Rolling Tachyon with Electric and Magnetic Fields
– T-duality approach – *

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abstract

We study the decay of unstable D$p$-branes when the world-volume gauge field is turned on. We obtain the relevant D$p$-brane boundary state with electric and magnetic fields by boosting and rotating the rolling tachyon boundary state of a D$(p−1)$-brane and then T-dualizing along one of the transverse directions. A simple recipe to turn on the gauge fields in the boundary state is given. We find that the effect of the electric field is to parametrically enhance coupling of closed string oscillation modes along the electric field direction and provide an intuitive understanding of the result in the T-dualized picture. We also analyze the system by using the effective field theory and compare the result with the boundary state approach.

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1 Introduction

Real-time dynamics of unstable D-brane has received considerable attention recently [1] - [17]. Assuming that the open string tachyon evolves homogeneously, Sen [1] has found that rolling of the tachyon down the potential hill toward the closed string vacuum can be studied in terms of solvable conformal field theory. Intriguingly, Sen’s analysis indicates that, at late time, the unstable Dp-brane converts into a pressureless tachyon matter [2] localized on the p-dimensional hyper-surface — a result bearing potentially interesting implications in the context of D-brane driven cosmology [18].

An interesting situation concerning rolling dynamics of the tachyon is when gauge fields are excited on the world-volume of decaying D-brane [6, 7, 11, 14]. On Dp-brane world-volume, electric field induces charge and current density of fundamental string, while magnetic field induces charge and current density of lower-dimensional D-brane. The physics we have in mind is, as a Dp-brane decays, how the fundamental strings and lower-dimensional D-branes are distributed, and whether they can be liberated out off the Dp-brane and move freely in the ambient space-time. A related question is how coupling of the decaying D-brane to massless and massive closed string modes are modified once the gauge field is turned on — an issue of direct relevance for non-commutative open string theory [19].

In this paper, we study the tachyon rolling in the electric and magnetic field background on the world-volume of unstable D-brane. The rolling tachyon boundary state with the electric field was derived in [11] by carefully considering the effect of the electric field to the boundary condition and the boundary interaction. Here, instead, we utilize T-duality and Lorentz transformation to turn on the constant gauge field and shed new light on derivation and structure of the rolling tachyon boundary state in the background gauge field. We find that the coupling to massive closed string modes is affected by the electric field in two notable ways. First, because of Lorentz contraction, coupling to the Lorentz transformed modes is amplified hierarchically — unstable D-brane is more strongly coupled to higher closed string modes. Second, the decay time-scale is Lorentz dilated, prolonging the lifetime of the unstable D-brane. Combined together, we conclude that the electric field imparts significant modification to the decay of unstable Dp-brane, especially, when the electric field becomes critical. Intuitively, we interpret this as a consequence of Lorentz enhancement of the D-brane energy density in the T-dualized picture.

This paper is organized as follows. In section 2, we recapitulate some of the results in [1, 2, 11, 12] relevant for foregoing analysis. In section 3, we show that tachyon rolling dynamics in the gauge field background can be studied by an elementary chain of maps involving Lorentz boost, rotation and T-duality. We present a simple recipe for the corresponding boundary state out of the boundary state for rolling tachyon. As a corollary, we also present, beginning with a
boundary state describing spatial modulation of tachyon, an analogous recipe for the boundary state describing tachyon modulation in world-volume magnetic field background. In section 4, we study explicitly how the coupling of the decaying D-brane to massless and massive closed string states is modified by the presence of the world-volume electric and magnetic field. In section 5, we analyze the system using the effective field theory and compare the result with those given in the previous sections.

2 Boundary State of Rolling Tachyon

In this section, we recapitulate aspects of rolling tachyon boundary state, as analyzed in [1, 2, 11, 12], relevant for the analysis we will make in the subsequent sections.

Consider a D25-brane in bosonic string theory. In boundary conformal field theory (CFT) description, classical dynamics of the D25-brane world-volume is described by turning on an appropriate boundary interaction. So, for the open string tachyon \( T(x) \) rolling down the world-volume potential hill toward the closed string minimum,

\[ T(x^0) \sim \lambda \cosh(x^0), \]

and the corresponding boundary interaction takes the form

\[ \Delta S = \tilde{\lambda} \int d\sigma \cosh X^0(\sigma), \tag{2.1} \]

where \( \tilde{\lambda} = \tilde{\lambda}(\lambda) \) is a real parameter and \( \sigma \) is the coordinate parameterizing the world-sheet boundary. Making the Wick-rotation of \( X^0 \rightarrow iX \), dynamics of the Euclidean time \( X(t, \sigma) \) described by the world-sheet action

\[ S_X = \int dt \int d\sigma \left[ (\partial_t X)^2 + (\partial_\sigma X)^2 + \delta(t) \tilde{\lambda} \cos X(\sigma) \right] \tag{2.2} \]

turns out a solvable boundary conformal field theory [20, 21, 22]. Making use of the results of these works and Wick-rotating back to the Minkowski time \( X^0 \), one can examine real-time rolling dynamics of the open string tachyon.

The boundary state for a D25-brane with the boundary interaction Eq.(2.1) is expressed as

\[ |D25⟩_T = |B⟩_{X^0} \otimes |N⟩_X \otimes |\text{ghost}⟩. \tag{2.3} \]

Here, the latter two parts are the spatial and the world-sheet ghost boundary-states for a flat D25-brane:

\[ |N⟩_X = \exp \left( -\sum_{i=1}^{25} \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{i-n}^i \bar{c}_{i-n} \right) |0⟩, \tag{2.4} \]

\[ |\text{ghost}⟩ = \exp \left( -\sum_{n=1}^{\infty} (\bar{b}_{-n} c_{-n} + b_{-n} \bar{c}_{-n}) \right) (c_0 + \bar{c}_0) c_1 \bar{c}_1 |0⟩, \]

\( ^{1}\)We set \( \alpha' = 1 \) throughout this paper, so that the tachyon mass-squared is given by \( m^2 = -1 \).
while the first part $|B\rangle_{X^0}$ is the boundary state describing real-time dynamics of the rolling tachyon.

In terms of the Wick-rotated variable $X$, the relevant boundary state has been constructed [20, 21, 22]. In case the $X$-coordinate is compactified on a circle of self-dual radius $R = 1$, the boundary state is given by acting SU(2) rotation on the unperturbed boundary state. The boundary state for non-compact $X$-coordinate is then obtained by projecting onto zero-winding subspace, and is given by

$$|B\rangle_X = \sum_{j = 0, \frac{1}{2}, \ldots, m = -j}^{+j} D^j_{m,-m}(R)|j; m, m\rangle.$$  \hfill (2.5)

Here, $R$ is the SU(2) rotation matrix

$$R(\bar{\lambda}) = \begin{pmatrix} \cos(\pi \bar{\lambda}) & i \sin(\pi \bar{\lambda}) \\ i \sin(\pi \bar{\lambda}) & \cos(\pi \bar{\lambda}) \end{pmatrix},$$  \hfill (2.6)

$D^j_{m,-m}(R)$ is the spin-$j$ representation matrix element for the rotation, and $|j; m, m\rangle$ is the Virasoro-Ishibashi state built over the primary state $|j; m, m\rangle = |j, m\rangle |j, m\rangle$.

The boundary state $|B\rangle_X$ is expressible in terms of $X$-coordinate oscillators by facilitating the fact that the state $|j, m\rangle$ belongs to the spin-$j$ representation of the SU(2) current algebra defined by

$$J^\pm = \oint \frac{dz}{2\pi i} e^{\pm 2iX_R(z)}, \quad J^3 = \oint \frac{dz}{2\pi i} i\partial_z X_R(z).$$

Explicitly,

$$|j, j\rangle = e^{2ijX(0)}|0\rangle \quad \text{and} \quad |j, m\rangle = N_{j,m}[J^-]^{(j-m)}|j, j\rangle,$$

where $N_{j,m}$ is the normalization constant. Virasoro-Ishibashi states, which preserves the diagonal part of the left- and the right-moving Virasoro symmetries, are then constructed as

$$|0; 0, 0\rangle = \left(1 + \frac{1}{2} \alpha^2_{-1} \alpha^2_{-1} + \cdots \right) |0\rangle$$

$$|\frac{1}{2}; \pm \frac{1}{2}, \pm \frac{1}{2}\rangle = \left(1 + \alpha_{-1} \alpha_{-1} + \frac{1}{6} \left(\alpha^2_{-1} \pm \sqrt{2} \alpha_{-2}\right) \left(\alpha^2_{-1} \pm \sqrt{2} \alpha_{-2}\right) + \cdots \right) e^{\pm iX(0)} |0\rangle$$

$$|1; 0, 0\rangle = \left(\alpha_{-1} \alpha_{-1} + \frac{1}{2} \alpha^2_{-1} \alpha^2_{-1} + \cdots \right) |0\rangle$$

$$|j; \pm j, \pm j\rangle = \left(1 + \alpha_{-1} \alpha_{-1} + \frac{1}{2} \alpha^2_{-2} \alpha^2_{-2} + \frac{1}{2} \alpha_{-2} \alpha_{-2} + \cdots \right) e^{\pm 2ijX(0)} |0\rangle \quad (j \geq 1)$$

$$|\frac{3}{2}; \pm \frac{1}{2}, \pm \frac{1}{2}\rangle = \left(\frac{1}{6} \left(\alpha_{-2} \pm \sqrt{2} \alpha^2_{-2}\right) \left(\alpha_{-2} \pm \sqrt{2} \alpha^2_{-2}\right) + \cdots \right) e^{\pm iX(0)} |0\rangle,$$

etc. Relative phase-factors among these Ishibashi states are determinable by demanding the physics that the boundary state represents an array of D-branes localized at $X = (2n + 1)\pi$.
when $\tilde{\lambda} = 1/2$ [1]. Multiplying the relevant matrix elements of the SU(2) rotation $R$:

$$
D^i \pm j \pm j(R) = \left(i \sin(\pi \tilde{\lambda})\right)^{2j} (j = 0, 1/2, 1, \ldots) \\
D^{3}_{0,0}(R) = \cos(2\pi \tilde{\lambda}) \\
D^{3}_{\pm \frac{1}{2}, \mp \frac{1}{2}}(R) = i \sin(\pi \tilde{\lambda}) \left(3 \cos^2(\pi \tilde{\lambda}) - 1\right) \\
D^{2}_{\pm 1,\pm 1}(R) = -\sin^2(\pi \tilde{\lambda}) \cos(2\pi \tilde{\lambda})
$$

to the Virasoro-Ishibashi states, and Wick-rotating back to the $X^0$-coordinate, the sought-for boundary state is obtained as

$$
|B\rangle_{X^0} = f(\tilde{x}^0) |0\rangle \\
+ g(\tilde{x}^0) \alpha_{-1}^0 \alpha_{-1}^0 |0\rangle \\
+ h_1(\tilde{x}^0) \alpha_{-2}^0 \alpha_{-2}^0 |0\rangle \\
+ h_2(\tilde{x}^0) (\alpha_{-1}^0)^2 (\alpha_{-1}^0)^2 |0\rangle \\
+ h_3(\tilde{x}^0) \left((\alpha_{-1}^0)^2 \alpha_{-2}^0 + \alpha_{-2}^0 (\alpha_{-1}^0)^2\right) |0\rangle \\
+ \ldots.
$$

(2.7)

The functions $f(\tilde{x}^0)$ is defined as

$$
f(\tilde{x}^0) = \left(1 + e^{x^0 \sin(\tilde{\lambda}\pi)}\right)^{-1} + \left(1 + e^{-x^0 \sin(\tilde{\lambda}\pi)}\right)^{-1} - 1,
$$

(2.8)

and the functions $g(\tilde{x}^0)$, $h_1(\tilde{x}^0)$, $h_2(\tilde{x}^0)$ and $h_3(\tilde{x}^0)$ are defined in terms of $f(\tilde{x}^0)$:

$$
g(\tilde{x}^0) = 1 + \cos(2\tilde{\lambda}\pi) - f(\tilde{x}^0) \\
h_1(\tilde{x}^0) = \frac{1}{2} \left(1 + \cos(2\tilde{\lambda}\pi)\right) - \sin(\tilde{\lambda}\pi) \cos(\tilde{\lambda}\pi) \cosh(x^0) - \frac{1}{2} f(\tilde{x}^0) \\
h_2(\tilde{x}^0) = 2 \sin(\tilde{\lambda}\pi) \cos^2(\tilde{\lambda}\pi) \cosh(x^0) + \frac{1}{2} f(\tilde{x}^0) \\
h_3(\tilde{x}^0) = -i \sqrt{2} \sin(\tilde{\lambda}\pi) \cos^2(\tilde{\lambda}\pi) \sinh(x^0).
$$

(2.10)

In the above expressions, we have omitted the overall normalization factor given by D25-brane tension $T_{25}$ for notational simplicity. We will recover the normalization factor when we compute the coupling to closed string states in section 4.

Notice that the function $f(\tilde{x}^0)$ exponentially vanishes at late time, while the function $g(\tilde{x}^0)$ converges to a finite constant. On the other hand, the functions $h_1(\tilde{x}^0)$, $h_2(\tilde{x}^0)$ and $h_3(\tilde{x}^0)$ contain terms depending on either $\cosh(x^0)$ or $\sinh(x^0)$, and hence blow up exponentially when $|x^0| \to \infty$. As such, the tachyon matter couples hierarchically strongly to higher-mass closed string states, hinting that unstable D25-brane would preferentially populate massive closed string modes rather than doing so massless (graviton, dilaton, anti-symmetric tensor) modes [12].
3 Tachyon Rolling in Constant Gauge Field Background

In this section, starting from the result given in the previous section, we construct the boundary state describing a D25-brane with both rolling tachyon and homogeneous gauge fields turned on. This is facilitated by making a chain of operations that leave solvability of the conformal field theory intact. To do so, we will first consider a D25-brane, on whose world-volume all excitations are set to zero except the rolling tachyon, and compactify a spatial direction on a circle. We T-dualize it to a D24-brane, and then boost the D24-brane rigidly along the compactified spatial direction. Subsequently, we T-dualize back along the boosted spatial direction. The final configuration we obtain is a D25-brane, whose world-volume excitation involves both rolling tachyon and constant electric field. This argument can be generalized to the cases with both electric and magnetic fields by combining Lorentz boost and rotation in the T-dualized picture.†

3.1 Turning on Constant Electric Field

We begin with our prescription. Consider a D25-brane § in $\mathbb{R}^{1,25}$ and let the tachyon field roll as in the previous section. We will compactify, say, $x^1$-direction on a circle $S_1$ of radius $R$, and wrap the D25-brane on $S_1$. In describing the open string dynamics in terms of the boundary states, it is convenient to decompose the world-sheet fields $X^a(z, \bar{z})$ into the left-moving part $X^a_L(z)$ and the right-moving part $X^a_R(\bar{z})$, respectively.

We now perform a chain of maps, which retains the solvability of the boundary conformal field theory. First, T-dualize the $x^1$-direction, so that the world-sheet fields $X^0, X^1$ are converted as

$$
\text{T-dual} : \begin{pmatrix} X^0_L \\ X^1_L \end{pmatrix} \rightarrow \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} X^0_L \\ X^1_L \end{pmatrix},$
$$
$$
\begin{pmatrix} X^0_R \\ X^1_R \end{pmatrix} \rightarrow \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} X^0_R \\ X^1_R \end{pmatrix}. \quad (3.11)
$$

At the same time, the D25-brane is turned into an array of localized D24-brane on the dual circle $\tilde{S}_1$ of radius $\tilde{R} = 2\pi/R$. We then take the decompactification limit $\tilde{R} \rightarrow \infty$, and isolate a localized D24-brane. In the boundary state description, the process maps the $X^1$-part of the Neumann state Eq.(2.4) into the following Dirichlet state:

$$
|D\rangle_{X^1} = \exp \left( + \sum_{n=1}^{\infty} \frac{1}{n} \alpha^1_n \overline{\alpha}^1_n \right) \delta(\tilde{x}^1) |0\rangle. \quad (3.12)
$$

†See for example [23] for a comprehensive review of boosted and rotated D-brane boundary states and their T-duality relations.

‡Extension to lower-dimensional D-branes is trivial. We will phrase the prescription so that it is applicable to all D$p$-branes ($p \geq 1$).
Notice that it involves Dirac delta-function of the zero-mode operator, $\hat{x}^1$. Next, we boost the D24-brane along $x^1$-direction with velocity $e$. The world-sheet fields $X_0^0, X_1^1$ are then mapped as

$$e - \text{boost} : \begin{pmatrix} X_0^0 \\ X_1^0 \\ X_R^0 \\ X_R^1 \end{pmatrix} \rightarrow \begin{pmatrix} Y_0^0 \\ Y_1^0 \\ Y_R^0 \\ Y_R^1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix} \begin{pmatrix} X_0^0 \\ X_1^0 \\ X_R^0 \\ X_R^1 \end{pmatrix},$$

(3.13)

where $\gamma = 1/\sqrt{1 - e^2}$ denotes the Lorentz factor. Notice that, the zero-mode constraint in Eq.(3.12) for the localized D24-brane renders two elementary but significant changes after the boost: because of Lorentz time dilation and length contraction effects, we find that

$$\hat{x}^0 = \gamma^{-1} \hat{y}^0 = \sqrt{1 - e^2} \hat{y}^0$$

$$\delta(\hat{x}^1) = \delta \left( \gamma (\hat{y}^1 + e \hat{y}^0) \right) = \gamma^{-1} \delta \left( \hat{y}^1 + e \hat{y}^0 \right).$$

(3.14)

We then compactify the $y^1$-direction on a circle $S'_1$ of radius $R'$, and arrange an array of D24-branes. Finally, T-dualize back along the $y^1$-direction, under which the world-sheet fields transform as

$$T - \text{dual} : \begin{pmatrix} Y_0^0 \\ Y_1^0 \\ Y_R^0 \\ Y_R^1 \end{pmatrix} \rightarrow \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} Y_0^0 \\ Y_1^0 \\ Y_R^0 \\ Y_R^1 \end{pmatrix},$$

(3.15)

Moreover, the zero-mode part Eq.(3.14) in the boundary state transforms as

$$\gamma^{-1} \delta(\hat{y}^1 + e \hat{y}^0) |0\rangle = \gamma^{-1} \sum_{n \in \mathbb{Z}} e^{in\pi (\hat{y}^1 + e \hat{y}^0)} |0\rangle \rightarrow \gamma^{-1} \sum_{m \in \mathbb{Z}} e^{im(\hat{y}^1 - \hat{y}^0 + e \hat{y}^0)R'} |0\rangle,$$

(3.16)

yielding D25-branes wrapped on a dual circle $S'_1$ of radius $R' = 2\pi / R'$. Decompactify the $y^1$-direction, $R' \rightarrow \infty$. Then, in Eq.(3.16), only the $m = 0$ term contributes to the boundary state, and yields the Born-Infeld factor: $\sqrt{1 - e^2} |0\rangle$ [24]. The resulting configuration is a D25-brane whose world-volume dynamics involves, in addition to the rolling tachyon field $T(\gamma^{-1} y^0)$, a homogeneous electric field $F_{01} = e$.

### 3.2 Recipe for the Boundary State Construction

Considerations of the previous subsection facilitates us an elementary recipe for construction of the boundary state describing tachyon rolling in a constant electric field background. Begin with the D25-brane boundary state Eq.(2.3) describing open string tachyon rolling dynamics.
Using the result Eq.(2.7) and expanding in powers of the string oscillators, the boundary state is expandable in oscillator-level:

\[
|D25\rangle_T = |D25\rangle_T^{\text{matter}} \otimes |\text{ghost}\rangle
\]  

(3.17)

\[
|D25\rangle_T^{\text{matter}} = |B\rangle_{X^0} \otimes |N\rangle_{X^1} \otimes_{i \neq 0,1} |N\rangle_{X^i}
\]

\[
= F(\vec{x}^0)|0\rangle + G_{ab}(\vec{x}^0) \alpha_{-1}^a \bar{\alpha}_{-1}^b |0\rangle \\
+ H_{ab}(\vec{x}^0) \alpha_{-2}^a \bar{\alpha}_{-2}^b |0\rangle + I_{abcd}(\vec{x}^0) \alpha_{-1}^a \alpha_{-1}^b \bar{\alpha}_{-1}^c \bar{\alpha}_{-1}^d |0\rangle + \cdots,
\]

(3.18)

where |0\rangle denotes the Fock-space vacuum of \(\alpha^a, \bar{\alpha}^i\) oscillators.

### 3.2.1 Turning on electric field

As explained in the previous subsection, the D25-brane boundary state describing rolling tachyon and electric field is then obtainable by applying sequentially the chain of maps, Eqs.(3.11, 3.13, 3.15), to Eq.(3.18). We find that the resulting boundary state is given by the following replacement to Eq.(3.18):

\[
|0\rangle \rightarrow \gamma^{-1}|0\rangle \\
\vec{x}^0 \rightarrow \gamma^{-1} y^0 \\
\begin{pmatrix} \alpha_{-n}^0 \\ \alpha_{-n}^1 \end{pmatrix} \rightarrow \Lambda^{-1} \begin{pmatrix} \beta_{-n}^0 \\ \beta_{-n}^1 \end{pmatrix} \\
\begin{pmatrix} \bar{\alpha}_{-n}^0 \\ \bar{\alpha}_{-n}^1 \end{pmatrix} \rightarrow \Lambda \begin{pmatrix} \bar{\beta}_{-n}^0 \\ \bar{\beta}_{-n}^1 \end{pmatrix},
\]

(3.19)

where

\[
\Lambda = \gamma \begin{pmatrix} 1 & +e \\ +e & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^{-1} = \gamma \begin{pmatrix} 1 & -e \\ -e & 1 \end{pmatrix}.
\]

(3.20)

Here, \(\beta_{-n}^a\) and \(\bar{\beta}_{-n}^a\) denote the oscillators of \(Y_{-L}^a\) and \(Y_{-R}^a\) after the T-duality (3.15), respectively. The resulting boundary state is that

\[
|D25\rangle_T^{\text{matter}}_{x,e} = F^e(\vec{y}^0)|0\rangle + G_{ab}^e(\vec{y}^0) \beta_{-1}^a \bar{\beta}_{-1}^b |0\rangle \\
+ H_{ab}^e(\vec{y}^0) \beta_{-2}^a \bar{\beta}_{-2}^b |0\rangle + I_{abcd}^e(\vec{y}^0) \beta_{-1}^a \beta_{-1}^b \bar{\beta}_{-1}^c \bar{\beta}_{-1}^d |0\rangle + \cdots
\]

where

\[
F^e(y^0) = \gamma^{-1} F(\gamma^{-1} y^0)
\]

\[
G_{ab}^e(y^0) = \gamma^{-1} \left( \Lambda^{-1} G(\gamma^{-1} y^0) \right)_{ab}
\]

\[
H_{ab}^e(y^0) = \gamma^{-1} \left( \Lambda^{-1} H(\gamma^{-1} y^0) \right)_{ab}
\]

\[
I_{abcd}^e(y^0) = \gamma^{-1} \left( \Lambda^{-1} \left( \Lambda^{-1} I(\gamma^{-1} y^0) \right)_{ac} \Lambda \right)_{bd},
\]

(3.21)
and so on. The physics behind the above transformation is clear. Nonzero electric field induces the Lorentz time dilation effect, and hence slowing down the tachyon-rolling time-scale.

### 3.2.2 Turning on magnetic field

It is straightforward to extend the above recipe to the boundary state describing tachyon modulation in a constant magnetic field background by replacing the $e$-boost Eq. (3.13) by a rotation. Begin with the D25-brane boundary state with tachyon modulation along the $x^1$-direction:

\[
|D25\rangle^{\text{matter}}_{T,b} = |B\rangle_{X^1} \otimes |N\rangle_{X^2} \otimes_{i \neq 1,2} |N\rangle_{X^i}
\]

\[
= F(\tilde{x}^1)|0\rangle + G_{ab}(\tilde{x}^1) \alpha_{-1}^a \alpha_{-1}^b |0\rangle
\]

\[
+ H_{ab}(\tilde{x}^1) \alpha_{-2}^a \alpha_{-2}^b |0\rangle + I_{abcd}(\tilde{x}^1) \alpha_{-1}^a \alpha_{-1}^b \alpha_{-1}^c \alpha_{-1}^d |0\rangle + \cdots .
\] (3.22)

Then, after a chain of T-dual map $\rightarrow$ rotation in $(1 - 2)$-plane $\rightarrow$ T-dual map, the boundary state describing tachyon modulation in a constant magnetic field background is given by the replacement in Eq. (3.22) by

\[
|0\rangle \rightarrow \tilde{\gamma}^{-1}|0\rangle
\]

\[
x^1 \rightarrow \tilde{\gamma}^{-1}y^1
\]

\[
\begin{pmatrix}
\alpha_{-1}^1 \\
\alpha_{-n}^1 \\
\alpha_{-2}^1 \\
\alpha_{-n}^2
\end{pmatrix}
\rightarrow
\Omega^{-1}
\begin{pmatrix}
\beta_{-1}^1 \\
\beta_{-n}^1 \\
\beta_{-2}^1 \\
\beta_{-n}^2
\end{pmatrix}
\]

\[
\begin{pmatrix}
\bar{\alpha}_{-1}^1 \\
\bar{\alpha}_{-n}^1 \\
\bar{\alpha}_{-2}^1 \\
\bar{\alpha}_{-n}^2
\end{pmatrix}
\rightarrow
\Omega
\begin{pmatrix}
\beta_{-1}^1 \\
\beta_{-n}^1 \\
\beta_{-2}^1 \\
\beta_{-n}^2
\end{pmatrix}
\]

(3.23)

where $\tilde{\gamma} = 1/\sqrt{1 + b^2}$, and

\[
\Omega = \tilde{\gamma}
\begin{pmatrix}
1 & -b \\
+ b & 1
\end{pmatrix}
\]

and

\[
\Omega^{-1} = \tilde{\gamma}
\begin{pmatrix}
1 & +b \\
- b & 1
\end{pmatrix}
\]

(3.24)

As a result, the sought-for boundary state is obtained as

\[
|D25\rangle^{\text{matter}}_{T,b} = F^b(\tilde{y}^1)|0\rangle + G_{ab}(\tilde{y}^1) \beta_{-1}^a \beta_{-1}^b |0\rangle
\]

\[
+ H_{ab}(\tilde{y}^1) \beta_{-2}^a \beta_{-2}^b |0\rangle + I_{abcd}(\tilde{y}^1) \beta_{-1}^a \beta_{-1}^b \beta_{-1}^c \beta_{-1}^d |0\rangle + \cdots ,
\] (3.25)

where

\[
F^b(y^1) = \tilde{\gamma}^{-1} F(\tilde{\gamma}^{-1} y^1)
\]

\[
G_{ab}^b(y^1) = \tilde{\gamma}^{-1} \left( \Omega^{-1} G(\tilde{\gamma}^{-1} y^1) \Omega \right)_{ab}
\]

\[
H_{ab}^b(y^1) = \tilde{\gamma}^{-1} \left( \Omega^{-1} H(\tilde{\gamma}^{-1} y^1) \Omega \right)_{ab}
\]

\[
I_{abcd}^b(y^1) = \tilde{\gamma}^{-1} \left( \Omega^{-1} I(\tilde{\gamma}^{-1} y^1) \Omega \right)_{ad}
\]

(3.26)

and so on.
3.2.3 Turning on electric + magnetic fields

We can also extend the recipes given in the previous subsections to the situation turning on both electric and magnetic field. This is achieved by T-dualizing, boosting and rotating in (2 + 1)-dimensional sub-space, and finally T-dualizing back the system.

Suppose that the matter part of the boundary state we start with is of the form

$$|\text{D25}\rangle_{\text{matter}} = (|B\rangle_{X^0} \otimes |B\rangle_{X^1} \otimes |N\rangle_{X^2}) \otimes_{i \neq 0,1,2} |N\rangle_{X^i}$$

$$= F(\tilde{x}^0, \tilde{x}^1)|0\rangle + G_{ab}(\tilde{x}^0, \tilde{x}^1) \alpha^a_{\gamma-1} \alpha^b_{\gamma-1} |0\rangle + H_{ab}(\tilde{x}^0, \tilde{x}^1) \alpha^a_{\gamma-1} \alpha^b_{\gamma-1} |1\rangle + I_{abcd}(\tilde{x}^0, \tilde{x}^1) \alpha^a_{\gamma-1} \alpha^b_{\gamma-1} \alpha^c_{\gamma-1} \alpha^d_{\gamma-1} |0\rangle + \cdots,$$  \hspace{1cm} (3.27)

describing the tachyon field, modulated along the \(x^1\)-direction, is rolling. Then the corresponding boundary state with both electric field \(e = F_{02}\) and magnetic field \(b = F_{12}\) turned on is given by the following replacement

\[
|0\rangle \rightarrow \bar{\gamma}^{-1} \gamma^{-1} |0\rangle \hspace{1cm} (3.28)
\]
\[
x^0 \rightarrow \gamma^{-1} y^0 \hspace{1cm} (3.29)
\]
\[
x^1 \rightarrow \bar{\gamma}^{-1} y^1 - b \bar{\gamma} y^0 \hspace{1cm} (3.30)
\]
\[
\begin{pmatrix}
\alpha^0_{\gamma-n} \\
\alpha^1_{\gamma-n} \\
\alpha^2_{\gamma-n}
\end{pmatrix}
\rightarrow
\Lambda^{-1} \Omega^{-1}
\begin{pmatrix}
\beta^0_{\gamma-n} \\
\beta^1_{\gamma-n} \\
\beta^2_{\gamma-n}
\end{pmatrix}
\hspace{1cm} (3.31)
\]
\[
\begin{pmatrix}
\bar{\alpha}^0_{\gamma-n} \\
\bar{\alpha}^1_{\gamma-n} \\
\bar{\alpha}^2_{\gamma-n}
\end{pmatrix}
\rightarrow
\Lambda \Omega
\begin{pmatrix}
\bar{\beta}^0_{\gamma-n} \\
\bar{\beta}^1_{\gamma-n} \\
\bar{\beta}^2_{\gamma-n}
\end{pmatrix}
\hspace{1cm} (3.32)
\]

where

\[
\Lambda = \begin{pmatrix}
\gamma & 0 & \gamma e' \\
0 & 1 & 0 \\
\gamma e' & 0 & \gamma
\end{pmatrix}
\hspace{1cm} \text{and} \hspace{1cm}
\Omega = \begin{pmatrix}
1 & 0 & 0 \\
0 & \bar{\gamma} & -\bar{\gamma} b \\
0 & +\bar{\gamma} b & \bar{\gamma}
\end{pmatrix}
\hspace{1cm} (3.33)
\]

in which

\[
\gamma = 1/\sqrt{1-e'^2} \hspace{1cm} \text{and} \hspace{1cm} \bar{\gamma} = 1/\sqrt{1+b^2}.
\hspace{1cm} (3.34)
\]

Here, we set \(e' = e/\sqrt{1+b^2}\). This redefinition of electric flux is necessary, since the velocity of the D24-brane along \(y^2\)-direction in the T-dualized picture is changed by the rotation. Note that we have reproduced correctly the Born-Infeld factor:

\[
\gamma^{-1} \bar{\gamma}^{-1} = \sqrt{1-e'^2 + b^2} = \sqrt{-\det(\eta_{ab} + F_{ab})}.
\hspace{1cm} (3.35)
\]

The recipe Eq.(3.29) and Eq.(3.30) follows from the following relation in the T-dualized picture:

\[
\begin{pmatrix}
y^0 \\
y^1 \\
y^2
\end{pmatrix}
= \Omega \Lambda
\begin{pmatrix}
x^0 \\
x^1 \\
x^2
\end{pmatrix}
\hspace{1cm} \text{with} \hspace{1cm} x^2 = 0,
\hspace{1cm} (3.36)
\]
where the right-hand-side refers to a D24-brane localized at \( x^2 = 0 \).

As a result, the matter part of the boundary state becomes

\[
| \text{D25} \rangle_{\text{matter}}^{e+b} = F^{e+b}(y^0, \tilde{y}^1) | 0 \rangle + G^{e+b}_{ab}(y^0, \tilde{y}^1) \beta^a_0 \beta^b_0 | 0 \rangle + H^{e+b}_{ab}(y^0, \tilde{y}^1) \beta^a_2 \beta^b_2 | 0 \rangle + I^{e+b}_{abcd}(y^0, \tilde{y}^1) \beta^a_1 \beta^c_1 \beta^b_1 \beta^d_1 | 0 \rangle + \cdots ,
\]

where

\[
F^{e+b}(y^0, y^1) = \gamma^{-1} \tilde{\gamma}^{-1} F(\gamma^{-1} y^0, \tilde{\gamma}^{-1} y^1 - be\tilde{\gamma} y^0), \tag{3.37}
\]

\[
G^{e+b}_{ab}(y^0, y^1) = \gamma^{-1} \tilde{\gamma}^{-1} \left( (\Omega \Lambda)^{-1} G(\gamma^{-1} y^0, \tilde{\gamma}^{-1} y^1 - be\tilde{\gamma} y^0) \Lambda \Omega \right)_{ab}, \tag{3.38}
\]

\[
H^{e+b}_{ab}(y^0, y^1) = \gamma^{-1} \tilde{\gamma}^{-1} \left( (\Omega \Lambda)^{-1} H(\gamma^{-1} y^0, \tilde{\gamma}^{-1} y^1 - be\tilde{\gamma} y^0) \Lambda \Omega \right)_{ab}, \tag{3.39}
\]

\[
I^{e+b}_{abcd}(y^0, y^1) = \gamma^{-1} \tilde{\gamma}^{-1} \left( (\Omega \Lambda)^{-1} t(\Omega \Lambda)^{-1} I(\gamma^{-1} y^0, \tilde{\gamma}^{-1} y^1 - be\tilde{\gamma} y^0) \Lambda \Omega \right)_{bc} \Lambda \Omega | 0 \rangle ,
\]

and so on.

As a check, we apply our prescription to the boundary state of a static D25-brane. The boundary state is given by the Neumann state:

\[
| \text{D25} \rangle_{\text{matter}} = \exp \left( - \sum_{n=1}^\infty \frac{1}{n} \eta_{ab} \alpha^a_n \alpha^b_{-n} \right) | 0 \rangle .
\]

From Eqs.(3.28), (3.31) and (3.32), we obtain the boundary state of a static D25-brane with electric and magnetic field turned on

\[
| \text{D25} \rangle_{e,b}^{\text{matter}} = \gamma^{-1} \tilde{\gamma}^{-1} \exp \left( - \sum_{k=1}^\infty \frac{1}{k} t(\Omega \Lambda)^{-1} \eta \Lambda \Omega \right)_{ab} \beta^a_k \beta^b_{-k} | 0 \rangle .
\]

Using the relation Eq.(3.35) and

\[
\left( t(\Omega \Lambda)^{-1} \eta \Lambda \Omega \right)_{ab} = \left( 1 - e^2 + b^2 \right)^{-1} \begin{pmatrix}
-(1 + e^2 + b^2) & -2eb & -2e \\
-2eb & 1 - e^2 - b^2 & -2b \\
+2e & +2b & 1 + e^2 - b^2
\end{pmatrix}_{ab}
\]

\[
= \left( \frac{1 - F}{1 + F} \right)_{ab},
\]

where \( F := (F^a_b) = (\eta^{ac} F_{cb}) \), we obtain the boundary state as

\[
| \text{D25} \rangle_{e+b}^{\text{matter}} = \sqrt{\det(1 + F)} \ exp \left( - \sum_{n=1}^\infty \frac{1}{n} \left( \frac{1 - F}{1 + F} \right)_{ab} \beta^a_{-n} \beta^b_{-n} \right) | 0 \rangle . \tag{3.40}
\]

The result is in perfect agreement with the Neumann boundary state with constant gauge fields, as computed previously in [25].
3.3 Generalization to super-string theory

It is straightforward to extend the derivation given in the previous subsections for the problem of rolling tachyon on an unstable $D^p$-brane in super-string theories. If we use the NSR-formulation, the Lorentz transformation and the T-duality map are given in the same way as the bosonic string case. Therefore, our recipe of turning on electric and magnetic fields given in the previous subsections are equally applicable to the super-string theories, with an understanding that the NSR fermionic oscillators transform the same as Eqs.(3.31, 3.32) under the $e$-boost and the $b$-rotation. In particular, this renders that the relations such as Eq.(3.37) and Eq.(3.38) continue to hold. In the absence of the world-volume gauge field, explicit form of the lower-level states in the rolling tachyon boundary state in super-string theory was obtained in [2, 12]. For the massless modes, compared with the bosonic string result, the only change is in the functional form of $f(x^0)$. Adopting the argument of [2], we can obtain the source of massless closed string states in the rolling tachyon boundary state for super-string by simply replacing the function $f(x^0)$ with

\[ f(x^0) = \left( 1 + e^{\sqrt{2}x^0} \sin^2(\tilde{\lambda}\pi) \right)^{-1} + \left( 1 + e^{-\sqrt{2}x^0} \sin^2(\tilde{\lambda}\pi) \right)^{-1} - 1 \]

in the corresponding results for bosonic string theory. It is then evident that, applying the argument given in the previous sub-sections, this is also the case even if we turn on constant electric and magnetic fields.

4 Coupling to Closed String States

Based on the recipe prescribed in the previous section, we now compute the boundary state representing the rolling tachyon with electric and magnetic field and extract the coupling strength of the unstable D-brane to lower-level closed string modes. In order to extract the coupling, it is useful to recall that the closed string wave function for the bosonic string is given as

\[ |\Phi\rangle = \{T(\tilde{x}) + h_{(ab)}(\tilde{x})\alpha_{-1}^a \bar{\alpha}_{-1}^b + B_{(ab)}(\tilde{x})\alpha_{-1}^a \bar{\alpha}_{-1}^b + \phi_s(\tilde{x})(\bar{c}_{-1}b_{-1} + c_{-1}\bar{b}_{-1}) \} c_1 \bar{c}_1 |0\rangle + \cdots \]  

(4.41)

up to the oscillator-level (1,1) states, where ellipses denote higher oscillator-level states. Here, $\phi_s(x)$ is Siegel’s dilaton field, and is related to the sigma model dilaton field $\varphi(x)$ as

\[ \varphi(x) = \phi_s(x) - \frac{1}{2} h^a_a(x), \]  

(4.42)

which couples to the world-sheet curvature scalar and behaves as a scalar under the linearized general coordinate transformation [25]. The boundary state couples to the closed string field through a term proportional to $\langle \Phi | (c_0 - \bar{\tau}_0) | B \rangle$ and hence it acts as a source for the closed
string field in the linearized equation of motion, \((Q_B + \overline{Q}_B) | \Phi \rangle = | B \rangle\), where \(Q_B\) is the BRST operator.

### 4.1 Rolling Tachyon with Electric Field

First, we demonstrate how to construct the boundary state for the rolling tachyon with electric field. We start with the boundary state representing the homogeneous tachyon rolling without gauge fields given in Eq.(2.3) with Eq.(2.4) and Eq.(2.7). The boundary state with electric field \(e = F_{01}\) is obtained by the replacement summarized in Eq.(3.19). The result is

\[
| D25 \rangle_{T,e} = | B \rangle_{Y^0} \otimes | B \rangle_{Y^1} \otimes_{i \neq 0,1} | N \rangle_{X^i} \otimes | \text{ghost} \rangle,
\]

where

\[
| B \rangle_{Y^0} = f(\gamma^{-1}y^0)|0\rangle + g(\gamma^{-1}y^0)\gamma^2 \left[ (\beta_{-1}^0 - e\bar{\beta}_{-1}) (\bar{\beta}_{-1}^0 + e\bar{\beta}_{-1}) \right] |0\rangle + h_1(\gamma^{-1}y^0)\gamma^2 \left[ (\beta_{-2}^0 - e\bar{\beta}_{-2}) (\bar{\beta}_{-2}^0 + e\bar{\beta}_{-2}) \right] |0\rangle + h_2(\gamma^{-1}y^0)\gamma^4 \left[ (\beta_{-1}^0 - e\bar{\beta}_{-1})^2 (\bar{\beta}_{-1}^0 + e\bar{\beta}_{-1})^2 \right] |0\rangle + h_3(\gamma^{-1}y^0)\gamma^3 \left[ (\beta_{-1}^0 - e\bar{\beta}_{-1})^2 (\bar{\beta}_{-2}^0 + e\bar{\beta}_{-2}) + (\beta_{-2}^0 - e\bar{\beta}_{-2}) (\bar{\beta}_{-1}^0 + e\bar{\beta}_{-1})^2 \right] |0\rangle + \cdots,
\]

and

\[
| B \rangle_{Y^1} = \gamma^{-1} \exp \left( - \sum_{n=1}^{\infty} \frac{\gamma^2}{n} (\beta_{-1}^0 - e\bar{\beta}_{-n}^0) (\bar{\beta}_{-n}^0 + e\bar{\beta}_{-n}^0) \right) |0\rangle.
\]

Here, \(f(x^0)\) and \(g(x^0)\) are the functions given in Eqs.(2.8, 2.9), and \(h_1(x^0), h_2(x^0), h_3(x^0)\) are the functions given in Eq.(2.10). The resulting boundary state is consistent with that given in [11]. Notice that, in Eqs.(4.44, 4.45), the power of the Lorentz factor \(\gamma\) is determined by the total number of \(\beta, \bar{\beta}\)-oscillators.

#### 4.1.1 Tachyon Coupling

The part of the boundary state that couples to the closed string tachyon is given by \(|0\rangle\), and is furnished by the first term in the expansion of \(| D25 \rangle_{T,e} \). Using Eqs.(4.44, 4.45), we find

\[
\rho_{\text{tachyon}}(y^0) = T_{25} \gamma^{-1} f(\gamma^{-1}y^0),
\]

where we have reinstated dependence of the boundary state on the D25-brane tension \(T_{25}\). We see that the tachyon coupling vanishes exponentially as \(|y^0| \to \infty\), finding qualitatively the same behavior as the situation without electric field. Time evolution of the tachyon coupling is
Figure 1: Behavior of $\rho_{\text{tachyon}} = \rho_{\text{dilaton}} = -T_{ii}(i \neq 0, 1)$ (vertical axis in arbitrary unit) in $y^0 = [0, 10]$ and $\tilde{\lambda} = [0, 1/2]$ parameter space. The left is for $e = 0$, while the right is for $e = 0.9$.

plotted in Fig.1 for various values of $\tilde{\lambda}$. The result exhibits that the tachyon coupling reaches vacuum value 0 faster at larger $\tilde{\lambda}$, and is compatible with Sen’s interpretation that $\tilde{\lambda}$ refers to the initial value of the open string tachyon field. Notice that, when compared with the situation without electric field, the decay process takes place slower by the Lorentz factor $\gamma$, and weaker by the inverse Lorentz factor $\gamma^{-1}$. The latter suppression factor is precisely the Born-Infeld factor.

### 4.1.2 Graviton Coupling

The boundary state that couples to the closed string graviton is given by the energy-momentum tensor of the D25-brane. A prescription of extracting energy-momentum tensor of the boundary state was provided in [2]. If the boundary state takes the form

$$|D25\rangle_{T,e}^{\text{matter}} = F^e(y^0)|0\rangle + G^e_{ab}(y^0)\beta_{2-1}^a\beta_{-1}^b|0\rangle + \cdots, \quad (4.47)$$

the energy-momentum tensor is given by

$$T^e_{ab}(y^0) := \frac{T_{25}}{2} \left( G^e_{(ab)}(y^0) - \eta_{ab}F^e(y^0) \right). \quad (4.48)$$

The second term in the right-hand side comes from the ghost contribution, which couples to the trace of the graviton field, $h^a_a$, via the dilaton field relation Eq.(4.42). In our case, we have

$$F^e(y^0) = \gamma^{-1}f(\gamma^{-1}y^0)$$

$$= \gamma(1 - e^2)f(\gamma^{-1}y^0)$$

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Putting them together, we obtain non-vanishing components of the energy-momentum tensor as

\begin{align}
T_{00}(y^0) &= +T_{25} \gamma \cos^2(\tilde{\lambda} \pi) \\
T_{11}(y^0) &= -T_{25} \gamma^2 \cos^2(\tilde{\lambda} \pi) - T_{25} \gamma^{-1} f(\gamma^{-1} y^0) \\
T_{ii}(y^0) &= -T_{25} \gamma^{-1} f(\gamma^{-1} y^0) (i = 2, \cdots, 25).
\end{align}

The results are in agreement with those obtained in [11] – the energy-momentum tensor consists of a linear superposition of the tachyon matter and the fundamental string fluid. We also confirm that the energy density is independent of time, and hence the energy-momentum tensor obeys the conservation condition \( \partial^a T_{ab} = 0 \) trivially.

### 4.1.3 Kalb-Ramond Coupling

The boundary state that couples to the closed string Kalb-Ramond field \( B_{[ab]} \) is given by the anti-symmetric part of \( G_{ab}^e \) in Eq.(4.49), and defines the fundamental string current density
tensor $Q_{[ab]}^e$:

$$Q_{01}^e := \frac{T_{25}}{2} C_{[01]}^e = T_{25} e \cos^2(\tilde{\lambda} \pi). \quad (4.53)$$

The result is again in agreement with that obtained in [11]. We notice that the fundamental string current density is a nonzero constant, and is seeded by the nonzero electric field $e$. As such, it obeys the conservation condition $\partial_a Q_{[ab]}^e = 0$ trivially. As we will see in section 5, the factor $\cos^2(\tilde{\lambda} \pi)$ is intimately related to the tachyon potential (as well as the world-volume displacement field). Notice also that the coupling is enhanced by the Lorentz factor $\gamma$.

From Eqs.(4.50, 4.51, 4.53), we observe that these level-(1,1) couplings obey the following inequality relations:

$$T_{00}^e \geq |Q_{01}^e| \geq -T_{11}^e \quad \text{at} \quad |y^0| \to \infty.$$

In the critical limit, $e \to 1$, we thus find that

$$T_{00}^e = -T_{11}^e = |Q_{01}^e|,$$

yielding precisely a sort of BPS-type equation obeyed by the fundamental closed string.

### 4.1.4 Dilaton Coupling

The boundary state that couples to the sigma model dilaton of the closed string is given by

$$(c_0 + \bar{c}_0)(c_{-1} c_1 + \bar{c}_{-1} \bar{c}_1)|0\rangle_{gh}.$$

Thus, the coupling of the decaying D25-brane to the closed string dilaton is given by

$$\rho_{\text{dilaton}}(y^0) = T_{25} \gamma^{-1} f(\gamma^{-1} y^0). \quad (4.54)$$

This is identical to the coupling to the closed string tachyon, so exhibits the same asymptotic behavior. In particular, despite the Lorentz time dilation and the Born-Infeld suppression effects, the dilaton tadpole vanishes exponentially as $|y^0| \to \infty$.

### 4.1.5 Massive State Coupling

Coupling of the decaying D25-brane to a massive closed string state can be read off from Eqs.(4.44, 4.45). As is evident from Eq.(4.44), power of the Lorentz factor $\gamma$ is fixed by the number of $\beta, \bar{\beta}$ oscillators for 0,1–coordinates. Hence, at $2n$-th level, among the boundary states created by these $\beta$-oscillators, the state created by

$$\beta^0_{-1} \cdots \beta^0_{-1} \beta^1_{-1} \cdots \beta^1_{-1} |0\rangle$$

with $n_0 + n_1 = n_0' + n_1' = n$
couples to the $2n$-th level closed string most strongly by the factor $\gamma^{2n-1}$. We thus find that two important consequences come about for the massive closed string state coupling of the D25-brane once nonzero electric field is turned on. Recall that, as was already noticed in [11, 12], the couplings to massive closed string modes blows up as $|y^0| \to \infty$. This may be interpreted as an indication of strong back-reaction effects. Once the electric field is turned on, characteristic time-scale of the back-reaction is prolonged by the (inverse) Born-Infeld factor $\gamma$, reflecting the familiar Lorentz dilation effect. In addition, the overall coupling strength is parametrically enhanced by $\gamma^{2n-1}$ at $2n$-th level. This is to be contrasted with parametric suppression of the tachyon and the sigma model dilaton couplings Eqs.(4.46, 4.54) by the factor $\gamma^{-1}$.

4.2 Rolling Tachyon with Electric + Magnetic Field

Next, we consider turning on both electric and magnetic fields on the D25-brane world-volume. We again start with the boundary state Eq.(2.3) and then perform $e$-boost and $b$-rotation. The matter part of Eq.(2.3) can be written as Eq.(3.27), where the first two coefficient functions are

$$ F(x^0) = f(x^0), \quad G_{00}(x^0) = g(x^0), \quad G_{ii}(x^0) = -f(x^0), \quad (i = 1, \cdots, 25). \quad (4.55) $$

Here, $f(x^0)$ and $g(x^0)$ are the functions defined in Eqs.(2.8, 2.9).

We next turn on constant electric field $e = F_{02}$ and magnetic field $b = F_{12}$. Following the prescription given in section 3.2.3, we obtain the matter part of the corresponding boundary state as

$$ |D25 \rangle_{T,e+b}^{\text{matter}} = F^{e+b}(\vec{y}^0) |0\rangle + G_{a b}^{e+b}(\vec{y}^0) \beta_{-1}^a \beta_{-1}^b |0\rangle + \cdots, \quad (4.56) $$
where the functions $F^{e+b}(y^0)$ and $G^{e+b}(y^0)$ are obtained by inserting Eq.(4.55) into the formulae Eq.(3.37) and Eq.(3.38). Explicitly,

$$F^{e+b}(y^0) = \frac{1}{\gamma\gamma} f(\gamma^{-1}y^0),$$

and

$$G^{e+b}_{00}(y^0) = +\gamma\gamma(1 + b^2)(1 + \cos(2\tilde{\lambda} \pi)) - \frac{1}{\gamma\gamma} f(\gamma^{-1}y^0),$$

$$G^{e+b}_{11}(y^0) = +\gamma\gamma \left( \frac{e^2b^2}{1 + b^2} \right) (1 + \cos(2\tilde{\lambda} \pi)) - \frac{1}{\gamma\gamma} \left( \frac{1 - b^2}{1+b^2} \right) f(\gamma^{-1}y^0),$$

$$G^{e+b}_{22}(y^0) = -\gamma\gamma \left( \frac{e^2}{1 + b^2} \right) (1 + \cos(2\tilde{\lambda} \pi)) - \frac{1}{\gamma\gamma} \left( \frac{1 - b^2}{1+b^2} \right) f(\gamma^{-1}y^0),$$

$$G^{e+b}_{ii}(y^0) = - \frac{1}{\gamma\gamma} f(\gamma^{-1}y^0) \quad (i = 3, \ldots, 25),$$

$$G^{e+b}_{01}(y^0) = +G^{e+b}_{10}(y^0) = \gamma\tilde{\gamma}eb (1 + \cos(2\tilde{\lambda} \pi)), $$

$$G^{e+b}_{02}(y^0) = -G^{e+b}_{20}(y^0) = \gamma\tilde{\gamma}e (1 + \cos(2\tilde{\lambda} \pi)), $$

$$G^{e+b}_{12}(y^0) = -G^{e+b}_{21}(y^0) = \gamma\tilde{\gamma} \left( \frac{e^2b}{1 + b^2} \right) (1 + \cos(2\tilde{\lambda} \pi)) + \frac{1}{\gamma\gamma} \left( \frac{2b}{1+b^2} \right) f(\gamma^{-1}y^0).$$

Thus the coupling to the closed string tachyon and the dilaton field can be immediately read off as

$$\rho_{\text{tachyon}}(y^0) = \rho_{\text{dilaton}}(y^0) = T_{25} \frac{1}{\gamma\gamma} f(\gamma^{-1}y^0).$$

The energy-momentum tensor defined by Eq.(4.48) is

$$T^{e+b}_{00}(y^0) = +T_{25} \gamma\tilde{\gamma}(1 + b^2) \cos^2(\tilde{\lambda} \pi), $$

$$T^{e+b}_{11}(y^0) = +T_{25} \gamma\tilde{\gamma} \left( \frac{e^2b^2}{1 + b^2} \right) \cos^2(\tilde{\lambda} \pi) - \frac{1}{\gamma\gamma} \left( \frac{1 - b^2}{1+b^2} \right) f(\gamma^{-1}y^0),$$

$$T^{e+b}_{22}(y^0) = -T_{25} \gamma\tilde{\gamma} \left( \frac{e^2}{1 + b^2} \right) \cos^2(\tilde{\lambda} \pi) + \frac{1}{\gamma\gamma} \left( \frac{1 - b^2}{1+b^2} \right) f(\gamma^{-1}y^0),$$

$$T^{e+b}_{ii}(y^0) = -T_{25} \frac{1}{\gamma\gamma} f(\gamma^{-1}y^0) \quad (i = 3, 4, \ldots, 25),$$

$$T^{e+b}_{01}(y^0) = T^{e+b}_{10}(y^0) = T_{25} \gamma\tilde{\gamma}eb \cos^2(\tilde{\lambda} \pi).$$

The fundamental string current density $Q^{e+b}_{[ab]} := \frac{1}{2} T_{25} G^{e+b}_{[ab]}$ is obtained as:

$$Q^{e+b}_{02}(y^0) = -Q^{e+b}_{20}(y^0) = T_{25} \gamma\tilde{\gamma}e \cos^2(\tilde{\lambda} \pi), $$

$$Q^{e+b}_{12}(y^0) = -Q^{e+b}_{21}(y^0) = T_{25} \gamma\tilde{\gamma} \left( \frac{e^2b}{1 + b^2} \right) \cos^2(\tilde{\lambda} \pi) + \frac{1}{\gamma\gamma} \left( \frac{b}{1+b^2} \right) f(\gamma^{-1}y^0).$$
The result may be understood heuristically as follows. The mutually orthogonal electric and magnetic fields give rise to nonzero field momentum on the Dp-brane world-volume (for instance, for \( p = 3 \), the field momentum is given by the Poynting vector, whose magnitude is \( eb \) and direction is perpendicular to the plane spanned by the electric and magnetic fields). As such, non-vanishing components of the energy-momentum tensor \( T_{01}^{e+b} \) and the string current density tensor \( Q_{02}^{e+b} \) are interpretable as arising from rigid flow of the fundamental string fluid. Indeed, it is possible to boost the system along 1-direction by

\[
V = \frac{T_{01}^{e+b}}{T_{00}^{e+b}} = \frac{eb}{1 + b^2}
\]

and

\[
\Gamma = 1/\sqrt{1 - V^2}
\]

into an inertial ‘rest’ frame, where the fluid is at rest. This rigid flow is also expected from the relation Eq.(3.30). From the requirement that \( e^2 \leq (1 + b^2) \), we deduce that the boost velocity \( V \) never exceeds the speed of light. Note that the magnetic field induces D23-brane density. However, as this D23-brane is also unstable and will eventually evaporate, we expect that the system behaves in a manner similar to that in pure electric field at late time. Indeed, in the ‘rest’ frame, non-vanishing components of the energy-momentum tensor are

\[
\tilde{T}_{00}^{e+b} = T_{25} \cos^2(\tilde{\lambda} \pi) \gamma \tilde{\gamma} \left( \frac{1 + b^2}{\Gamma^2} \right)
\]

\[
\tilde{T}_{22}^{e+b} = -T_{25} \cos^2(\tilde{\lambda} \pi) \gamma \tilde{\gamma} \left( \frac{e^2}{1 + b^2} \right) = -v_s^2 \tilde{T}_{00}^{e+b},
\]

while that of the fundamental string current density tensor is

\[
\tilde{Q}_{02}^{e+b} = T_{25} \cos^2(\tilde{\lambda} \pi) \gamma \tilde{\gamma} \left( \frac{e}{\Gamma} \right) = v_s \tilde{T}_{00}^{e+b},
\]

Here, we have denoted the sound velocity of the fundamental string fluid as

\[
v_s^2 := \frac{\tilde{T}_{22}}{\tilde{T}_{00}} = \frac{e^2}{(1 + b^2)^2 - e^2 b^2},
\]

and suppressed \( f(\gamma^{-1} y^0) \)-dependent contributions, as they drop out at late time or in the critical limit.

Since the gauge field strengths are constrained \( e^2 \leq (1 + b^2) \), we find that the sound velocity cannot exceed the speed of light

\[
v_s^2 \leq \frac{e^2}{1 + b^2} \leq 1.
\]

Therefore, it is now evident that the BPS-like inequality comes about as

\[
\tilde{T}_{00}^{e+b} \geq |\tilde{Q}_{02}^{e+b}| \geq -\tilde{T}_{22}^{e+b}.
\]

For sub-critical background \( e^2 < (1 + b^2) \), the fundamental string fluid behaves as a non-relativistic medium in that the sound velocity of the fluid is given by \( v_s \). One readily finds that the two inequalities Eqs.(4.62, 4.61) are saturated precisely in the critical limit \( e^2 = (1 + b^2) \).
5 Comparison with Effective Field Theory Approach

Finally, we shall be considering the rolling tachyon with electric and magnetic fields in the effective field theory approach. Consider the $D_p$-brane effective action of the form

$$S_{DBI} = -T_p \int_{\Sigma_{p+1}} d^{p+1}x \, V(T) \sqrt{-\det(\eta + F)} \mathcal{F}(z), \quad (5.63)$$

where

$$z = \left( \frac{1}{\eta + F} \right)^{(ab)} \partial_a T \partial_b T.$$  

We will not make use of the explicit form of the function $\mathcal{F}(z)$ and the tachyon potential $V(T)$ in the following discussion. If we choose

$$\mathcal{F}(z) = \frac{z^4 \Gamma(z)^2}{2 \Gamma(2z)} \quad \text{and} \quad V(T) = e^{-\frac{1}{4}r^2},$$

the action Eq.(5.63) coincides with the background-independent string field theory action for a non-BPS D-brane in super-string theory given in [26, 27]. We can also get the action proposed in [28] with an alternative choice $\mathcal{F}(z) = \sqrt{1+z}$. A similar treatment can be found in [14].

We consider the situation considered in section 4.2, and assume that we can safely restrict our consideration with homogeneous tachyon rolling with constant electric field $e = F_{02}$ and magnetic field $b = F_{12}$. Then, the D-brane world-volume Lagrangian density becomes

$$\mathcal{L} = -T_p V(T) \sqrt{1 - e^2 + b^2} \mathcal{F}(z),$$

in which

$$z = -\left( \frac{1 + b^2}{1 - e^2 + b^2} \right) \dot{T}^2(t).$$

The canonical conjugate momenta are

$$\pi \equiv \frac{\delta \mathcal{L}}{\delta \dot{e}} = T_p \frac{e}{\sqrt{1 - e^2 + b^2}} V(T) \mathcal{D}(z), \quad (5.64)$$

$$P_T \equiv \frac{\delta \mathcal{L}}{\delta \dot{T}} = T_p \frac{2(1 + b^2)}{\sqrt{1 - e^2 + b^2}} V(T) \mathcal{F}'(z) \dot{T},$$

where we have defined $\mathcal{D}(z) := \mathcal{F}(z) - 2z \mathcal{F}'(z)$. The Hamiltonian density is then obtained as

$$\mathcal{H} = (e\pi + \dot{T}P_T - \mathcal{L}) = T_p \frac{1 + b^2}{\sqrt{1 - e^2 + b^2}} V(T) \mathcal{D}(z). \quad (5.65)$$

This Hamiltonian density should be compared with the energy density $T_{00}^{e+b}$ computed in the previous section from the corresponding boundary state. We find a complete agreement between Eq.(4.59) and Eq.(5.65) provided we identify

$$V(T) \mathcal{D}(z) \longleftrightarrow \frac{1}{2} \left( 1 + \cos(2\pi \tilde{\lambda}) \right). \quad (5.66)$$
For example, with $F_{ab} = \dot{T} = 0$, $D(z) = 1$ and $V(T) \rightarrow 0$ as $\lambda \rightarrow 1/2$. This explains Sen’s identification of the closed string vacuum with $\lambda = 1/2$.

Coupling to the Kalb-Ramond field $B_{ab}$ can be introduced by replacing $F_{ab}$ with the gauge-invariant combination $(F_{ab} + B_{ab})$ in Eq.(5.63). As such, the source for $B_{ab}$ field is obtained by varying the Lagrangian density with respect to $F_{ab}$. Therefore, the electric displacement field $\pi$ in Eq.(5.64) ought to correspond to $Q^{02}_{e+b} = -Q^{e-b}_{02}$ in Eq.(4.60). In fact, they agree each other once the identification Eq.(5.66) is made. Similarly, the source current tensor that couples to the $B_{12}$ field is obtained as

$$\frac{\partial \mathcal{L}}{\partial b} = - T_p \left( \bar{\gamma} \bar{\gamma} \left( \frac{e^2 b}{1 + b^2} \right) V(T) D(z) + \frac{1}{\bar{\gamma}} \left( \frac{b}{1 + b^2} \right) V(T) \mathcal{F}(z) \right).$$

Again, this agrees with $Q^{e+b}_{12}$ in Eq.(4.60) provided, in addition to the previous identification Eq.(5.66), we also make the identification

$$V(T) \mathcal{F}(z) \longleftrightarrow f(\gamma^{-1} y^0). \quad (5.67)$$

The energy-momentum tensor is computable from the Lagrangian density Eq.(5.63) by replacing the flat metric $\eta_{ab}$ with a curved one $g_{ab}$ and keep the terms which are linear with respect to the virtual variation $\delta g_{ab} = g_{ab} - \eta_{ab}$. Some useful relations are

$$\delta \sqrt{-\det(g + F)} = \sqrt{-\det(\eta + F)} \frac{1}{\eta + F} \delta g_{ab} \quad (ab)$$

$$= \gamma \bar{\gamma} \left( -(1 + b^2)\delta g_{00} + eb(\delta g_{01} + \delta g_{10}) + (1 - e^2)\delta g_{11} + \delta g_{22} \right)$$

and

$$\delta z = - \left( \frac{1}{\eta + F} \delta g \frac{1}{\eta + F} \right) (ab) \bar{T} \partial_a T \partial_b T$$

$$= -\gamma^4 \bar{\gamma}^4 \left[ (1 + b^2)\delta g_{00} - eb(1 + b^2)(\delta g_{01} + \delta g_{10}) + b^2 e^2 \delta g_{11} - e^2 \delta g_{22} \right] \bar{T}^2.$$

As a result, we obtain

$$\frac{\delta \mathcal{L}}{\delta g_{00}} = \frac{T_p}{2} \gamma \bar{\gamma} (1 + b^2) V(T) D(z),$$

$$\frac{\delta \mathcal{L}}{\delta g_{11}} = \frac{T_p}{2} \left( +\gamma \bar{\gamma} \frac{e^2 b^2}{1 + b^2} V(T) D(z) - \frac{1}{\bar{\gamma}} \frac{1}{1 + b^2} V(T) \mathcal{F}(z) \right),$$

$$\frac{\delta \mathcal{L}}{\delta g_{22}} = \frac{T_p}{2} \left( -\gamma \bar{\gamma} \frac{e^2}{1 + b^2} V(T) D(z) - \frac{1}{\bar{\gamma}} \frac{1}{1 + b^2} V(T) \mathcal{F}(z) \right),$$

$$\frac{\delta \mathcal{L}}{\delta g_{i i}} = - \frac{T_p}{2} \gamma \bar{\gamma} \left( i = 3, \cdots, p \right),$$

$$\frac{\delta \mathcal{L}}{\delta g_{01}} = \frac{\delta \mathcal{L}}{\delta g_{10}} = - \frac{T_p}{2} \gamma \bar{\gamma} eb V(T) D(z).$$
We confirm that all these reproduce correctly the energy-momentum tensor given in Eq.(4.59), once we make use of the relations Eq.(5.66) and Eq.(5.67).

The coupling to the sigma model dilaton field is obtainable by multiplying $e^{-\varphi}$ factor to the $D_p$-brane world-volume Lagrangian density. As such, the dilaton tadpole is given by

$$
\rho_{\text{dilaton}} = -\mathcal{L}(T) = T_p V(T) \sqrt{1 - e^2 + b^2 \mathcal{F}(z)}.
$$

This agrees with the result Eq.(4.58) obtained from the boundary state formalism in the previous section provided the identification Eq.(5.67) is again used.

How does the coupling to the closed string modes look like in the Einstein frame? Take, for instance, the massless modes. The dilaton, metric, and Kalb-Ramond fields in the Einstein frame, $\varphi^E, g^E_{ab}, B^E_{ab}$, are related to those in the sigma model frame as

$$
\varphi^E = \varphi, \quad g^E_{ab} = e^{-\alpha \varphi} g_{ab}, \quad B^E_{ab} = e^{-\alpha \varphi} B_{ab},
$$

where $\alpha = 1/6$ for bosonic string and $\alpha = 1/2$ for super-string theories, respectively. Expanding the $D_p$-brane world-volume Lagrangian density up to the linear order with respect to the fluctuation of these fields around the flat background, we obtain

$$
\delta \mathcal{L} \sim \rho_{\text{dilaton}} \delta \varphi + \frac{1}{2} T^{ab} \delta g_{ab} + \frac{1}{2} Q^{ab} \delta B_{ab}
$$

$$
\sim \left( \rho_{\text{dilaton}} + \frac{\alpha}{2} \eta_{ab} T^{ab} \right) \delta \varphi^E + \frac{1}{2} T^{ab} \delta g^E_{ab} + \frac{1}{2} Q^{ab} \delta B^E_{ab}.
$$

It shows that the energy-momentum tensor and the string current tensor remains unchanged, while the dilaton coupling is modified in the Einstein frame to:

$$
\rho^E_{\text{dilaton}} = \rho_{\text{dilaton}} + \frac{\alpha}{2} \eta_{ab} T^{ab}.
$$

As the trace of the energy-momentum tensor asymptotes to a non-vanishing constant value at late time (see Eq.(4.59)), we see that the dilaton coupling remains finite in the Einstein frame, though it vanishes in the sigma model frame.

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