The geometric phase in open systems

\[ i |\psi(t)\rangle = \sum_{\{\alpha\}} W(\alpha) \langle \alpha | \psi(t) \rangle, \]

where \( W(\alpha) = 1 - i \Pi(\alpha) \) and \( W(\alpha) = \frac{1}{2} \mathcal{L}(\alpha) |\psi(t)\rangle \).

For a small time interval \( \Delta t \), we can describe the time evolution of the density matrix by

\[ \rho(t + \Delta t) \approx \rho(t) + i \Delta t \mathcal{L}(\alpha) \rho(t) \Delta t. \]

The quantum jumps model is particularly suitable for the case of quantum phases in open systems, as for example in the case of an open quantum system, we can always define and calculate a geometrical phase for the system's unitary evolution, regardless of whether the jumps are included. Therefore, we avoid the problem of finding the proper parallel transport, which is a system evolving according to the following geometric phase.

Let us take a system evolving according to the following geometric phase.

In this case, the geometric phase is proportional to the number of jumps, which is independent of the number of jumps determined by the dephasing operator.

\[ H = \frac{1}{2} \sum_{\{\alpha\}} \mathcal{L}(\alpha) |\psi(t)\rangle \]

Note that the operators \( \mathcal{L}(\alpha) \) fulfill the completeness relation

\[ \sum_{\{\alpha\}} |\alpha\rangle \langle \alpha | = 1. \]

In this description, the dynamics of the system is described by

\[ \rho(t) = e^{-i H \Delta t} \rho(t) e^{i H \Delta t}. \]

The geometric phase is given by

\[ \Phi = \frac{1}{2} \sum_{\{\alpha\}} \mathcal{L}(\alpha) \langle \alpha | \psi(t) \rangle. \]

The geometric phase in open systems is then calculated for a set of possible trajectories, occurring each with a finite probability, \( \langle \alpha | \psi(t) \rangle \). The time-evolved state is then obtained by summing over all possible trajectories, i.e.,

\[ |\psi(t + \Delta t)\rangle = \sum_{\{\alpha\}} |\alpha\rangle \langle \alpha | \psi(t) \rangle. \]

The geometric phase is calculated by

\[ \Phi = \frac{1}{2} \sum_{\{\alpha\}} \mathcal{L}(\alpha) \langle \alpha | \psi(t) \rangle, \]

where \( \mathcal{L}(\alpha) \) is the geometric phase associated to the \( \{\alpha\} \)-th trajectory.
is given by the Pancharatnam formula: 

\[ \gamma_\phi = - \arg \left\{ \langle \psi_1 | \psi_2 | \psi_3 | \ldots | \psi_N \rangle \langle \psi_N | \psi_1 \rangle \right\} \]  

(5)

Therefore, we are able to associate a meaningful geometrical phase to each trajectory \( \Gamma \) described by the system, as the continuous limit of Eq. (4) for the sequence \( \{ \psi_1, \psi_2, \ldots, \psi_N \} \).

As an example, let us consider the “no-jump” trajectory for a completely general master equation. The evolution of a quantum state along this trajectory is obtained by the repeated action of the operator \( W_\delta \). At the time \( t = m \Delta t \), the quantum state will be approximately given by:

\[ |\psi_m\rangle = (W_\delta)^m |\psi_0\rangle = \left( 1 - i \frac{T}{N} H \right)^m |\psi_0\rangle. \]  

(6)

which in the continuous limit \( N \to \infty \) yields to a dynamics governed by the complex effective Hamiltonian \( \hat{H} \):

\[ i \frac{d}{dt} |\psi^0(t)\rangle = \hat{H} |\psi^0(t)\rangle \quad |\psi^0(0)\rangle = |\psi_0\rangle \]  

(7)

Thus, the evolution corresponding to this trajectory is given by a smooth chain of (non-normalized) states \( |\psi(t)\rangle \), in which case \( \gamma_\phi \) converges to:

\[ \gamma = - \text{Im} \int_0^T \frac{\langle \psi(t) | \hat{H} | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle} dt - \arg \left\{ \langle \psi(T) | \psi(0) \rangle \right\}. \]  

(8)

Substituting Eq. (4) into Eq. (6), we obtain the geometric phase for a no-jump trajectory, which is given by:

\[ \gamma^0 = \int_0^T \frac{\langle \psi^0(t) | \hat{H} | \psi^0(t) \rangle}{\langle \psi^0(t) | \psi^0(t) \rangle} dt - \arg \left\{ \langle \psi^0(T) | \psi^0(0) \rangle \right\} \]  

(9)

This is the geometric phase associated to a non-unitary evolution of a system \( \Gamma \), when there are no jumps. The first term is clearly the opposite of the dynamical phase associated to the non-unitary evolution, as it is given by the average of the Hamiltonian (up to a minus sign) along the path traversed by the system. The second term is the total phase difference between the final and the initial state, according to Pancharatnam’s definition of distant parallelism \( \Gamma \). Thus the geometric phase is obtained as the difference between total and dynamical phase associated to a given evolution of pure states \( \Gamma \).

Note that, in the special case in which \( \sum_{j=1}^n \Gamma_j^\dagger \Gamma_j \propto 1 \) (which is a unitary evolution), the geometric phase associated with the no-jump trajectory is the same as the one acquired by an isolated system evolving under the same Hamiltonian \( \hat{H} \). This becomes clear when one notes that, in this case, \( W_\delta = (1 - \alpha) I + i \hat{H} \Delta t \) and the evolution of state \( |\Psi(t)\rangle \) is the same as its isolated counterpart up to a global normalization factor \( e^{-\alpha \omega} \). In other words, for this particular source of decoherence, if the reservoir is permanently measured and no jump is detected, there is no gain of information on the system, which simply projects it back into its unitary evolution.

Note, also, that following the idea of \( \Gamma \), it is possible to represent the geometric phase as the integral of the Berry connection form:

\[ d\omega = \text{Im} \left\{ \frac{\langle \psi(t) | \hat{H} | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle} \right\} dt \]  

(10)

along a closed path. This path is formed by the trajectory \( \psi(t) \) followed by the states along the Hilbert space during the dynamical evolution and the shortest geodesic connecting final and initial states \( \psi(T) \) and \( \psi(0) \). Thus the second term of equation (9) can be regarded as the path integral of the Berry connection along this geodesic.

Suppose, now, that there is only one jump in the trajectory at an arbitrary time \( t_1 \), which occurs in a time much shorter than any other characteristic time of the system. Then, we can separate the evolution into two parts (before and after the jump) and, the continuous limit of equation (4) leads to the following expression:

\[ \gamma_j^1 = \int_0^{t_1} \frac{\langle \psi(t) | \hat{H} | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle} dt - \arg \left\{ \langle \psi(T) | \psi(0) \rangle \right\} + \int_{t_1}^T \frac{\langle \psi(t) | \hat{H} | \psi(t) \rangle}{\langle \psi(t) | \psi(t) \rangle} dt - \arg \left\{ \langle \psi''(T) | \psi''(0) \rangle \right\} \]  

(11)

where \( W_j \) is the operator associated to the occurred jump, and \( \psi(t) \) and \( \psi''(t) \) are the states evolving under the effective Hamiltonian \( \hat{H} \), before and after the jump respectively. They are given by the equation (4) with initial conditions \( \psi(0) = \psi_0 \) and \( \psi(t_1) = W_j \psi(t_1) \), respectively.

The first and third terms represent the dynamical phase given by the effective evolution (4), before and after the jump occurs. The last term is the phase difference between initial and final state of the total evolution. The second term is a phase associated to the occurrence of a jump at time \( t_1 \). Analogously to the total phase associated to final and initial state, this term represents the phase difference between the states after and before the jump, and geometrically, it can be regarded as the path integral of the Berry connection along the shortest geodesic joining them.

This result can be easily generalized to any trajectory, allowing for a more complicated sequence of jumps and no-jump evolutions. The geometric phase is then represented as the sum of terms of the form \( \arg \left\{ \langle \psi(t_1) | \Gamma_j | \psi(t_1) \rangle \right\} \) regarded as the phase associated to the jump \( \Gamma_j \) occurring at the instant \( t_1 \), and terms of the form (4) for the no jump evolutions. And clearly all these phases can be regarded as the integrals of Berry connection along a complex path composed of geodesics joining initial and final state of the jumps, and the paths traversed by the state during the evolution under \( \hat{H} \).
Let us apply this general quantum jumps procedure to a well known physical system. First, let us consider
the simplest example of decoherence: a two levels system
to under the free Hamiltonian $H = \frac{\hbar}{2} \sigma_z$ and
subjected to dephasing, which can be described by the
Master equation \( \Gamma \) with $\Gamma = \lambda \sigma_z$, where $\lambda$ is the the coefficient giving the probability per unit time of a “phase
jump”.

Since this is a decoherence model for which $\Gamma \Gamma = \sigma_z^2 \propto 1$, which is a simple instant of a unital evolution,
according to the previous considerations, the geometric
phase associated to the no-jump case is given by
the standard geometric phase associated to the unitary
evolution of a spin 1/2 linearly coupled to a constant
magnetic field. For instance, after a time $t = 2\pi/\omega$,
$\gamma_\theta = \pi(1 - \langle \psi_0 | \sigma_z | \psi_0 \rangle) = \pi(1 - \cos \theta)$, where $\psi_0$ is the
initial state and $\theta$ is its azimuthal angle in the Bloch
sphere representation.

Although the no-jump case may seem trivial, this
system has a much more remarkable property: the geometric
phase is actually robust against dephasing, in this simple,
but very useful example. In fact, we show below that the
final geometric phase is unaffected by any number of
jumps for any particular trajectory. To show that, let
us consider first the case of a single jump, in which the
phase is given by:

$$
\gamma_1 = -\int_0^{t_1} \frac{\omega}{2} \langle \psi | \sigma_z | \psi \rangle dt - \text{arg} \{ \langle \psi | \sigma_z | \psi \rangle \}
$$

where the fact that $H$ and $\Gamma$ commute has been used. 
This result is easily generalized to any number $k$ of jumps:

$$
\gamma_k = -\int_0^{2\pi/\omega} \frac{\omega}{2} \langle \psi | \sigma_z | \psi \rangle dt - \text{arg} \{ \langle \psi | \sigma_z | \psi \rangle^k \} = \pi(1 - \cos \theta),
$$

Thus, no matter how many jumps occur in the chosen
trajectory, we can associate the same geometric evolution
to the system. There is a simple geometrical explanation for
this effect. Dephasing is a special source for decoherence
because it does not change the projection of the spin vector
on the direction of the magnetic field, i.e. it does not change
the relative angle $\theta$ between the directions of the
magnetic field and the spin. After each jump, the spin
is still precessing around the magnetic field alongside the
same curve. As a result, the total area covered by its
trajectory remains the same, and so does the geometric
phase acquired by the spin state, which is proportional
to this area. Therefore, in the end, the geometric phase
acquired by the spin state will be the same, no matter
how diffused its total phase may be. That does not
mean that dephasing will not affect the measurement of
this phase. Indeed, it will lower the visibility of any inter-
ference measurement made on the spin, because the
visibility of the state is lowered when its mixedness is
increased (we will address this in more details in a sepa-
rate publication). However, as the calculations above
show, the reduced visibility will be caused by a random-
ization of the dynamical phase, and not the geometrical
one, which proves to be much more robust in this case.

A more realistic example includes spontaneous decay
as a source of decoherence for the spin 1/2 system. In
this case, it is only worth analyzing the no-jump case,
since any jump causes immediate and complete loss of
phase information of the quantum state. Spontaneous
decay \( \gamma = \alpha \sigma_- \) is a decoherence source that cannot be
associated to a unital map \( \sigma_+ \sigma_- \neq 1 \) and, therefore,
the phase will be affected even if no jump is detected.
However, as we show in figure 2, the no-jump trajectory
is a smooth spiral converging to the lower state, which
still allows us to calculate the phase using Eq. 8. We
obtained \( \gamma = \pi + \frac{\omega}{2} \ln \{ \langle \psi | e^{-i\frac{\omega}{2} - \sigma_z \tau} | \psi \rangle \} \), which in the
limit $\omega \gg \alpha$ leads to

$$
\gamma \approx \pi(1 - \cos \theta) + (4\pi)^2 \frac{\alpha^2}{\omega} \sin^2 \theta + \alpha \left( \frac{\alpha}{\omega} \right)^2
$$

Again, this result has a very simple geometrical explana-
tion: as we observe the reservoir and detect no jump, the
probability that the system is in the lower state smoothly
increases, changing $\theta$ and, therefore, the element of area
covered by the spin trajectory in each infinitesimal time
interval, as shown in figure 2.

Another simple case that can be analyzed is the spin
flip along a direction arbitrary $\Gamma = \sigma_\theta$. In this case,
the no jump situation is again trivial and similar to the
dephasing reservoir, since $\sigma_z^2 = 1$. When one or more
jumps occur, we can use Eq. 11 (or its generalization to
many jumps) to easily calculate the final phase, which
will be a sum of the partial areas covered in each trajec-
tory with plus or minus sign depending on the respective
coupling energy of the spin with the magnetic field. Our
treatment is, of course, applicable even when the master
 equation contains many different sources of errors act-
ing simultaneously on the system, since we can use the
generalized form of equation 11 to calculate the phase.

In conclusion, in this paper, we present a method to
calculate geometric phases in open systems. Our method
is general and can be applied as long as the system dy-
namics is described by a master equation in the form of
Eq. 11, which is the most general completely positive
trace preserving continuous evolution. By using the
quantum jumps approach we avoid the problem of defin-
ing Berry’s phases for mixed states: in each trajectory,
the quantum state of the system remains pure and the
phase can be calculated through usual procedures. In
particular, we show that it is always possible to calcu-
late this phase, either for the nojump trajectories or
for the ones in which one or more jumps occur. We also show that, for special unitary decoherence sources, the phase remains unaffected for the no jumps trajectories. As a direct application of our method, we calculate the geometric phases of spin 1/2 systems coupled to different reservoirs. We show that these phases are totally robust against phase diffusion, in which case the lower visibility observed due to the non-unitary evolution may be attributed solely to a randomization of the dynamical phase. This property may be interesting for possible applications, especially in quantum computing, since dephasing may be difficult to monitor and correct, in general. Therefore, it is interesting noticing that geometric phases are robust against this decoherence source. We also present a nice geometrical explanation to this effect, as well as to the effect on the geometric phase when spontaneous emission is present, but no jump is detected. We also briefly comment on other typical decoherence effects on the system, like arbitrary spin flips. The method presented here is completely general and can be applied to many other physical systems.

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