HTL-RESUMMED THERMODYNAMICS OF HOT AND DENSE QCD: AN UPDATE

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We review the proposal to resum the physics of hard thermal loops in the thermodynamics of the quark-gluon plasma through nonperturbative expressions for entropy and density obtained from a Φ-derivable two-loop approximation. A comparison with the recently solved large-$N_f$ limit of hot QCD is performed, and some updates, in particular on quark number susceptibilities, are made.

1. Introduction

Even at temperatures several orders of magnitude higher than $\Lambda_{\text{QCD}}$ the perturbative series for the thermodynamic potentials of hot QCD\textsuperscript{1,2,3} does not appear to converge and thus seems to be devoid of predictive power. While the first correction at order $g^2$ gives a reasonable estimate for the results obtained in lattice gauge theory a few times above the transition temperature, everything breaks down as soon as collective phenomena such as Debye screening come into the play and produce formally higher-order contributions suppressed by only single powers of $g$ (see Fig. 1).

Since similar difficulties have been observed in simple scalar models\textsuperscript{6,7}, the reason for this failure does not have to do so much with the fact that QCD has a nonperturbative sector even in its deconfined phase which limits the number of computable perturbative coefficients. It rather seems that screening effects should better not be treated in a strictly perturbative way. A first encouraging attempt for improving the situation was put forward by Karsch et al.\textsuperscript{8}, who proposed to keep a screening mass unexpanded at any given order of the loop expansion and to fix this mass by a stationarity principle. Technically, this corresponds to rewriting the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_{\text{int}} + \frac{1}{2} m^2 \phi^2. \quad (1)$$
This optimization of thermal perturbation theory was successfully applied to $\phi^4$ theory to three-loop order\textsuperscript{9} and extended to QCD by Andersen et al.\textsuperscript{10,11}. There they proposed replacing the simple mass term by the gauge-invariant hard-thermal-loop (HTL) action\textsuperscript{12,13}. This method, which they termed HTL perturbation theory (HTLPT), is in principle a systematic and manifestly gauge invariant scheme. From a physical point of view, it has, however, the somewhat unsatisfactory feature that HTL’s are used uniformly for soft and hard momenta although the HTL effective action is accurate only for soft momenta (and soft virtuality) $\ll T$. In the following we shall aim at a resummation that captures more closely the physics of the collective excitations of the quark-gluon plasma.

2. $\Phi$-derivable approximations and HTL resummation

Our proposal\textsuperscript{14,15,16} is to implement HTL resummations in the spirit of the so-called $\Phi$-derivable approximations\textsuperscript{17} of the 2PI skeleton expansion. In the latter, the thermodynamic potential is expressed in terms of dressed propagators ($D$ for bosons, $S$ for fermions) according to

$$\Omega[D, S] = \frac{1}{2} T \text{Tr} \log D^{-1} - \frac{1}{2} T \text{Tr} \Pi D - T \text{Tr} \log S^{-1} + T \text{Tr} \Sigma S + T \Phi[D, S]$$  \hspace{1cm} (2)

where $\Phi[D, S]$ is the sum of 2-particle-irreducible “skeleton” diagrams. The self-energies $\Pi = D^{-1} - D^{-1}_0$ and $\Sigma = S^{-1} - S^{-1}_0$, where $D_0$ and $S_0$ are bare propagators, are themselves functionals of the full propagators, determined
by the stationarity property

$$\frac{\delta \Omega[D,S]}{\delta D} = 0 = \frac{\delta \Omega[D,S]}{\delta S}. \quad (3)$$

according to

$$\frac{\delta \Phi[D,S]}{\delta D} = \frac{1}{2} \Pi, \quad \frac{\delta \Phi[D,S]}{\delta S} = \Sigma. \quad (4)$$

The \( \Phi \)-derivable two-loop approximation consists of keeping only the two-loop skeleton diagrams, which leads to a dressed one-loop approximation for the self-energies (4). In a gauge theory this introduces gauge dependences (which are however parametrically suppressed\(^\text{18}\)), but we shall construct further approximations which are manifestly gauge independent.

A self-consistent two-loop approximation for \( \Omega \) has a remarkable consequence for the first derivatives of the thermodynamic potential, the entropy and the number densities:

$$S = - \left. \frac{\partial (\Omega/V)}{\partial T} \right|_\mu, \quad N = - \left. \frac{\partial (\Omega/V)}{\partial \mu} \right|_T. \quad (5)$$

Because of the stationarity property (3), one can ignore the \( T \) and \( \mu \) dependences implicit in the spectral densities of the full propagators, and differentiate exclusively the statistical distribution functions \( n \) and \( f \) in (2). Now the derivative of the two-loop functional \( T\Phi[D,S] \) at fixed spectral densities of the propagators \( D \) and \( S \) turns out to cancel part of the terms \( \text{Im} (\Pi D) \) and \( \text{Im} (\Sigma S) \) in (2) leading to the remarkably simple formulae\(^\text{19,15,16}\)

$$S = - \text{tr} \int \frac{d^4k}{(2\pi)^4} \frac{\partial n(\omega)}{\partial T} \left[ \text{Im} \log D^{-1} - \text{Im} \Pi \text{Re} D \right]$$

$$-2 \text{tr} \int \frac{d^4k}{(2\pi)^4} \frac{\partial f(\omega)}{\partial T} \left[ \text{Im} \log S^{-1} - \text{Im} \Sigma \text{Re} S \right], \quad (6)$$

$$N = -2 \text{tr} \int \frac{d^4k}{(2\pi)^4} \frac{\partial f(\omega)}{\partial \mu} \left[ \text{Im} \log S^{-1} - \text{Im} \Sigma \text{Re} S \right]. \quad (7)$$

Through these formulae, all interactions below order \( g^4 \) are summarized by spectral data only, which shows that entropy and density are the preferred quantities for a quasiparticle description.

The leading-order interaction terms \( \propto g^2 \) arise from hard loop momenta involving only the light-cone projection of the self-energies, e.g. in pure glue QCD

$$S_2 = 2N_g \int \frac{d^4k}{(2\pi)^4} \frac{\partial n}{\partial T} \text{Re} \Pi_T \text{Im} \frac{1}{\sqrt{\omega^2 - k^2}}, \quad (8)$$
where $\Pi_T$ is the transverse component of the gluon self-energy, which on
the light-cone is accurately (to order $g^2$) given by its HTL value $\Pi_T(k,k) = \hat{m}_\infty^2 = \frac{1}{2} \hat{m}_D^2$ even though $k$ is no longer soft\textsuperscript{20,21}.

The more critical $g^3$ ("plasmon") term in the thermodynamic potentials,
on the other hand, arises in an unusual manner: while in the pressure it is
 determined by the soft momentum regime of the one-loop contribution, in
the above expressions for the entropy, there are two distinct origins. One
part comes from order-$g$ corrections to $\Pi_T(k,k)$ at hard momenta $k \sim T$
and are to be interpreted as a correction to the entropy of hard gluons.
Only a fraction ($< 1/4$) arises as the entropy of soft gluons in the HTL
approximation\textsuperscript{14}. The required correction for the hard excitations is

$$\delta m_\infty^2(k) = \text{Re} \delta \Pi_T(k,k) = \text{Re} \left( \sum_{\text{loop}} \right)$$

and this can be calculated by standard\textsuperscript{22} HTL perturbation theory\textsuperscript{a}.

These corrections to the asymptotic thermal masses are, in contrast to
the latter, nontrivial functions of the momentum, and can be evaluated
only numerically. However, as far as the generation of the plasmon term is
concerned, these functions contribute in a certain averaged form which can be
calculated analytically,

$$\bar{\delta m}_\infty^2 = \frac{1}{2\pi^2} g^2 N T \hat{m}_D.$$  \quad (10)

This result pertains only to the hard excitations; corrections to the various
thermal masses of soft excitations are known to differ substantially from
(10). For instance, the relative correction to the gluonic plasma frequency\textsuperscript{23}
at $k = 0$, $\delta m_{\text{pl}}^2 / \hat{m}_{\text{pl}}^2$, is only about a third of $\delta m_\infty^2 / m_\infty^2$; the NLO correction to the nonabelian Debye mass on the other hand is even positive for
small coupling and moreover logarithmically enhanced\textsuperscript{24}.

For an estimate of the effects of a proper incorporation of the next-
to-leading order corrections we have therefore proposed to include the latter
only for hard excitations and to define our next-to-leading approximation
(for gluons) through

$$S_{\text{NLA}} = S_{\text{HTL}} \big|_{\text{soft}} + S_{\hat{m}_\infty^2} \big|_{\text{hard}},$$

where $\hat{m}_\infty^2$ includes (10) and where $S_{\text{HTL}}$ refers to evaluating the \Phi-derivable 2-loop
entropy in the HTL approximation exactly. To separate soft ($k \sim \hat{m}_D$)
and hard ($k \sim 2\pi T$) momentum scales, we introduce the intermediate scale

$$\Lambda = \sqrt{2\pi T \hat{m}_D c_\Lambda}$$

and consider a variation of $c_\Lambda = \frac{1}{2} \ldots 2$ as part of our
theoretical error estimate.

\textsuperscript{a}Because the external momentum is hard, no HTL vertices are needed, and only one
propagator has to be dressed by HTL.
Another crucial issue concerns the definition of the corrected asymptotic mass $\bar{m}_\infty$. For the range of coupling constants of interest ($g > 1$), the correction $|\delta m_\infty^2|$ is greater than the LO value $m_\infty^2$, leading to tachyonic masses if included in a strictly perturbative manner.

However, this problem is not at all specific to QCD. In the simple $g^2\varphi^4$ model, one-loop resummed perturbation theory gives

$$m^2 = g^2 T^2 (1 - \frac{3}{\pi} g)$$

(11)

which also turns tachyonic for $g > 1$. On the other hand, the solution of the one-loop gap equation is a monotonic function in $g$, and it turns out that the first two terms in a $(m/T)$-expansion of this gap equation,

$$m^2 = g^2 T^2 - \frac{3}{\pi} g^2 T m,$$

(12)

which is perturbatively equivalent to (11), has a solution that is very close to that of the full gap equation (for MS renormalization scale $\bar{\mu} \approx 2\pi T$).

In QCD, where the non-local gap equations are too complicated to be attacked directly we adopted (12) as a model to include $\delta m_\infty^2$, which is needed for the completion of the plasmon effect. This leads to

$$\bar{m}_\infty^2 = \frac{1}{6} (N + \frac{3}{2} N_f) g^2 T^2 - \frac{1}{\sqrt{2\pi}} g^2 N T \bar{m}_\infty.$$

(13)

Up to a single integration constant the resulting entropy expression determines the thermodynamic pressure. Choosing this (strictly nonperturbative) input such that e.g. $P(T_c) \approx 0$, where $T_c$ is taken from the lattice, this ambiguity in fact becomes quickly negligible for larger $T/T_c$, because the contribution of the bag constant thus introduced drops like $T^{-4}$ in the normalized pressure $P/P_0$, where $P_0$ is the ideal-gas limit.

The main uncertainty rather comes from the choice of the renormalization point $\bar{\mu}$. In the following we always consider varying $\bar{\mu}$ by a factor of 2 around a central value of $2\pi T$ and determine the strong coupling constant from the 2-loop renormalization group equation.

The result of this procedure is displayed in Fig. 2 and compared with lattice data from the Bielefeld and PC-PACS groups, and also with the recent two-loop calculation in HTLPT. This shows a clear improvement compared to the perturbative result to order $g^3$, and remarkable agreement with lattice data for $T \gtrsim 3T_c$.

\textsuperscript{b}Numerically, this differs only slightly from the Padé approximants that we employed in our first publications\textsuperscript{14,15}.
Compared to the HTLPT calculation, an important difference of our approach is the separate treatment of hard and soft contributions, but the HTLPT result also has a large $g^5$-contribution\textsuperscript{11} with opposite sign from that obtained in 3-loop perturbation theory.

3. Inclusion of fermions and the large-$N_f$ limit

When fermions are included ($N_f \neq 0$), part of the plasmon effect in the pressure (and all of the plasmon effect in the fermion density) is contributed by next-to-leading order corrections to the asymptotic thermal mass of the fermions, whose leading-order (HTL) value is $\tilde{M}_\infty^2 = g^2 C_f T^2/4$, with $C_f = 2N_f/N_g$. These can be calculated in standard HTL perturbation theory according to

$$\frac{1}{2k} \delta M^2_{\infty}(k) = \delta \Sigma_+ (\omega = k) = \text{Re} \left( \delta \Sigma_+ + \delta \Sigma_- \right) \big|_{\omega = k},$$

which is again a function of $k$ that can be evaluated only numerically. However in the plasmon effect it enters only in the analytically calculable averaged form

$$\tilde{\delta} M^2_{\infty} = \frac{\int dk \ k \ f'(k) \ Re \ 2k \delta \Sigma_+ (\omega = k) }{\int dk \ k \ f'(k) } = - \frac{1}{2\pi} g^2 C_f T \tilde{m}_D.$$  

In our previous work\textsuperscript{16} we have incorporated this correction in complete analogy to the gluonic asymptotic mass (13), that is, we have adopted a
gap equation quadratic in $\bar{M}_\infty$ that is perturbatively equivalent to (15).

However, the recent work on the large-$N_f$ limit of QCD by Moore\textsuperscript{25} has triggered us to reconsider this procedure, because it turns out that a quadratic gap equation for the fermions does not comply with the large-$N_f$ limit.

In the large-$N_f$ limit, $N_f \to \infty$, $g^2 \to 0$ such that $g^2_{\text{eff.}} = g^2 N_f/2 \sim 1$, the quadratic gap equation for the gluons has the correct behaviour that $\bar{m}_\infty^2 \to g^2_{\text{eff}} T^2/6 + O(1/N_f)$.

Fermion self-energies are suppressed by a factor $1/N_f$, but we still need to consider them in our expressions for entropy and density because there are $N_f$ fermions which together produce a $N_f^0$ contribution to the fermionic entropy. This precisely equals $-N N_f \bar{M}_\infty^2 T/6$, where $\bar{M}_\infty^2$ represents the average appearing in (15). The latter involves the correction term calculated in (15), but without further (rainbow-like) corrections on the internal fermion line. The fermionic “gap equation” thus has to remain linear in $\bar{M}_\infty^2$. An equation for $\bar{M}_\infty^2$ with the correct behaviour in the large-$N_f$ limit is given by

$$\bar{M}_\infty^2 = g^2 C_f T^2/4 - \frac{1}{\sqrt{2\pi}} g^2 C_f T \bar{m}_\infty. \quad (16)$$

There is then no negative feedback from the fermion mass itself, it only inherits higher-order terms from the solution to $\bar{m}_\infty$ (when $N_f$ is finite).

In Fig. 3 we compare the 2-loop $\Phi$-derivable result in our approximations in the limit of large $N_f$ with the exact result\textsuperscript{c}. The latter has the curious behaviour of being nonmonotonic as $g^2_{\text{eff.}}$ is increased, and this behaviour is in fact qualitatively reproduced in our approach with (16). However, when $g^2_{\text{eff.}} \gtrsim 7.4$, $\bar{M}_\infty^2$ becomes negative (dashed lines in Fig. 3). This is not a problem for the $\Phi$-derivable expressions at order $N_f^0$, but it means that for large but still finite $N_f$ the fermionic quasiparticles (at least in our approximation) cease to exist.

However, in our applications to real QCD the revised fermionic gap equation (16) does not have the problem of giving rise to tachyonic masses even close to the transition temperature. If we therefore compare the results of our approximations in the large $N_f$ limit with the exact one only in the region where $\bar{M}_\infty^2$ remains positive, the outcome is in fact encouraging: the agreement below the point where $\bar{M}_\infty^2$ vanishes is remarkably good even though the coupling is no longer small and $\hat{m}_{\text{D}}/T \equiv g_{\text{eff}}/\sqrt{3} \sim 1$.

\textsuperscript{c}The result published in Ref. \textsuperscript{25} has recently been found to be in error; the corrected exact result can be found in Ref. \textsuperscript{26}.\textsuperscript{26}
By a curious coincidence, for \( N = 3 \) and \( N_f = 3 \) the revised gap equation (16) together with (13) has exactly the same solutions as the uncoupled quadratic gap equations we used previously. Only for \( N_f > 3 \) there is at all a coupling where the fermionic mass ceases to grow monotonically with \( g \); for \( N_f \leq 3 \) this never happens. Because of this coincidence, the numerical changes in our previous results are almost completely negligible. As an example, Fig. 4 updates the results published previously\(^{16} \) for the entropy with flavour numbers \( N_f = 0, 2, 3 \) in comparison with the estimated continuum extrapolation of Ref. \(^{27} \). Only the NLA result for \( N_f = 2 \) changes at all from the gray dash-dotted line to the one slightly below it.

![Figure 3](image3.png)  
**Figure 3.** The exact result\(^{26} \) for the pressure in the limit of large \( N_f \) compared with the \( \Phi \)-derivable 2-loop result in the HTL approximation (full lines) and in the next-to-leading approximation (full lines ending in dashed lines), for \( \bar{\mu} = T \) and \( 4\pi T \). The gray lines denote the next-to-leading approximation with quadratic fermionic gap equation considered in Ref. \(^{16} \), but which we argue needs to be replaced by Eq. (16).

![Figure 4](image4.png)  
**Figure 4.** The entropy in the HTL approximation (full lines, \( N_f = 0, 3, 2 \) from bottom to top) and in the next-to-leading approximation (dash-dotted lines, \( N_f = 0, 2, 3 \) from bottom to top) with \( \bar{\mu} = 2\pi T \) and compared with an estimated continuum extrapolation\(^{27} \) of the lattice result for \( N_f = 2 \). Only the NLA \( N_f = 2 \) result changes by switching to the fermionic gap equation (16) from the gray dash-dotted line to the black one just below.

More interestingly, with the new fermionic gap equation (16) we can narrow down somewhat our predictions\(^{28} \) for the quark number susceptibilities. Previously we have determined our estimated theoretical errors for the latter by combining the results obtained from a quadratic gap equation with the results from a Padé approximant. To be compatible with the large-\( N_f \) limit, we can now restrict to (16) and produce somewhat narrower error bands (which still are dominated by the \( \bar{\mu} \) dependence). Fig. 5 shows our prediction for \( N_f = 2 \) in comparison with available lattice data\(^{29} \). Fig. 6 shows the results for quenched QCD together with recent results for two
different continuum extrapolations\(^{30}\) (both are higher than the previous lattice results\(^{31}\)).

It would be interesting to compare our predictions with HTLPT to two-loop order; the one-loop HTL-resummed results from charge correlators\(^{32}\) suffer from severe overcounting while missing out the plasmon term\(^{33}\) so that one should not perform a comparison yet.\(^d\)

![Figure 5](image.jpg)  
Figure 5. Comparison of our results for \(\chi/\chi_0\) in massless \(N_f=2\) QCD with the lattice results of Ref.\(^{29}\) (no continuum extrapolation). Full lines refer to the HTL approximation, dash-dotted lines to NLA. (Gray dash-dotted denotes our previous NLA estimate\(^{28}\).)

![Figure 6](image.jpg)  
Figure 6. Comparison of our updated results for \(\chi/\chi_0\) in the formal limit \(N_f=0\) with the previous lattice results for quenched QCD of Ref.\(^{31}\) (gray data points) and two recent continuum extrapolations\(^{30}\) (black data points).

4. Outlook

The HTL-based quasiparticle description of the thermodynamics of hot QCD that we have developed can be straightforwardly extended to finite chemical potential. First steps in this direction that go beyond the simple quasiparticle models of Ref.\(^{36}\) are indeed encouraging\(^{37}\). Further refinements, in particular a full inclusion of the momentum dependence of the next-to-leading order asymptotic masses, is work in progress.

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\(^d\)For results and conjectures based on the recent determination of the \(g^6 \ln(1/g)\) term in the thermodynamic potential\(^{34}\) see Ref.\(^{35}\).
References

37. P. Romatschke, these proceedings [hep-ph/0210331].