New Relations for Excited Baryons in Large $N_c$ QCD

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We show that excited baryons in large $N_c$ QCD form multiplets, within which masses are first split at $O(1/N_c)$. The dominant couplings of resonances to various mesons are highly constrained: The $N(1535)$ decays at leading $1/N_c$ order exclusively to $\eta$-$N$ rather than $\pi$-$N$, and vice versa for the $N(1650)$. This multiplet structure is reproduced by a simple large $N_c$ quark model, well studied in the literature, that describes resonances as single-quark excitations.

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During the past two decades there have been two approaches to using large $N_c$ QCD in the description of excited baryons. The first approach, developed in the mid-1980s in the context of Skyrme models, was based on the study of meson-baryon scattering with the identification of excited baryons as resonances (as indeed they are in nature). The second approach, developed more recently, was cast in quark model language and used operator counting rules in a manner similar to those developed to describe the large $N_c$ properties of the ground-state band of $I = J$ baryons. However, unlike for the case of ground-state band baryons, where the quark model is a shorthand for simply implementing trivial dynamical assumptions, in particular, the method is based on matrix elements of operators between excited baryon states, which is strictly only sensible if the excited states are stable, at least at large $N_c$. While this assumption is true in quark models, it is not true in large $N_c$ QCD. This raises the question of the extent to which results for excited baryons in such treatments are in fact results of large $N_c$ QCD.

Here we show: i) At leading order in the large $N_c$ expansion of QCD the baryons form multiplets, within which masses and widths are degenerate up to splittings of $O(1/N_c)$; this result can be derived from large $N_c$ consistency rules with no additional model assumptions. ii) Certain multiplets have no coupling to the $\pi$-$N$ decay channel, while others have no coupling to the $\eta$-$N$ channel up to corrections of $O(1/N_c)$; phenomenological evidence of this is seen in the decays of the $N(1535)$ and $N(1650)$. The degeneracy structure of result i) was previously derived in the quark model language by Pirjol and Yan. We reproduce these quark model results via the technology of Refs. 8, 10. We note that the fact that large $N_c$ quark model reproduces the multiplet structure of large $N_c$ QCD helps to justify these models.

In this work we focus on the lowest-lying negative-parity states. In the $N_c = 3$ quark model they correspond to a 70-plet of SU(6). For simplicity we look here at nonstrange baryons. However, the generalization to positive-parity states and states with nonzero strangeness is straightforward and will be considered in the future.

We start by working directly in a large $N_c$ world (neglecting $1/N_c$ corrections) in order for our analysis to be consistent with results based on treatments of meson-baryon scattering. Of course, in comparing with the physical world of $N_c = 3$, one must bear in mind possible $1/N_c$ corrections (which can in principle be parameterized in a consistent and constrained way), as well as artifacts such as states existing in large $N_c$ but not in the physical world.

Our principal tools are the linear relations between meson-baryon scattering amplitudes in various partial waves that become exact in the large $N_c$ limit. We limit our attention to the scattering of one of the physical world. Recall that in the large $N_c$ limit the $\Delta$ is stable, and thus it is perfectly meaningful to discuss scattering off the $\Delta$. The relevant $S$ matrix formulas for such scattering are given by:

\[ S_{LL'}^{R'R} = \sum_K (-1)^{R' - R} \sqrt{(2R + 1)(2R' + 1)(2K + 1)} \times \left\{ K \begin{array}{c} I \ J \ 
\end{array} \right\} \left\{ K \begin{array}{c} R \ 
\end{array} L \begin{array}{c} 1 \ 
\end{array} \right\} s_{KL'L}^{R'}, \] (1)

\[ S_{LRJ} = \sum_K \delta_{KL} \delta(LRJ) s_{K}^{R}. \] (2)

For $\pi$ scattering, we denote the incoming baryon spin (=isospin) by $R$, the final baryon spin as $R'$, the orbital angular momentum of the incident (final) pion by $L (L')$, and $J$ represents the total isospin (angular momentum) of the state. $S_{LL'R'R'J}^{R'R}$ denotes the $S$ matrix for this channel (isospin- and angular momentum-reduced in the Wigner-Eckart theorem sense), the factors in braces are $6j$ coefficients, and $s_{KL'L}^{R}$ are universal amplitudes.
that are independent of $I$, $J$, $R$, and $R'$. In $\eta$-meson scattering, the isospin $R$ of the baryon cannot change and equals the total isospin $I$ of the state. The orbital angular momentum $L$ of the $\eta$ remains unchanged in the process due to large $N_c$ constraints. The total angular momentum $J$ of the state is then constrained by the triangle rule $\delta(LRJ)$. $S^\eta_{LRJ}$ is the reduced scattering amplitude, and $s^\eta_K$ are universal amplitudes independent of $R$ and $J$. The structure of Eqs. (1) and (2) imply that the scattering amplitudes in different channels are linearly related. There are more $S$ matrix amplitudes $S^\eta_{LRRL}J$ than there are $s^\eta_K$ functions; thus there are linear constraints between the $S^\eta_{LRRL}J$ (that hold to leading order in the $1/N_c$ expansion). Analogously, there are fewer $s^0_K$ amplitudes than $S^\eta_{LRJ}$ amplitudes.

Equations (1) and (2) were originally derived in the Skyrme model. Despite these initial derivations, the validity of Eqs. (1) and (2) does not depend on model assumptions about chiral solitons; rather the equations follow directly from large $N_c$ QCD. A derivation of these relations directly from the large $N_c$ consistency rules, based on the formalism of Ref. 17, and which exploits the noted $I_1 = J_1$ rule, can be found in a more detailed report of our work.[21] We note that the only assumptions needed in this derivation are i) Baryonic quantities in large $N_c$ QCD scale according to the generic large $N_c$ rules of Witten,[21] or more slowly (if there are cancellations); ii) there exists a hadronic description that reproduces the large $N_c$ QCD results; iii) the $\pi$-$N$ coupling scaling is generic (without cancellations) scaling as $N_c^{-1/2}$; and iv) nature is realized in the most symmetric representation of the contracted $SU(2N_f)$ group that emerges from the previous assumptions.

To make concrete predictions about degeneracy patterns, one needs a method to identify the position of a resonance. We use a common theoretical definition of resonance position: The partial-wave scattering amplitudes are functions of energy. By analytically continuing the energies into the complex plane one can identify the resonance position as the location of the pole, with the real part being identified as the mass and the imaginary part as the width. This theoretical definition is unambiguous but has the disadvantage of not being directly determined by scattering with physical kinematics. Fortunately, in the limit of vanishingly small widths, the theoretical definition agrees with the result extracted from scattering data via any sensible prescription.

This resonance position definition implies the existence of degeneracies. Consider the structure of Eqs. (1) and (2). A pole in the partial wave amplitude for some channel on the left-hand side of Eq. (1) implies a pole in one of the $s^\eta_{KL'L'}$ amplitudes on the right-hand side. But since all the $s^\eta_{KL'L'}$ amplitudes contribute to more than one channel, a pole in one channel implies the existence of poles in others, i.e., states with different $I, J$ quantum numbers but with the same mass and width. An analogous argument exists for $\eta$-meson scattering.

To isolate multiplets of degenerate baryon states, one must first determine whether degeneracies could occur in poles of the $s^\eta_{KL'L'}$ and $s^0_K$ amplitudes themselves. A priori one expects the $K$ sectors ought to be dynamically disconnected at large $N_c$, as it would be unnatural to suppose otherwise. In Skyrme models, this is quite clear—the various $K$ sectors are completely separate. However, one may reasonably expect degeneracies in the poles of amplitudes with identical $K$ but distinct values of $L$ or $L'$. $L$ is not a conserved quantum number; one can have a resonance with given $I$ and $J$ but that can reached in scattering by more than one $L$ (e.g., by scattering off a $\Delta$ rather than $N$). In such a case, scattering amplitudes in channels with different $L$ have degenerate poles. Similarly, partial-wave amplitudes involving the scattering of different mesons, but which couple to the same $K$ channel, may produce degenerate poles (e.g., $s^\eta_{222}$ and $s^0_2$).

Degenerate multiplets can be found quite simply. Start by assuming a pole in one of the $s^\eta_{KL'L'}$ or $s^0_K$ amplitudes and use selection rules implicit in the $6j$ coefficients in Eq. (1) and the triangle constraint in Eq. (2) to find all partial-wave amplitudes that couple. Note in doing this one must consider all possible $L, L', R,$ and $R'$ as well as the conserved $I$ and $J$ (which label the quantum numbers of the state). Since we focus on negative-parity states, we restrict attention to the case of $L$ even. From this exercise one identifies the following multiplets of negative-parity baryon states with degenerate masses and widths (modulo splitting at order $1/N_c$) and the associated $s^\eta_{KL'L'}$ or $s^0_K$ amplitudes:

$$\begin{align*}
N_{1/2}, & \Delta_{3/2}, \cdots (s^0_0), \\
N_{1/2}, & \Delta_{1/2}, N_{3/2}, \Delta_{5/2}, \cdots (s^\eta_{100}, s^\eta_{722}), \\
\Delta_{1/2}, & N_{3/2}, \Delta_{3/2}, N_{5/2}, \Delta_{5/2}, \cdots (s^\eta_{222}, s^\eta_0), \\
\Delta_{3/2}, & N_{5/2}, \Delta_{5/2}, \Delta_{7/2}, \cdots (s^\eta_{322}).
\end{align*}$$

These multiplets are formally infinite in size, as we are working in the large $N_c$ limit and thus some states are large $N_c$ artifacts. Note that all amplitudes with the same $K$ have exactly the same multiplet structure: For example, $s^\eta_{100}$ and $s^\eta_{722}$ have the same states in their multiplets, as do $s^\eta_{222}, s^\eta_0$. As discussed above, it is natural to interpret this to mean that any given multiplet of baryon states is accessible via more than one scattering channel provided it has the same $K$.

We note that the same degeneracy patterns were found using the quark model assumption of stable excited baryons.[7] However, as noted above this assumption is inconsistent with large $N_c$ QCD where the width of excited states in generically $O(N_c^0)$. We stress that our prediction of these degenerate multiplets is model independent, following directly from large $N_c$ consistency conditions and that it predicts degenerate masses and widths. We do not predict a priori whether or not large $N_c$ QCD generates low-lying resonances that are narrow enough
to identify. Rather we show that if any such resonances do exist, they fall into multiplets labeled by $K$. The fact that we find multiplets of degenerate states at large $N_c$ is not surprising; it reflects the contracted SU(2$N_f$) symmetry emerging for baryons as $N_c \to \infty$. It is somewhat surprising that these multiplets were not identified previously in the context of large $N_c$ meson-baryon scattering (excepting the degeneracy between the $N_{1/2}$ and the $\Delta_{1/2}$), particularly since studies of the baryon spectrum in models using Eq. [1] were done nearly two decades ago [11,13]. In those works, however, the resonance position was fixed “experimentalist-style” via motion in the Argand plots rather than by looking directly at poles in the complex plane; this obscured the underlying degeneracies.

We note that Eqs. [3–6] may be of more theoretical interest than phenomenological import in the real world of $N_c=3$. The phenomenological difficulty is that the states in nature one may wish to identify with the negative-parity 70-plet (in quark model language) lie in a 200 MeV-wide band (from 1520 MeV ($N_{3/2}$) to 1700 MeV ($\Delta_{3/2}$), as listed by the Particle Data Group [22]). Such states may be associated with three distinct $K$’s, all of which are split by $O(N^0_c)$. On the other hand, the $\Delta-N$ mass splitting is an $O(1/N_c)$ effect but is $\sim 290$ MeV. Since the expected $1/N_c$ corrections that split the multiplets are of the same scale or larger than the distance between the multiplets, it may be very difficult to identify states clearly as belonging to given $K$ multiplets on the basis of spectroscopy alone.

However, the multiplet structure does have nontrivial phenomenological implications for the decays of excited baryons. Note that $K = 0$ negative-parity states do not couple to the $\pi$-$N$ channel. This follows from the fact [Eq. [1]] that $K$ is a vector sum of $L$ and the $\pi$ isospin (=1), implying that $K = 0$ only occurs for $L = 1$, which gives the wrong parity. Similarly, the negative-parity states in the $K = 1$ multiplet cannot couple to the $\eta$-$N$ channel. Assuming that there exist well isolated resonances in both the $K = 0$ and $K = 1$ channels, at large $N_c$ the states associated with the $K = 1$ channel decay into $\pi$-$N$ but not $\eta$-$N$, and vice versa for the $K = 0$ states. For the $N_c = 3$ world there are subleading $1/N_c$ effects, which can cause some mixing of the $K$ modes and hence give nonzero but suppressed decays into the “forbidden” channels. When looking for this effect in phenomenology one must avoid large $N_c$ artifacts. In particular, the existence of more than one low-lying negative-parity $\Delta_{3/2}$ state appears to be an artifact (only one is present in the $N_c = 3$ quark model), so this prediction for decay modes only applies to the $N_{1/2}$ in the real world. The decay patterns of the $N(1535)$ and $N(1650)$ are consistent with the $N(1535)$ being predominantly the $K = 0$ state, while the $N(1650)$ is predominantly $K = 1$. The Particle Data Group [22] lists the decay fraction of $N(1535)$ into $\pi$-$N$ (35–55%) as being essentially the same as the decay fraction into $\eta$-$N$ (35–50%), despite having a substantially larger phase space for decay (nominally by a factor of $\sim 2.6$ evaluated at the mass peak, but effectively higher since the $\eta$-$N$ channel is only 50 MeV from threshold). This indicates the coupling to $\eta$-$N$ is much larger than to $\pi$-$N$. In contrast, the decay fractions for the $N(1650)$ are listed as $\sim 55$–90% into $\pi$-$N$ and only 3–10% into $\eta$-$N$. Since the phase space for decay is only $\sim 1.6$ times greater for the $\pi$-$N$ channel, this indicates a much stronger coupling of the state to $\pi$-$N$.

We now turn to the other major approach that has been used to study excited baryons in the large $N_c$ limit [1,2,3,6,7,8,9,10,11,12,13]. This approach uses quark model language and expresses the large $N_c$ constraints via identification of the $N_c$ scaling of operators. Formally, this is very similar to the techniques of Ref. [17]. An important distinction, however, is that Ref. [11] used quark model language only as simple tool for doing group theory and without any dynamical assumptions of the quark model, while the work of Refs. [1,2,3,6,7,8,9,10,11,12,13] depends in part on quark model dynamics: it neglects coupling to decay channels and hence the widths of the baryons. Moreover, the analyses done so far are based on the dynamics of the simplest version of the quark model, i.e., the lowest-lying negative-parity states are described as being single-quark excitations without configuration mixing. In this picture the states discussed here have one quark carrying a single unit ($\ell = 1$) of orbital excitation about an $(N_c-1)$-quark core symmetric under spin$x$flavor. For $N_c = 3$ this produces the familiar SU(6) 70-plet, but for larger $N_c$ generates additional states.

Previous derivations of the multiplets were based directly on implementation of the large $N_c$ consistency rules [2] which, though intellectually elegant, is rather cumbersome. Here we show how to derive them us the simple operator methods of Refs. [4,6,11]. Up to $O(N^0_c)$, 3 operators contribute to the Hamiltonian in this quark picture, denoted by $H = c_1 \mathbb{1} + c_2 \mathbb{L} + c_3 \ell(2) \sigma G_c/N_c$, where lowercase indicates operators acting upon the excited quark, subscript $c$ indicates those acting upon the core, $G^{\alpha\beta}$ denotes the combined spin-flavor operator $\propto q^\dagger \sigma^\alpha \pi^\sigma q$, and $\ell(2)$ is the $\Delta \ell = 2$ tensor operator. References [6,11] elucidate this notation. We find that to leading order in $1/N_c$ all of the mass eigenvalues are given by only three linear combinations of these parameters:

$$m_0 \equiv c_1 N_c - (c_2 + 5/24 c_3),$$
$$m_1 \equiv c_1 N_c - 1/2 (c_2 - 5/24 c_3),$$
$$m_2 \equiv c_1 N_c + 1/2 (c_2 - 1/24 c_3).$$

For example, the Hamiltonian up to $O(N^0_c)$ for the two $N_{1/2}$ states is

$$H_{N_{1/2}} = (\mathbb{N}_{1/2} \mathbb{N}_{1/2}') M_{N_{1/2} N_{1/2}}^T,$$

$$M_{N_{1/2}} = \begin{pmatrix}
c_1 N_c - 5/2 c_2 & -1/3 \sqrt{2} c_2 - 5/24 c_3 \\
-1/3 \sqrt{2} c_2 + 5/24 c_3 & c_1 N_c - 5/6 c_2 - 5/48 c_3
\end{pmatrix}. $$
where $N_{1/2}$ and $N'_{1/2}$ refer to unmixed negative-parity spin-1/2 nucleon states in the initial quark model basis. This may be obtained from Eqs. (A6)–(A8) or Table II of Ref. [10] [again including only contributions up to $O(N_c^0)$]. Diagonalizing, one finds the masses of the two physical states are given by $M_{N_{1/2}}^{(1)} = m_0$ and $M_{N_{1/2}}^{(2)} = m_1$. Note the surprising absence of square roots of terms quadratic in $c_2$ and $c_3$, which also implies simple analytic results for mixing angles as will be presented in our longer article on this work [20].

Using analogous notation for the other states, we find $M_{N_{3/2}}^{(1)} = m_2$, $M_{N_{3/2}}^{(2)} = m_1$. The $N_{5/2}$ state is unmixed but also has a degenerate eigenvalue: $M_{N_{5/2}} = m_2$. For large $N_c$, the relevant $\Delta$ states of low spin are two $\Delta_{1/2}$'s (masses $m_1$ and $m_2$), three $\Delta_{3/2}$'s ($m_0$, $m_1$, and $m_2$), two $\Delta_{5/2}$'s ($m_0$ and $m_1$), and one $\Delta_{7/2}$ ($m_2$) (In comparison, the only $\Delta$'s for $N_c = 3$ are a single $\Delta_{1/2}$ and $\Delta_{3/2}$). These simple analytic expressions for the masses were not noted in previous work using this approach; those studies included the $O(1/N_c)$ terms in the Hamiltonian and then diagonalized numerically, obscuring the leading $O(N_c^0)$ result.

Clearly, the fact that all of the masses described by the model are given by either $m_0$, $m_1$, or $m_2$ to leading order in the $1/N_c$ expansion implies that at large $N_c$ the various states fall into degenerate multiplets. These multiplets, labeled by the mass, are given by

$$
N_{1/2}, \Delta_{3/2}, \cdots (m_0), \quad (9)
$$

$$
N_{1/2}, \Delta_{1/2}, N_{3/2}, \Delta_{3/2}, \Delta_{5/2}, \cdots (m_1), \quad (10)
$$

$$
\Delta_{1/2}, N_{3/2}, \Delta_{3/2}, N_{5/2}, \Delta_{5/2}, \Delta_{7/2}, \cdots (m_2). \quad (11)
$$

This multiplet structure is striking. Comparing the multiplet structures in Eqs. (9)–(11) predicted for the simple quark model with single-quark excitations to the structure seen for resonances directly from large $N_c$ QCD in Eqs. (3)–(6), it is apparent that the two are identical. The interpretation is simply that $m_0$ represents a pole mass appearing in the $K = 0$ scattering amplitude, $m_1$ in $K = 1$, and $m_2$ in $K = 2$. The fact that these two pictures have the same multiplet structure at large $N_c$ implies that that even this simplest version of the quark model manages to capture at least some nontrivial aspects of QCD dynamics. This might help explain, in part, the surprising phenomenological successes of simple quark models.

In summary, we have shown, in a fully model-independent way, that in large $N_c$ QCD the masses and widths of excited baryons (as measured by their pole positions) form multiplets of states with degenerate masses and widths labeled by $K$. The quantum numbers of states in these multiplets are given in Eqs. (9)–(11). The $K = 0$ ($K = 1$) multiplet decouples from the $\pi$–$N$ ($\eta$–$N$) channel as $N_c \to \infty$, suggesting that $N(1535)$ is associated with $K = 0$ while $N(1650)$ is associated with $K = 1$. Simple quark models describing the lowest-lying excited baryons as single-quark excitations reproduce the same pattern of degenerate multiplets in the large $N_c$ limit as that seen directly in large $N_c$ QCD.

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