II. Equations

\[ \frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi - V = 0 \]

where

\[ \frac{\partial \phi}{\partial t} = f \]

\[ (c^2 - \Delta + \theta \phi) \phi = c^2 - \Delta f + \theta \phi f = \phi^2 \]

These equations are used to describe cosmological black holes. They are somewhat analogous to the standard equations used in quantum mechanics but with additional terms that account for the gravitational field.

III. Introduction

The transversal modes of the near extremal Schwarzschild-de Sitter black hole
where \( r_c > r_h \). The function \( f \) has zeroes at \( r_h, r_c \), and \( r_\Box = -(r_h + r_c) \). In terms of these quantities, \( f \) can be expressed as

\[
f = \frac{1}{a^2 r} (r - r_h)(r - r_c)(r - r_\Box).\tag{3}
\]

It is useful to regard \( r_h \) and \( r_c \) as the two fundamental parameters of the SdS spacetime, and to express \( M \) and \( a^2 \) as functions of these variables. The appropriate relations are

\[
a^2 = r_h r_c + r_h r_c + r_c^2 \tag{4}
\]

and

\[
2M a^2 = r_h r_c (r_h + r_c). \tag{5}
\]

We also introduce the surface gravity \( \kappa_b \) associated with the black hole horizon \( r = r_h \), as defined by the relation \( \kappa_b = \frac{1}{2} \frac{df}{dr_{\Box}} \). Explicitly, we have

\[
\kappa_b = \frac{(r_c - r_h)(r_h - r_\Box)}{2a^2 r_h}. \tag{6}
\]

After a Fourier decomposition in frequencies and a multipole expansion, the scalar, electromagnetic and gravitational perturbations all obey a wave equation of the form \( \Box \)

\[
\frac{\partial^2 \phi(\omega, r)}{\partial r^2} + \left[ \omega^2 - V(r) \right] \phi(\omega, r) = 0, \tag{7}
\]

where the tortoise coordinate is given by

\[
r_* \equiv \int f^{-1} dr,
\]

and the potential \( V \) depends on the kind of field under consideration. Explicitly, for scalar perturbations

\[
V_s = f \left[ \frac{l(l + 1)}{r^2} + \frac{2M}{r^3} - \frac{2}{a^2} \right],
\]

while for electromagnetic perturbations

\[
V_{el} = f \left[ \frac{l(l + 1)}{r^2} \right].
\]

The gravitational perturbations decompose into two sets \( \Box \), the odd and the even parity one. We find however that for this spacetime, they both yield the same quasinormal frequencies, so it is enough to consider one of them, the odd parity ones say, for which the potential is \( \Box \)

\[
V_{grav} = f \left[ \frac{l(l + 1)}{r^2} - \frac{6M}{r^3} \right]. \tag{11}
\]

In all cases, we denote by \( l \) the angular quantum number, that gives the multipolarity of the field.

### A. The Near Extremal SdS Black Hole

Let us now specialize to the near extremal SdS black hole, which is defined as the spacetime for which the cosmological horizon \( r_c \) is very close (in the \( r \) coordinate) to the black hole horizon \( r_h \), i.e. \( \frac{r_c - r_h}{r_h} \ll 1 \). For this spacetime one can make the following approximations

\[
r_\Box \sim -2r_h^2; \ a^2 \approx 3r_h^2; \ M \sim \frac{r_h}{3}; \ \kappa_b \sim \frac{r_c - r_h}{2r_h}. \tag{12}
\]

Furthermore, and this is the key point, since \( r \) is constrained to vary between \( r_h \) and \( r_c \), we get \( r - r_\Box \sim r_h - r_\Box \sim 3r_\Box \) and thus

\[
f \sim \frac{(r - r_h)(r - r_c)}{r_\Box}. \tag{13}
\]

In this limit, one can invert the relation \( r_\Box(r) \) of \( \Box \) to get

\[
r = \frac{r_h r_c (r_h + r_c)}{1 + e^{2\kappa_b r}}. \tag{14}
\]

Substituting this on the expression \( \Box \), for \( f \) we find

\[
f = \frac{(r_c - r_h)^2}{4r_h^2 \cosh(\kappa_b r)^2}. \tag{15}
\]

As such, and taking into account the functional form of the potentials \( \Box \), we see that for the near extremal SdS black hole the wave equation \( \Box \) is of the form

\[
\frac{\partial^2 \phi(\omega, r)}{\partial r^2} + \left[ \omega^2 - \frac{V_0}{\cosh(\kappa_b r_*)^2} \right] \phi(\omega, r) = 0, \tag{16}
\]

with

\[
V_0 = \begin{cases} 
\kappa_b^2 l(l + 1), & \text{scalar and electromagnetic} \text{ perturbations.} \\
\kappa_b^2 (l + 2)(l + 1), & \text{gravitational} \text{ perturbations.}
\end{cases} \tag{17}
\]

The potential in \( \Box \) is the well-known Pöschl-Teller potential \( \Box \). The solutions to \( \Box \) were studied and they are of the hypergeometric type, (for details see Ferrari and Mashhoon \( \Box \)). It should be solved under appropriate boundary conditions:

\[
\phi \sim e^{-i\omega r_*}, \ r_* \to -\infty \tag{18}
\]

\[
\phi \sim e^{i\omega r_*}, \ r_* \to \infty. \tag{19}
\]

These boundary conditions impose a non-trivial condition on \( \omega \) \( \Box \), and those that satisfy both simultaneously are called quasinormal frequencies. For the Pöschl-Teller potential one can show \( \Box \) that they are given by

\[
\omega = \kappa_b \left[ -\left( n + \frac{1}{2} \right) + \sqrt{\frac{V_0}{\kappa_b^2} - \frac{1}{4}} \right], \ n = 0, 1, \ldots \tag{20}
\]
Thus, with \( n \) one has
\[
\frac{\omega}{k_b} = -(n + \frac{1}{2}) + \sqrt{f(l+1) - \frac{1}{4}}, \quad n = 0, 1, \ldots
\] (21)
for scalar and electromagnetic perturbations. And
\[
\frac{\omega}{k_b} = -(n + \frac{1}{2}) + \sqrt{f(l+1)(l-1) - \frac{1}{4}}, \quad n = 0, 1, \ldots
\] (22)
for gravitational perturbations. Our analysis shows that Eqs. (21–22) are correct up to terms of order \( O(r_c - r_b) \) or higher. Moss and Norman [3] have studied the quasinormal frequencies in the SdS geometry numerically and also analytically, by fitting the potential to a Pöschl-Teller potential. Their analytical results (see their Figs 1 and 2) were in excellent agreement with their numerical results, and this agreement was even more remarkable for near extremal black holes and for high values of the angular quantum number \( l \). We can now understand why: for near extremal black holes the true potential is indeed given by the Pöschl-Teller potential! Furthermore for near extremal SdS black holes and for high \( l \) our formula (21) is approximately equal to formula (19) of Moss and Norman [3]. With their analytical method of fitting the potential one can never be sure if the results obtained will continue to be good as one increases the mode number \( n \). But we have now proved that if one is in the near extremal SdS black hole, the Pöschl-Teller is the true potential, and so Eq. (21) is exact. For example, Moss and Norman obtain numerically, and for gravitational perturbations with \( l = 2 \) of nearly extreme SdS black holes, the result
\[
\frac{\omega_{\text{num}}}{k_b} = 1.93648 - i(n + \frac{1}{2}),
\] (23)
and we obtain, from (24)
\[
\frac{\omega}{k_b} = 1.936492 - i(n + \frac{1}{2}).
\] (24)
For \( l = 3 \) Moss and Norman [3] obtain
\[
\frac{\omega_{\text{num}}}{k_b} = 3.12249 - i(n + \frac{1}{2}),
\] (25)
and we obtain, from (24)
\[
\frac{\omega}{k_b} = 3.122499 - i(n + \frac{1}{2}).
\] (26)
So this remarkable agreement allows us to be sure that (21–22) are indeed correct.

III. CONCLUSIONS

We have found an analytical expression for the quasinormal modes and frequencies of a nearly extreme Schwarzschild-de Sitter black hole. This expression, Eqs. (21–22) are correct up to terms of order \( O(r_c - r_b) \) or higher for all \( n \). This means that we can be confident that for high overtones, i.e. large \( n \), our expression is still valid. One can see that the real part of the quasinormal frequency does not depend on the integer \( n \) labelling the mode. Therefore, frequencies with a large imaginary part still have a real part given by
\[
\sqrt{f(l+1) - \frac{1}{4}}
\] for scalar and electromagnetic perturbations and by
\[
\sqrt{f(l+1)(l-1) - \frac{1}{4}}
\] for gravitational perturbations. Can one explain an highly damped quasinormal frequency with an \( l \)-dependent real part in light of the recent conjectures relating it to the Barbero-Immirzi parameter? We think it is too early to answer this, and much more work is still necessary, specially in higher dimensional spacetimes and on Anti-de Sitter spacetimes [14], where the AdS/CFT conjecture may have a word to say about this.

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