Some Remarks on Dirac’s Contributions to General Relativity

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Abstract

I provide a very brief sketch of some of Dirac’s interests and work in gravity, particularly his Hamiltonian formulation of Einstein’s theory and its relation to his earlier research.

It is a great honor for any theoretical physicist to speak at a Memorial for Professor Dirac (like many others, I cannot bring myself to call him Paul!), whom I last saw when I gave a Colloquium here in Tallahassee shortly before he died. He went through the canonical behavior of sleeping during my talk, then awakening at the end with a perfectly reasonable question. He had expressed a desire to see me afterwards in his office; there, he immediately inquired what was new in physics. I told him that he probably wouldn’t be pleased to hear that there had appeared a finite quantum field theory, whereupon he asked – unhappily – what it was. When told it was $N=4$ supersymmetric Yang–Mills, he denied knowledge of any of those words. After hearing a suitable translation, his reaction was simply that either there was an error in the calculations or that the model was really non-interacting, an opinion then also held by some pros. It was clear that his mind was made up about the bankruptcy of QFT, but then again, he was also one of the inventors of extended objects.

It is de rigueur to include a Dirac story in any lecture of which he is the subject, so that by now there are very few unknown ones. My favorite, because physically fraught, comes from Abdus Salam’s introduction to Dirac’s talk at the famous 1968 Trieste Conference. When Dirac first came to St. John’s, there was a traditional Christmas event at which the Maths tutors would pose a riddle to the incoming students; here’s what he drew: Three exhausted fishermen wash up on the beach with their haul, but are too sleepy to split it. At dawn, the first fisherman awakens, takes his 1/3 share and goes home, throwing the one leftover fish back into the sea. Next, number two, unaware of the first, acts identically, as does, finally the third one. What is the minimum (integer) size $N$ of the original pile? Dirac’s lightning-fast solution: $N=–2$! Not only does it foreshadow that other Dirac sea, but displays a fixed point invariance besides – every fisherman sees the same pile! [To keep my audience from calculating, here is the equation: $4N = 9P + 10$ where $P$ is the pile faced by the late sleeper. Of course, $N = 25$ is the “correct”, and so much duller, answer.]

What is not needed from me is an encomium: Dirac was a true Martian, a Hungarian (-in-law) one at that. Feynman and Schwinger, neither otherwise overly impressionable, regarded him with awe, and he was right near Einstein and Bohr on Landau’s famous logarithmic rankings. Instead, in the short time available, I will give a brief (but necessarily incomplete) appreciation of some of Dirac’s work in General Relativity (GR), which I was in a position to observe. Here, I only cite the original papers, but not later lectures or reprises.1

1There exists [1] a very useful “Dalitz plot”, referencing Dirac’s complete works, and reprinting those published
Let me begin by setting the historical stage. After its rapid initial successes, GR was very much a stepchild of theoretical physics research for three decades, until the early fifties. Even then, the renaissance to which Dirac’s work belonged was primarily disconnected from the (indifferent or hostile) field and particle theory mainstream. Indeed, this separation was traditional. While GR was understood quickly after its discovery, neither Bohr nor Heisenberg, for example, ever ventured there, although the former did use the equivalence principle against Einstein in a famous debate on quantum mechanics at the 1927 Solvay Conference, and the latter was the first, in the late thirties, to understand why perturbative quantization of theories, with positive dimensional (self-)coupling constant would fail. Pauli of course started life writing a text on GR, but despite continued interest, he never really contributed to it at the “Pauli” level. In later years, Schrödinger did venture into the field with some brilliant pedagogical expositions, but alas mostly into the morass of “unification by nonsymmetric metric” that occupied Einstein’s own late years. Born explicitly wrote that once he understood GR, he vowed never to work on it. Thus (apart perhaps from Jordan and Klein), Dirac was unique among the creators of quantum mechanics to work seriously on GR.

Looking at Dirac’s own earlier work, one is struck first by some seemingly disconnected, but indicative, themes. The first, physics in deSitter (dS) space [4], comes (as usual) out of nowhere. There he argued that “masslessness” means a non-vanishing mass parameter for spin 1/2 (see also [5]), to some extent foreshadowing the spin 3/2 properties of cosmological supergravity [6, 7]. [Amusingly, Dirac worked only in dS rather than in AdS, and so required this spin 1/2 mass to be imaginary, \( m \approx \sqrt{-\Lambda} \), instead of real as in the natural SUGRA domain, AdS!] More formally, Dirac exploited the Weitzenbock identity for gravity, just as he first did to discover the \( g = 2 \) factor for the Dirac electron in a background magnetic field. To analyze propagation, one must first square the Dirac equation to get a wave operator. When the partial derivative is replaced by a covariant one, the square is more complicated: apart from factors of 1/2, \( i \) etc., one has schematically

\[
\mathcal{P}^2 \equiv \frac{1}{2} \{\gamma, \gamma\} DD + \frac{1}{4}[\gamma, \gamma][D, D].
\]

The last term vanishes for ordinary derivatives, but in general the commutator \([D, D]\) defines a curvature (“field strength”) and in the gravitational case, its net effect is to become the scalar curvature, which (in AdS) is proportional to the cosmological constant \( \Lambda \). Consequently,

\[
(\mathcal{P} + m)(\mathcal{P} - m) = D^2 - (\Lambda + m^2)
\]

and propagation seems to be on the light cone only if \( \Lambda + m^2 = 0 \); actually for \( s=1/2 \), things are a bit more subtle and \( m=0 \) is in fact the counterpart of the conformally improved scalar there. For higher (also integer) spins, things get even more interesting, as described in [8]. Dirac was to return to this theme, but it really belongs to QFT in a (constant curvature) gravitational background rather than to dynamic gravity.

A more direct lead to GR is Dirac’s abiding interest in Lagrangian/Hamiltonian dynamics, locality and “unusual” systems. Two [9, 10] of the great articles on formulations of dynamical systems are well-known. However, there is one other which is significant because he did not use it, namely introduction [11] of what Dirac calls “homogenous coordinates”. This is nothing but a before 1949.

\footnote{The first, and completely isolated, attempts at treating (linearized) GR as a dynamical system were probably those of the early thirties by Rosenfeld [2], and by Bronstein [3] in the USSR. Tragically, the latter’s name coincided with Trotsky’s real one, and he disappeared early in the Stalin purges.}
variant of the Jacobi form of the traditional action principle for normal systems in flat space in which time and Hamiltonian form a new, “n + first”, conjugate \((q, p)\) pair. In terms of the extended set, and of the Lagrange multiplier \(\lambda\) that keeps the “true number” of degrees of freedom to be the original \(n\), the initial Lagrangian

\[
L = \sum_{i=1}^{n} p_i \dot{q}_i - H(p, q)
\]  

becomes

\[
L = \sum_{i=1}^{n+1} p_i \dot{q}_i + \lambda \tilde{H}(p, q) .
\]

where the constraint \(\tilde{H} = 0\) has a root \(p_{n+1} = -H(p, q)\). As it happens, in our (ADM) contemporaneous and independent development of Einstein theory [12], understanding that the Einstein–Hilbert (or any other diffeo-invariant) action was necessarily an “already parametrized” system \(a la\) (3b) was an essential, beautiful, confirmation of our canonical formulation: The big difference from the traditional Jacobi applications is that there is no \(a priori\) passage in GR back from (3b) to (3a): indeed, there is no (3a)\(^3\)! Instead one could fix a gauge, \(q_{n+1} = t\) and solve the constraint for its conjugate \(p_{n+1}\) almost arbitrarily (within physical bounds). [We learned about Jacobi from the Lanczos book on variational principles, rather than from [11].] In any case, looking back, it is hard to understand why Dirac never explicitly invoked the Jacobi formulation for GR, given that one of his motivations was surely to apply his general formalisms to this challenging system!

Dirac’s results were concentrated in three papers [13, 14, 15].\(^3\) Let us briefly summarize the salient points, starting from Dirac’s realization that a 3+1 decomposition of the gravitational field variables (rather than maintaining manifest 4-covariance) is essential to any Hamiltonian – and hence quantization – description. He accordingly used projections with respect to a \(t=\)const. surface and noted that the metric components \(g_{\mu0}\) are only lapses and shifts (as they were later called) and not dynamical. He thus inferred that the time development of any dynamical \((i.e.,\) not involving \(g_{\mu0}\)) gravitational variable \(\eta\) is governed by an evolution equation of the form

\[
\dot{\eta} = \int d^3x [N\xi_L + N_i \xi_i^i] , \quad N \equiv (-g^{00})^{-1/2} , \quad N_i \equiv g_{0i}
\]

and hence must be derived from a Hamiltonian

\[
H = \int d^3x (NH_L + N_i H^i) ; \quad H_L \equiv \mathcal{R} + (p_{ij}^2 - p_{ij}^2/2) , \quad H^i = -D_j p^{ij}
\]

where \(H_L, H^i\) depend as shown on the conjugate \((but not independent)\) pairs of spatial tensors \((g_{ij}, p^{ij})\), which (“weakly”) obey the standard set of Poisson brackets. Here \(\mathcal{R}\) is the intrinsic 3-space curvature scalar and \(p^{ij}\) is essentially the second fundamental form. The \((H_L, H^i)\) are four (“weak” or “secondary”) constraints on the \((g_{ij}, p^{ij})\), and they are understood to be combinations of the \(G_\mu^0\) components of the Einstein tensor, because those do not depend on second time derivatives. That property is easily noted from the Bianchi identities, \(\partial_0 G^0_\mu \sim (\partial_i + \Gamma) G: the right side has only second time derivatives since the \(G_{\mu\nu}\) there are only spatially differentiated, so the left, \(G_\mu^0\) must only have first time derivatives since it is time-differentiated. At this point, the natural – but hardly

\(^3\)The density of competing groups in the field was so low that the dilute approximation can be used in assessing Dirac’s contribution, which like ours was also foreign to that of most relativists. For an idea of the changing research directions that characterized the era, the Proceedings [16] of several conferences bracketing this time are instructive: the first, in Bern, was in 1955, then came Chapel Hill in 1957, Royaumont in 1959 and Warsaw in 1962.
unique – coordinate choice \( g_{\mu 0} = -\delta_{\mu 0} \) is invoked to yield the “true” Hamiltonian \( H_{\text{MAIN}} \), which is just \( \int d^3 x H_L \) after dropping the total 3-divergence that is the linear part of the 3-scalar curvature, \( i.e., \) the (ADM) energy. [Dirac’s third work [15] is concerned with establishing the form of the field’s energy, and evaluate it for some special cases.] Dirac then went on to discuss general issues of constrained Hamiltonian dynamics, in the spirit of [9, 10], in order to understand coordinate fixing and reduction of the apparent number of variable pairs by the constraints. The choice of time is the obvious minimal surface one, \( p_i = 0 \), leaving 5 pairs and finally a spatial harmonic gauge is invoked for the remainder, and the resulting P.B. are then discussed in a perturbative way. The second paper ends with a short section entitled “Quantization”, which simply says that since a complete set of commuting variables is formed by the unimodular, divergenceless part of the spatial metric (together with matter variables), the wave function(al) that depends only on this reduced space is unconstrained, and thus obeys a simple Schrödinger equation. As can be expected, Dirac worries about quantum problems: he notes that the \( t=\text{const.} \) surfaces must remain spacelike, which means that \( g_{ij} \) has to maintain a positive signature – \( i.e., \) positive determinant. Since this quantity is for him also related to the “energy density”, he states that violation could occur very near point sources, basically due to negative gravitational self-energy. The concluding sentence is: “The gravitational treatment of point particles thus brings in one further difficulty, in addition to the usual ones in the quantum theory.” This is rather curious coda since the above problems are really as relevant classically, and of course they are very different from the perturbative nonrenormalizability issues that have dominated all subsequent studies. After this pioneering foray, Dirac’s original publications in the field waned, apart from one later paper [17] on conformally invariant extensions of GR.

I close by emphasizing that, in an era when geometry was (for “real” relativists) the vital guiding thread to the mysteries of GR, the Hamiltonian approaches provided an alternate framework, common to nonabelian gauge QFT, in which the gravitational field is regarded as a regular dynamical system with degrees of freedom, asymptotic boundary conditions and global conserved quantities correlated to the chosen asymptotics. The Hamiltonian approach also serves well when the cosmological constant does not vanish, not to mention its role in elucidating supergravity, which is rightly described as the “Dirac square root” of Einstein theory! Although Dirac lived to see both supergravity and (pre-revolution) string theory, he was, like Moses, unable to enter the promised land, to which he had been our guide in so many ways. We all stand on Dirac’s shoulders.

References