, line b). This reduces the continuum (often neglected as “background”), see Fig. 5, line c. Moreover, if these poles emerge onto the physical sheet when tuning a parameter (say, $a$), the total QNM and continuum contributions now become separately continuous.

In terms of $t'$, also the QNMs can be handled for $x, -y \to \infty$. The various relative amplitudes are then completely fixed—unlike usually in (Q)NM analysis. Figure 5 includes the contribution of the least damped
QNMs, at \( \omega_1 = \pm 0.747 - 0.178i \). Since \( \omega_1 \) dominates in signal analysis, the comparison shows that the “background” is small—even smaller if \( \omega_\pm \) are regarded as QNMs, cf. Fig. 1, line b and above. This will be relevant once gravitational waves are detected [2].

All of the above refers to \( \ell = 2 \). The much larger \( \gamma = \Gamma(\ell \geq 3) \) are still unattainable numerically, but since any unconventional poles must be further away from \( \Omega \) [1], their influence on the cut should be smaller.

The inverse problem of recovering \( V(x) \) from eigenvalues is well-studied for closed systems [25]: two real spectra suffice for a finite interval. For open systems, we conjecture one only needs one complex spectrum—provided the discrete QNMs are complete. The cut (rendering them incomplete) is likely to hamper inversion, but, intriguingly, the extended family of QNMs (including the unconventional ones) may be complete for the dynamics (apart from the prompt signal) and permit inversion, even though \( V \) is not finitely supported [18]. The present at least shows that one pair of nearby poles on the unphysical sheet already dominates the cut.

These questions may be explored through solvable models with potential tails. Some aspects of the RWE can also be analyzed asymptotically [21]. Numerical algorithms valid on the NIA and even into the unphysical region would also be useful, allowing QNMs there to be studied directly rather than through extrapolation.

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[20] If the OWC would be imposed at \( x = L \), one needs accuracy \( e^{-2x_L} \) to exclude an \( O(1) \) admixture of \( g(+\gamma) \). Since \( V \) is not finitely supported, \( L \to \infty \), so the KGE cannot be integrated directly. Instead, one continues from the upper to the lower half \( \omega \)-plane—implicit in all analytic formulas [22] and in the OWC itself. See further P.T. Leung et al., Phys. Lett. A 247, 253 (1998).