II. MASSIVE PARTICLES

Complementary physical laws are only one approximation of a correct description of a quantum system. In this section, we show the advantages of a quantum description of a particle in terms of localization, coherence, interference, and quantum entanglement. The next two sections present detailed calculations on the quantum nature of a particle in a magnetic field.

An open question is how to describe a single particle in a magnetic field. A magnetic field is a source of spin-orbit coupling, which is responsible for the spin-dependent polarizability of a particle. In a magnetic field, the spin and orbital angular momenta are not independent, and the resulting magnetic moment is given by the spin-orbit coupling.

The quantum mechanical description of a particle in a magnetic field is a powerful tool for understanding the behavior of particles at the quantum level. It allows us to calculate the magnetization of a particle, which is a measure of the magnetic moment.

1. INTRODUCTION

The phenomenon of quantum coherence is a fundamental aspect of quantum mechanics. In the quantum world, the properties of a system are described by a wave function, which is a mathematical object that can have complex values. The wave function is a probability amplitude, and its square gives the probability of finding the system in a certain state.

In quantum mechanics, the Schrödinger equation governs the evolution of a quantum system. The equation describes the time-dependent behavior of the wave function, and it is a central equation in quantum mechanics.

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spinor,

$$\psi(p) = \begin{pmatrix} a_1(p) \\ a_2(p) \end{pmatrix},$$

(3)

where the amplitudes $a_i$ satisfy $\sum_i \int |a_i(p)|^2 dp = 1$. The normalization of these amplitudes is a matter of convenience, depending on whether we prefer to include a factor $p_0 = (m^2 + p^2)^{1/2}$ in it, or to have such factors in the transformation law below. Here we shall use the second alternative, because it is closer to the nonrelativistic notation which appears in the usual definition of entropy.

We emphasize that we consider normalizable states, in the momentum representation, not momentum eigenstates as usual in textbooks on particle physics. The latter are chiefly concerned with the computation of (in-out) matrix elements needed to obtain cross sections and other asymptotic properties. However, in general particles have no definite momentum. For example, if an electron is elastically scattered by some target, the electron state after the scattering is a superposition that involves momenta in all directions. In that case, it is still formally possible to ask, in any Lorentz frame, what is the value of a spin component in a given direction (this is a legitimate Hermitian operator).

Let us define a reduced density matrix, $\tau = \int dp \psi(p)\psi^\dagger(p)$, giving statistical predictions for the results of measurements of spin components by an ideal apparatus which is not affected by the momentum of the particle. The spin entropy is

$$S = -\text{tr}(\tau \log \tau) = - \sum_j \lambda_j \log \lambda_j,$$

(4)

where $\lambda_j$ are the eigenvalues of $\tau$.

As usual, ignoring some degrees of freedom leaves the others in a mixed state. What is not obvious is that in the present case the amount of mixing depends on the Lorentz frame used by the observer. Indeed consider another observer (Bob) who moves with a constant velocity with respect to Alice who prepared state $\psi$. In the Lorentz frame where Bob is at rest, the same spin $\frac{1}{2}$ particle has a state

$$\psi'(p) = \begin{pmatrix} a_1'(p) \\ a_2'(p) \end{pmatrix}.$$

(5)

The transformation law is

$$a'_i(p) = [(\Lambda^{-1})_i]_j \sum_j D_{rs}[\Lambda, (\Lambda^{-1})_j] a_j(\Lambda^{-1})_r,$$

(6)

where $D_{rs}$ is the Wigner rotation matrix for a Lorentz transformation $\Lambda$.

As an example, take a particle prepared by Alice with spin in the $z$ direction, so that $a_1(p) = 0$, and

$$a_2(p) = N \exp(-p^2/2\Delta^2),$$

(7)

where $N$ is a normalization factor. Spin and momentum are not entangled, and the spin entropy is zero. When that particle is described in Bob’s Lorentz frame, moving with velocity $\beta$ in a direction at an angle $\theta$ with Alice’s $z$-axis, a detailed calculation shows that both $a_1'$ and $a_2'$ are nonzero, so that the spin entropy is positive. This phenomenon is illustrated in Fig. 1. It can be shown that a relevant parameter, apart from the angle $\theta$, is the leading order in momentum spread,

$$\Gamma = \frac{\Delta}{m} 1 - \frac{(1 - \beta^2)^{1/2}}{\beta},$$

(8)

where $\Delta$ is the momentum spread in Alice’s frame. The entropy has no invariant meaning, because the reduced density matrix $\tau$ has no covariant transformation law, except in the limiting case of sharp momenta. Only the complete density matrix transforms covariantly.

![Fig. 1: Dependence of the spin entropy $S$, in Bob’s frame, on the values of the angle $\theta$ and a parameter $\Gamma = \frac{1 - (1 - \beta^2)^{1/2}}{\beta}\Delta/m\beta$.](image)

It is noteworthy that a similar situation arises for a classical system whose state is given in any Lorentz frame by a Liouville function. Recall that a Liouville function expresses our probabilistic description of a classical system — what we can predict before we perform an actual observation — just as a quantum state is a mathematical expression used for computing probabilities of events.

Consider now a pair of orthogonal states that were prepared by Alice. How well can moving Bob distinguish them? We shall use the simplest criterion, namely the probability of error $P_E$, defined as follows: an observer receives a single copy of one of the two known states and performs any operation permitted by quantum theory in order to decide which state was supplied. The probability of a wrong answer for an optimal measurement is

$$P_E(p_1, p_2) = \frac{1}{2} - \frac{1}{3} \text{tr} \sqrt{(p_1 - p_2)^2}.$$  

(9)

In Alice’s frame $P_E = 0$. It can be shown that in
Bob’s frame, $P_E^I \propto \Gamma^2$, where the proportionality factor depends on the angle $\theta$ defined above.

An interesting problem is the relativistic nature of quantum entanglement when there are several particles. For two particles, an invariant definition of the entanglement of their spins would be to compute it in the Lorentz “rest frame” where $\langle \sum \mathbf{p} \rangle = 0$. However, this simple definition is not adequate when there are more than two particles, because there appears a problem of cluster decomposition: each subset of particles may have a different rest frame. This is a difficult problem, still awaiting for a solution. We shall mention only a few partial results.

Alsing and Milburn considered bipartite states with well-defined momenta. They showed that while Lorentz transformations change the appearance of the state in different inertial frames and the spin directions are Wigner rotated, the amount of entanglement remains intact. The reason is that Lorentz boosts do not create spin-momentum entanglement when acting on eigenstates of momentum, and the transformations on the pair are implemented on both particles as local unitary transformations which are known to preserve the entanglement. The same conclusion is also valid for photon pairs.

However, realistic situations involve wave packets. For example, a general spin-$\frac{1}{2}$ two-particle state may be written as

$$|Y_{12}\rangle = \sum_{\sigma_1, \sigma_2} \int dp_1 dp_2 \frac{\mathcal{p}_1 \cdot \mathcal{p}_2}{(2\pi)^3 2p^3} |\mathbf{p}_1, \sigma_1 \rangle \otimes |\mathbf{p}_2, \sigma_2 \rangle,$$

where

$$d\mu(p) = \frac{d^3p}{(2\pi)^3 2p^3}$$

is a Lorentz-invariant measure. For particles with well defined momenta, $\mathbf{p}$ is sharply peaked at some values $\mathbf{p}_{10}$, $\mathbf{p}_{20}$. Again, a boost to any Lorentz frame $S'$ will result in a unitary $U(A) \otimes U(A)$, acting on each particle separately, thus preserving the entanglement. However, if the momenta are not sharp, so that the spin-momentum entanglement is frame dependent, then the spin-spin entanglement is frame-dependent as well.

Gingrich and Adami investigated the reduced density matrix for $|Y_{12}\rangle$ and made explicit calculations for the case where $g$ is a Gaussian, as in $\mathcal{P}$. They showed that if two particles are maximally entangled in a common (approximate) rest frame (Alice’s frame), then the degree of entanglement, as seen by a Lorentz-boosted Bob, decreases when the boost parameter $\beta \to 1$. Of course, the inverse transformation from Bob to Alice will increase the entanglement. Thus, we see that the spin-spin entanglement is not a Lorentz invariant quantity, exactly as spin entropy is not a Lorentz scalar.

III. PHOTONS

Relativistic effects that we describe in this section are essentially different from those for massive particles that were discussed above, because photons have only two linearly independent polarization states. The properties that we discuss are kinematical, not dynamical. At the statistical level, it is not even necessary to involve quantum electrodynamics. Most formulae can be derived by elementary classical methods $\mathcal{P}$. It is only when we consider individual photons, for cryptographic applications, that quantum theory becomes essential. The diffraction effects mentioned above lead to superposition rules which make it impossible to define a reduced density matrix for polarization. As shown below, it is still possible to have “effective” density matrices; however, the latter depend not only on the preparation process, but also on the method of detection that is used by the observer.

In applications to secure communication, the ideal scenario is that isolated photons (one particle Fock states) are emitted. In a more realistic setup, the transmission is by means of weak coherent pulses containing on the average less than one photon each. A basis of the one-photon space is spanned by states of definite momentum and helicity,

$$|k, \epsilon^\pm \rangle \equiv |k\rangle \otimes |\epsilon^\pm \rangle,$$

where the momentum basis is normalized by $\langle q|k \rangle = (2\pi)^3 (2\hbar)^3 \theta(q-k)$, and helicity states $|\epsilon^\pm \rangle$ are explicitly defined by Eq. (12) below.

As we know, polarization is a secondary variable: states that correspond to different momenta belong to distinct Hilbert spaces and cannot be superposed (an expression such as $|\epsilon^+ \rangle + |\epsilon^- \rangle$ is meaningless if $\mathbf{k} \neq \mathbf{q}$). The complete basis $\mathcal{M}$ does not violate this superposition rule, owing to the orthogonality of the momentum basis. Therefore, a generic one-photon state is given by a wave packet

$$|\Psi\rangle = \int d\mu(k)|\mathbf{k}\rangle|\mathbf{k}, \alpha(\mathbf{k})\rangle.$$

The Lorentz-invariant measure is $d\mu(k) = d^3k/(2\pi)^3 2k^0$, and normalized states satisfy $\int d\mu(k)/|\mathbf{k}|^2 = 1$. The generic polarization state $|\alpha(\mathbf{k})\rangle$ corresponds to the geometrical 3-vector

$$\alpha(\mathbf{k}) = \alpha_+|\mathbf{k}\rangle \epsilon_+^\pm + \alpha_-|\mathbf{k}\rangle \epsilon_-^\pm,$$

where $|\alpha_+|^2 + |\alpha_-|^2 = 1$, and the explicit form of $\epsilon_{\pm}^\pm$ is given below.

Lorentz transformations of quantum states are most easily computed by referring to some standard momentum, which for photons is $\mathbf{p}' = (1, 0, 0, 1)$. Accordingly, standard right and left circular polarization vectors are $\epsilon_{\pm}^\pm = (1, \pm i, 0)/\sqrt{2}$. For linear polarization, we take Eq. (12) with $\alpha_+ = (\alpha_-)^*$, so that the 3-vectors $\alpha(\mathbf{k})$ are real. In general, complex $\alpha(\mathbf{k})$ correspond to elliptic polarization.

Under a Lorentz transformation $A$, these states become $|k_A, \alpha(\mathbf{k}_A)\rangle$, where $\mathbf{k}_A$ is the spatial part of a four-vector $k_A = A\mathbf{k}$, and the new polarization vector can be obtained by an appropriate rotation $\mathcal{P}$:

$$\alpha(\mathbf{k}_A) = R(\mathbf{k}_A) R(\mathbf{k})^{-1} \alpha(\mathbf{k}),$$
where $\mathbf{k}$ is the unit vector in the direction of $\mathbf{k}$. Finally, for each $\mathbf{k}$ a polarization basis is labeled by the helicity vectors,

$$\epsilon_{\mathbf{k}}^\pm = R(\mathbf{k})\epsilon_{\mathbf{p}}^\pm. \quad (16)$$

Let us try to define a reduced density matrix in the usual way,

$$\rho = \int d\nu(\mathbf{k})|f(\mathbf{k})|^2 |\mathbf{k}, \alpha(\mathbf{k})\rangle \langle \mathbf{k}, \alpha(\mathbf{k})|? \quad (17)$$

The superselection rule that was mentioned above does not forbid this definition, because only terms with the same momentum $\mathbf{k}$ are summed. However, since polarization is a secondary variable, this object cannot have definite transformation properties under boosts. This deficiency is familiar to us from the analysis of reduced density matrices of massive particles. However, for massless particles, the situation is worse: POVMs that are given by $2 \times 2$ matrices represent measurement devices and should transform under a representation of the rotation group $O(3)$. On the other hand, even for the complete photon state, ordinary rotations of the reference frame correspond to elements of $E(2)$, so that probabilities would not be invariant under rotations.

Therefore, let us find a more physical definition of a reduced density matrix for polarization. The labelling of polarization states by Euclidean vectors $\epsilon_{\mathbf{k}}^\pm$ suggests the use of a $3 \times 3$ matrix with entries labelled $x$, $y$, and $z$. Classically, they correspond to different directions of the electric field. For example, a reduced density matrix $\rho_x$ would give the expectation values of operators representing the polarization in the $x$ direction, seemingly irrespective of the particle’s momentum.

To have a momentum-independent polarization is to tacitly admit longitudinal photons. Unphysical concepts are often used in intermediate steps in theoretical physics. Momentum-independent polarization states thus consist of physical (transversal) and unphysical (longitudinal) parts, the latter corresponding to a polarization vector $\epsilon^l = \mathbf{k}$. For example, a generalized polarization state along the $x$-axis is

$$|x\rangle = x_+(\mathbf{k})|\epsilon_{\mathbf{k}}^+\rangle + x_-(\mathbf{k})|\epsilon_{\mathbf{k}}^-\rangle + x_\ell(\mathbf{k})|\epsilon_{\mathbf{k}}^\ell\rangle, \quad (18)$$

where $x_+(\mathbf{k}) = \epsilon^+_{\mathbf{k}} \cdot \mathbf{x}$, and $x_\ell(\mathbf{k}) = \mathbf{x} \cdot \mathbf{k} = \sin \theta \cos \phi$. It follows that $|x_+|^2 + |x_-|^2 + |x_\ell|^2 = 1$, and we thus define

$$\epsilon_x(\mathbf{k}) = \frac{x_+(\mathbf{k})|\epsilon_{\mathbf{k}}^+\rangle + x_-(\mathbf{k})|\epsilon_{\mathbf{k}}^-\rangle}{\sqrt{|x_+|^2 + |x_-|^2}} \quad (19)$$

as the polarization vector associated with the $x$ direction. It follows from (18) that $\langle x|\mathbf{x}\rangle = 1$ and $\langle x|\mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y} = 0$, and likewise for other directions, so that

$$|x\rangle \langle x| + |y\rangle \langle y| + |z\rangle \langle z| = \mathbb{I}_p, \quad (20)$$

where $\mathbb{I}_p$ is the unit operator in momentum space.

We can now define an “effective” reduced density matrix adapted to a specific method of measuring polarization, as follows [16]. To the direction $\mathbf{x}$ corresponds a projection operator

$$P_x = |x\rangle \langle x| \otimes \mathbb{I}_p = |x\rangle \langle x| \otimes \int d\nu(\mathbf{k})|\mathbf{k}\rangle \langle \mathbf{k}|. \quad (21)$$

The action of $P_x$ on $|\Psi\rangle$ follows from Eq. (18) and $\langle \epsilon^+_\mathbf{k}\epsilon^-_{\mathbf{k}} \rangle = 0$. Only the transversal part of $|x\rangle$ appears in the expectation value:

$$\langle \Psi| P_x |\Psi\rangle = \int d\nu(\mathbf{k})|f(\mathbf{k})|^2 |x_+(\mathbf{k})\alpha^+_\mathbf{k}(\mathbf{k}) + x_-(\mathbf{k})\alpha^-_{\mathbf{k}}(\mathbf{k})|^2.$$

It is convenient to write the transversal part of $|x\rangle$ as

$$|b_x(\mathbf{k})\rangle \equiv (|\epsilon^+_\mathbf{k}\rangle \langle \epsilon^+_\mathbf{k}| + |\epsilon^-_{\mathbf{k}}\rangle \langle \epsilon^-_{\mathbf{k}}|)|x\rangle = x_+(\mathbf{k})|\epsilon^+_\mathbf{k}\rangle + x_-(\mathbf{k})|\epsilon^-_{\mathbf{k}}\rangle. \quad (22)$$

Likewise define $|b_y(\mathbf{k})\rangle$ and $|b_z(\mathbf{k})\rangle$. These three state vectors are neither of unit length nor mutually orthogonal. For $\mathbf{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ we have

$$|b_x(\mathbf{k})\rangle = c(\theta, \phi)|\alpha(\mathbf{k}, \epsilon_x(\mathbf{k}))\rangle, \quad (24)$$

where $\epsilon_x(\mathbf{k})$ is given by Eq. (19), and $c(\theta, \phi) = \sqrt{x_+^2 + x_-^2}.$

Finally, a POVM element $E_x$ which is the physical part of $P_x$, namely is equivalent to $P_x$ for physical states (without longitudinal photons) is

$$E_x = \int d\nu(\mathbf{k})|\mathbf{k}, b_x(\mathbf{k})\rangle \langle \mathbf{k}, b_x(\mathbf{k})|, \quad (25)$$

and likewise for other directions. The operators $E_x$, $E_y$, and $E_z$ indeed form a POVM in the space of physical states, owing to Eq. (18). The above derivation was, admittedly, a rather circuitous route for obtaining a POVM for polarization. This is due to the fact that the latter is a secondary variable, subject to superselection rules. Unfortunately, this is the generic situation.

To complete the construction of the density matrix, we introduce additional directions. Similarly to the standard practice of channel matrices reconstruction [17], we consider $E_{x+z,x+z}$, $E_{x+z,x-iz}$ and similar combinations. For example,

$$E_{x+z,x+z} = \frac{1}{2}(|x\rangle \langle x| + |z\rangle \langle z|) \otimes \mathbb{I}_p. \quad (26)$$

Let us denote $|x\rangle \langle x| \otimes \mathbb{I}_p$ as $E_{xz}$, even though this is not a positive operator. We then get a simple expression for the reduced density matrix corresponding to the polarization state $|\alpha(\mathbf{k})\rangle$:

$$\rho_{mn} = \langle \Psi|E_{mn}|\Psi\rangle = \int d\nu(\mathbf{k})|f(\mathbf{k})|^2 \langle \alpha(\mathbf{k})|b_m(\mathbf{k})\rangle \langle b_n(\mathbf{k})|\alpha(\mathbf{k})\rangle. \quad (27)$$
where $m, n = x, y, z$. It is interesting to note that this derivation gives a direct physical meaning to the naive definition of a reduced density matrix,

$$\rho_{mn}^{\text{naive}} = \int d\mu(k) |f(k)|^2 \alpha_m(k) \alpha_n^*(k) = \rho_{mn} \quad (28)$$

Our basis states $|k, \epsilon_k\rangle$ are direct products of momentum and polarization. Owing to the transversality requirement $\epsilon_k \cdot \mathbf{k} = 0$, they remain direct products under Lorentz transformations. All the other states have their polarization and momentum degrees of freedom entangled. As a result, if one is restricted to polarization measurements as described by the above POVM, there do not exist two orthogonal polarization states. An immediate corollary is that photon polarization states cannot be cloned perfectly, because the no-cloning theorem forbids an exact copying of unknown non-orthogonal states. In general, any measurement procedure with finite momentum sensitivity will lead to the errors in identification. First we present some general considerations and then illustrate them with a simple example.

Let us take the $z$-axis to coincide with the average direction of propagation so that the mean photon momentum is $k_A z$. Typically, the spread in momentum is small, but not necessarily equal in all directions. Usually the intensity profile of laser beams has cylindrical symmetry, and we may assume that $\Delta x / \Delta z \approx \Delta y / \Delta z$, where the index $r$ means radial. We may also assume that $\Delta r \gg \Delta z$. We then have

$$f(k) \propto f_{1}^{|k_z - k_A|/\Delta z} f_{2}(k_r/\Delta r). \quad (29)$$

We approximate

$$\theta \approx \tan \theta \equiv k_r/k_z \approx k_r/k_A. \quad (30)$$

In pictorial language, polarization planes for different moments are tilted by angles up to $\sim \Delta r/k_A$, so that we expect an error probability of the order $\Delta r^2/k_A^2$. In the density matrix $\rho_{mn}$ all the elements of the form $\rho_{rnrn}$ should vanish when $\Delta r \to 0$. Moreover, if $\Delta z \to 0$, the non-vanishing $xy$ block goes to the usual (monochromatic) polarization density matrix.

As an example we consider two states which, if the momentum spread could be ignored, would be $|k_A z, \epsilon_{k_A z}\rangle$. To simplify the calculations we assume a Gaussian distribution:

$$f(k) = N e^{-\left|k_z - k_A\right|^2/2\Delta z^2} e^{-k_r^2/2\Delta r^2}, \quad (31)$$

where $N$ is a normalization factor and $\Delta z \ll \Delta r$. Moreover, we take the polarization components to be $\epsilon_{k_A z} = e_{k_A z}$. That means we have to analyze the states

$$|\Psi_\pm\rangle = \int d\mu(k) f(k) |\epsilon_{k_A z}\rangle |\epsilon_{k_A z}\rangle,$$

where $f(k)$ is given above.

It can be shown that in the leading order

$$P_E(\rho_+ , \rho_-) = \frac{\Delta r^2}{4k_A^2}. \quad (33)$$

Now we turn to the distinguishability problem from the point of view of a moving observer, Bob. The probability of an error by Bob is still given by Eq. 4. The distinguishability of polarization density matrices depends on the observer’s motion. We again assume that Bob moves along the $z$-axis with a velocity $v$. Detailed calculations show that

$$P_E' = \frac{1 + v}{1 - v} P_E, \quad (34)$$

which may be either larger or smaller than $P_E$. As expected, we obtain for one-photon states the same Doppler effect as in the classical calculations.

Since maximally entangled states are pure, the fact that all polarization density matrices are mixed implies that maximal EPR-type correlations shall never be observed, and that maximal attainable value will depend on the momentum spread of the states.

IV. COMMUNICATION CHANNELS

Although reduced polarization density matrices have no general transformation rule, the above results show that such rules can be established for particular classes of experimental procedures. We can then ask how these effective transformation rules, $\mathbf{T}' = T(\mathbf{T})$, fit into the framework of general state transformations. Equation 6 gives an example of such a transformation.

$$\rho' = \rho(1 - \frac{\Gamma^2}{4}) + (\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y) \frac{\Gamma^2}{8}. \quad (35)$$

It relates the reduced density matrix $\rho$, obtained from Alice’s Eq. 4, to Bob’s density matrix $\rho'$. This particular transformation is completely positive; however, it is not so in general.

It can be proved that distinguishability, as expressed by natural measures like $P_E$, cannot be improved by any completely positive transformation $\mathbf{T}$. It is also known that the complete positivity requirement may fail if there is a prior entanglement with another system. Since in the two previous sections we have seen that distinguishability can be improved, we conclude that these transformations are not completely positive. The reason is that the Lorentz transformation acts not only on the “interesting” discrete variables, but also on the “hidden” momentum variables that we elected to ignore and to trace out, and its action on the interesting degrees of freedom depends on the hidden ones.

This technicality has one important consequence. The notion of a channel is fundamental both in classical and quantum communication theory. Quantum channels are described as completely positive maps that act on qubit states. Qubits themselves are realized as particles’
discrete degrees of freedom. If relativistic motion is important, then not only does the vacuum behave as a noisy quantum channel, but the very representation of a channel by a completely positive map fails.