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Abstract

Using Ehlers and unitary transformations, from Kerr solution of the Ernst equation, we build a parametrized Kerr solution depending on three parameters.

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1 Introduction

In a preceding paper [1] we have presented a new three parameters axisymmetric stationary solution of Ernst equation by using Ehlers and unitary transformations on the Bonanos [2] solution. The parameters are associated with the total mass M of the source and its angular momentum per unit mass a . The third parameter produces a topological deformation of the ergosphere.

Here we apply these transformations to the Kerr metric and we obtain again a solution with three parameters. The two parameters, M and a , are present here too, and we call the third parameter m_1 . We show that with a simple transformation the solution reduces to that of Kerr again, but notwithstanding, m_1 has a peculiar role. It allows to classify the three topological families of Kerr solutions obtained for the three cases $M > a$, $M = a$ and $M < a$, which until now were discussed separately ([3] see p 375). Varying the parameter m_1 we can pass continuously from one family to the other. We illustrate clearly this passage by drawing the ergospheres and horizons for different values of m_1 .

2 Method of solution of Ernst equation

The element of a general axisymmetric stationary spacetime is the so called Papapetrou metric, which in cylindrical coordinates, ρ , z and ϕ , reads

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (1)$$

where the gravitational potentials, f , ω and γ , are functions of ρ and z only. The canonical coordinates of Weyl, ρ and z , can be given in terms of prolate spheroidal coordinates, λ and μ , by the relations

$$\rho = k(\lambda^2 - 1)^{1/2}(-\mu^2)^{1/2}, \quad z = k\lambda\mu, \quad (2)$$

where $k > 0$ is a constant, λ a radial coordinate and $-1 \leq \mu \leq 1$ is an angular coordinate. The metric (1) with relations (2) can be rewritten like

$$ds^2 = f(dt - \omega d\phi)^2 - \frac{k^2}{f} \left[e^{2\gamma}(\lambda^2 - \mu^2) \left(\frac{d\lambda^2}{\lambda^2 - 1} + \frac{d\mu^2}{1 - \mu^2} \right) + (\lambda^2 - 1)(1 - \mu^2)d\phi^2 \right], \quad (3)$$

where the potentials are now functions of λ and μ . Einstein vacuum field equations reduce to the Ernst equation [4],

$$(\xi\bar{\xi} - 1)\nabla^2\xi = 2\bar{\xi}\nabla\xi \cdot \nabla\xi, \quad (4)$$

where ∇ and ∇^2 are the gradient and the three-dimensional Laplacian operators respectively, $\bar{\xi}$ is the conjugated complex potential of ξ , and in general its solution can be expressed as

$$\xi(\lambda, \mu) = P(\lambda, \mu) + iQ(\lambda, \mu), \quad (5)$$

where P and Q are real functions of λ and μ . To determine the potentials f , ω and γ of the metric (3), the method consists to use the following relation between f , the twist potential Φ and ξ ,

$$f + i\Phi = \frac{\xi - 1}{\xi + 1}, \quad (6)$$

which implies, with (5),

$$f = \frac{P^2 + Q^2 - 1}{R^2}, \quad \Phi = \frac{2Q}{R^2}, \quad (7)$$

where

$$R^2 = (P + 1)^2 + Q^2. \quad (8)$$

In (6), Φ is a twist potential defined up to a constant and related to the dragging ω by the following differential equations,

$$\frac{\partial \omega}{\partial \lambda} = \frac{k(1 - \mu^2)}{f^2} \frac{\partial \Phi}{\partial \mu}, \quad \frac{\partial \omega}{\partial \mu} = -\frac{k(\lambda^2 - 1)}{f^2} \frac{\partial \Phi}{\partial \lambda}. \quad (9)$$

The potential ω is obtained by integration of (9), and γ is determined by quadratures. Any solution of Ernst equation is a solution of Einstein equations.

3 Parametrized Kerr solution

We start with the Kerr solution ([5] see p 382),

$$\xi_K = P_K + iQ_K, \quad (10)$$

with

$$P_K = p\lambda, \quad Q_K = q\mu, \quad (11)$$

where p and q are real constants satisfying

$$p^2 + q^2 = 1. \quad (12)$$

By means of the following particular Ehlers transformation on (10),

$$\xi_1 = \frac{c_1 \xi_K + d_1}{\bar{d}_1 \xi_K + \bar{c}_1}, \quad (13)$$

where c_1 and d_1 are complex constants satisfying

$$\begin{pmatrix} c_1 & d_1 \\ \bar{d}_1 & \bar{c}_1 \end{pmatrix} \in SU(1, 1), \quad |c_1|^2 - |d_1|^2 = 1. \quad (14)$$

We choose for c_1 and d_1

$$c_1 = 1 + i\alpha_1, \quad d_1 = i\alpha_1, \quad (15)$$

being α_1 a real constant. Then, a second step consists to perform an unitary transformation on ξ_1 ,

$$\xi_2 = -e^{i\theta_0}\xi_1 = (m + in)\xi_1, \quad m^2 + n^2 = 1, \quad (16)$$

with θ_0 an arbitrary real constant, and m and n real constants. Now we find with (10,13,16)

$$\xi_2 = \frac{A + iB}{C + iD}, \quad (17)$$

where

$$\begin{aligned} A &= P_K(m - \alpha_1 n) - Q_K(\alpha_1 m + n) - \alpha_1 n, \\ B &= P_K(\alpha_1 m + n) + Q_K(m - \alpha_1 n) + \alpha_1 m, \\ C &= -\alpha_1 Q_K - 1, \\ D &= \alpha_1(P_K + 1). \end{aligned} \quad (18)$$

Choosing

$$\alpha_1 = -\frac{n}{2(1+m)} \quad (19)$$

and applying the method recalled in section 2, we find the potentials corresponding to the solution (1), with (7), of the Ernst equation,

$$f = \frac{p^2\lambda^2 + q^2\mu^2 - 1}{(p\lambda - 1)^2 + q^2\mu^2} \cos^{-2} \frac{\theta_0}{2}, \quad (20)$$

$$\Phi = -\frac{2p\mu}{(p\lambda - 1)^2 + q^2\mu^2} \cos^{-2} \frac{\theta_0}{2}, \quad (21)$$

$$\omega = \frac{2kq(p\lambda - 1)(1 - \mu^2)}{p(p^2\lambda^2 + q^2\mu^2 - 1)} \cos^2 \frac{\theta_0}{2}. \quad (22)$$

We observe that for the calculation of Φ in (21) an adding constant that can be transformed away since has no role for the calculation of ω in (22). The factor $\cos^{-2}(\theta_0/2)$ in (20) can be absorbed by a rescaling process, such as $ds_2^2 = \cos^{-2}(\theta_0/2)ds^2$. Introducing the coordinates r and θ through the Boyer-Lindquist transformation

$$\lambda = \frac{r - M}{k}, \quad \mu = \cos \theta, \quad (23)$$

we obtain asymptotically $r \rightarrow \infty$ for (20) and (22)

$$f \approx 1 + \frac{2k}{p} \frac{1}{r} + O\left(\frac{1}{r^2}\right), \quad (24)$$

$$\omega \approx 2 \left(\frac{k}{p}\right)^2 \frac{\sin^2 \theta}{r} \cos^2 \frac{\theta_0}{2} + O\left(\frac{1}{r^2}\right), \quad (25)$$

which shows that the solution is asymptotically flat. From (24,25), differently from Kerr metric, we have

$$p = -\frac{k}{M}, \quad q = \frac{a}{M} \cos^{-2} \frac{\theta_0}{2} \quad (26)$$

and from (12) we have,

$$k^2 = M^2 - m_1^2 a^2, \quad (27)$$

where

$$m_1 = \cos^{-2} \frac{\theta_0}{2}. \quad (28)$$

with $m_1 \in [1, \infty)$.

Finally, with the above choice for the constants we can rewrite the potentials like

$$f = 1 - \frac{2Mr}{r^2 + m_1^2 a^2 \cos^2 \theta}, \quad (29)$$

$$\omega = \frac{2aMr \sin^2 \theta}{r^2 - 2Mr + m_1^2 a^2 \cos^2 \theta}, \quad (30)$$

$$e^{2\gamma} = \frac{r^2 - 2Mr + m_1^2 a^2 \cos^2 \theta}{r^2 - 2Mr + m_1^2 a^2 \cos^2 \theta + M^2 \sin^2 \theta}. \quad (31)$$

We see from (29-31) that when $m_1 = 1$ the Kerr metric is reobtained. Asymptotically, $r \rightarrow \infty$, (29,30) now becomes

$$f \approx 1 - \frac{2M}{r} + \frac{2m_1^2 a^2 M \cos^2 \theta}{r^3} + O\left(\frac{1}{r^5}\right), \quad (32)$$

$$\begin{aligned} \omega &\approx \frac{2aM \sin^2 \theta}{r} + \frac{6aM^2 \sin^2 \theta}{r^2} \\ &+ \frac{2aM \sin^2 \theta (4M^2 - m_1^2 a^2 \cos^2 \theta)}{r^3} + O\left(\frac{1}{r^4}\right). \end{aligned} \quad (33)$$

Now writing the metric (29-31) in the form

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2, \quad (34)$$

we obtain for the metric coefficients,

$$g_{tt} = 1 - \frac{2Mr}{r^2 + m_1^2 a^2 \cos^2 \theta}, \quad (35)$$

$$g_{t\phi} = -\frac{2Mar \sin^2 \theta}{r^2 + m_1^2 a^2 \cos^2 \theta}, \quad (36)$$

$$g_{rr} = -\frac{r^2 + m_1^2 a^2 \cos^2 \theta}{r^2 - 2Mr + m_1^2 a^2}, \quad (37)$$

$$g_{\theta\theta} = -r^2 - m_1^2 a^2 \cos^2 \theta, \quad (38)$$

$$\begin{aligned} g_{\phi\phi} = & - \left\{ r^2 + m_1^2 a^2 + \frac{2Ma^2 r \sin^2 \theta}{r^2 + m_1^2 a^2 \cos^2 \theta} \right. \\ & \times \left[m_1^2 + \frac{(m_1^2 - 1)2Mr}{r^2 - 2Mr + m_1^2 a^2 \cos^2 \theta} \right] \Big\} \sin^2 \theta. \end{aligned} \quad (39)$$

On (29-31) we can make the transformation

$$M_1 = \frac{M}{m_1}, \quad r_1 = \frac{r}{m_1} \quad (40)$$

producing

$$f = 1 - \frac{2M_1 r_1}{r_1^2 + a^2 \cos^2 \theta}, \quad (41)$$

$$\omega = \frac{2aM_1 r_1 \sin^2 \theta}{r_1^2 - 2M_1 r_1 + a^2 \cos^2 \theta}, \quad (42)$$

$$e^{2\gamma} = \frac{r_1^2 - M_1 r_1 + a^2 \cos^2 \theta}{r_1^2 - 2M_1 r_1 + a^2 \cos^2 \theta + M_1^2 \sin^2 \theta}, \quad (43)$$

which shows that solution (29-31) corresponds to a Kerr solution with parameters M_1 and a . We see too from (41-43) that this solution is valid too for $0 < m_1 < 1$, hence the solution (29-31) is valid for

$$m_1 \in]0, \infty[. \quad (44)$$

The mapping $m_1 = 0$ is not allowed since then (40) has no meaning.

From (32,33) we see that the difference between classic Kerr solution and (29,30) lies in the third order asymptotical behaviour for f and for ω . The parameters M and a can be deduced from observations implying the potentials f and ω up to second order in $1/r$. Once these parameters are determined, we can find m_1 if the observations are refined up to third order in $1/r$ for f and ω , which is the postpost Newtonian approximation.

For given M and a , $m_1 \neq 1$ changes the geometry of spacetime, so we can classify the solutions like,

$$\begin{aligned} i) \quad m_1 &> \frac{M}{a}, \quad \text{Kerr black hole,} \\ ii) \quad m_1 &= \frac{M}{a}, \quad \text{extreme Kerr black hole,} \\ iii) \quad m_1 &< \frac{M}{a}, \quad \text{Kerr without event horizon.} \end{aligned}$$

In figure 1 we have plotted the ergospheres and event horizons, when they are defined, for different values of m_1 . We obtain the following interesting features.

- The topology of the exterior ergospheres change according to the values of m_1 : a) For $0 < m_1 < M/a$ (curves a and b) we have the usual spherical topology of Kerr. b) For $m_1 = M/a$ (curve c) we have the topology of an extreme black hole of Kerr. c) For $m_1 > M/a$ (curves d , e and f) the topology of the ergospheres is toroidal. When $m_1 \rightarrow \infty$ the ergosphere tends to a disc in the equatorial plane.
- The radius of the event horizon decreases and the radius of the Cauchy horizon increases when $m_1 (< M/a)$ increases, until its limiting value when $m_1 = M/a$. For $m_1 > M/a$ the horizons are no more defined.
- The interior ergosphere, for $m_1 < M/a$, increases with increasing m_1 and is disconnected of the exterior ergosphere (curves a and a). When m_1 attains the value $m_1 = M/a$, the interior ergosphere joins the exterior ergosphere continuously (curve

- c). For $m_1 > M/a$, the interior and exterior ergospheres are connected forming one ergosphere (curves d , e and f).
- For $m_1 > M/a$, from the fact that both ergospheres are connected and the horizons disappear, the singularity, forming a ring, becomes naked for certain values of the angle aperture centered at the axis z . The angle increases with increasing values of m_1 . This configuration is identical to Kerr spacetime for $M < a$ ([3] see p 375).

4 Concluding remarks

The solution that we obtained (29-31), in spite of the transformation (40) leading to the Kerr solution, is different from it for the following reasons. For a given source (M, a) , the solution (29-31) can produce a toroidal topology for its exterior ergosphere even in the case $M > a$, which is never the case for the Kerr solution with the same (M, a) : to be produced, it is sufficient that its third parameter satisfies $m_1 > M/a$. Likewise, if $M < a$, the solution (29-31) can produce a spherical topology for its exterior ergosphere, - it suffices that $m_1 < M/a$ -, which is never the case for the Kerr solution with that source (M, a) . If, far from the source, we can associate the potentials f and ω to observable quantities, then up to the post-Newtonian approximation, $O(r^{-2})$, we can determine the parameters M and a . The metric up to this order is indiscernible from that of Kerr. But if we make observations up to $O(r^{-3})$, then, in principle, we can determine m_1 , and if $m_1 \neq 1$ then we can conclude that the metric is different from Kerr with a topology depending upon its value.

Furthermore, if we compare the solution (29-31) to the one found in [1] we observe that the topology of the ergospheres are different. The angle aperture in [1] appeared in the equatorial plane, while here the equatorial plane includes, on both sides the axis z , the ergosphere for any value of m_1 .

For (29-31) the horizons do not exist when $m_1 > M/a$, while the singularity is naked for increasing angle apertures, centered at the axis z , while m_1 increases. While in the solution found in [1] the singularity is always dressed because the event horizon is always defined.

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Figure 1: Plots of the exterior and interior ergospheres (*a* to *f*), and the corresponding event horizons (dashed lines) for $M = 4$, $a = 2$ and different values of the parameter m_1 : i) $m_1 < M/a$, $m_1 = 1$ (Kerr) curves *a*, $m_1 = 1.8$ curves *b*; ii) $m_1 = M/a = 2$ (extreme black hole where the two horizons are the same), curve *c*; iii) $m_1 > M/a$ (the horizons are no more defined), $m_1 = 2.1$ curve *d*, $m_1 = 2.5$ curve *e*, $m_1 = 10$ curve *f*.

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