Global-fidelity limits of state-dependent cloning of mixed states

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By relevant modifications, the known global-fidelity limits of state-dependent cloning are extended to mixed quantum states. We assume that the ancilla contains some a priori information about the input state. As it is shown, the obtained results contribute to the stronger no-cloning theorem.

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I. INTRODUCTION

It is well known that a quantum state seeks to escape the observer. One of the manifestations of this sensitivity is expressed by the uncertainty relation. Other striking consequence is revealed by the no-cloning theorem of Wootters, Zurek [1] and Dieks [2]. In effect, an arbitrary unknown pure state cannot be cloned. With relevant modifications, this result is generalized and extended to mixed quantum states [3]. That is, noncommuting mixed states cannot be broadcast. Pati and Braunstein formulated the no-deleting principle [4], which is complementary to the no-cloning theorem. In Ref. [5] the stronger no-cloning theorem was established. Let \( \{ |s_1\rangle, |s_2\rangle \} \) be any pair of nonorthogonal pure states and \( \{ \Upsilon_1, \Upsilon_2 \} \) be any pair of mixed states. According to the stronger no-cloning theorem, there is a physical operation \( |s_j\rangle \otimes \Upsilon_j \mapsto |s_j\rangle |s_j\rangle \) if and only if there is a physical operation \( \Upsilon_j \mapsto |s_j\rangle \). In other words, the full information of the clone must a priori be provided in the ancilla state \( \Upsilon_j \) alone [5].

Thus, it is impossible to copy an arbitrary unknown quantum state perfectly. However, nothing prevents us from finding a close approach to the ideality. The approximate quantum copying was originally considered by Bužek and Hillery [6]. In addition, they examined approximate cloning machines operating on prescribed two nonorthogonal pure states [7]. In Ref. [8] such devices were called ”state-dependent cloners”. It is clear that evaluation of an accuracy of cloning is dependent on the notion of ”closeness” to ideality. The authors of Ref. [8] introduced the notions of ”global fidelity” and ”local fidelity”. The state-dependent cloning was studied by optimizing both the global fidelity and the local fidelity. Chefles and Barnett generalized the notion of the global fidelity to the case, where the states to be cloned have arbitrary a priori probabilities [9]. They obtained the least upper bound on the global fidelity for cloning of two pure states with arbitrary a priori probabilities. Ref. [10] considered state-dependent \( N \rightarrow L \) cloning with respect to both ”global fidelity” criterion and ”local fidelity” criterion. The other category of cloners contains universal cloning machines which copy arbitrary state equally well. First such example was given by Bužek and Hillery [6]. Refs. [8,11] constructed the universal qubit cloner that maximizes the local fidelity. Analogous problem for multi-level quantum system was solved in Refs. [12,13].

All the above results examine the pure-state cloning. Ref. [14] introduced the single qubit purification procedure that was used in extending of the acceptable input for the optimal cloners to mixed quantum states. However, the described in Ref. [14] scenario is not equivalent to the standard statement of cloning problem.

Ref. [15] presented a new approach to the problem of state-dependent cloning. It examined the relative error which is complementary to the criterion of the global fidelity. The method of Ref. [15] lively uses the notion of the angle between two vectors. Ref. [16] extended the notion of the angle to mixed quantum states, that allows to widen the global-fidelity limit of state-dependent cloning deduced in Ref. [8] to the mixed-state cloning. For this extension the squared modulus of the inner product should be replaced by the fidelity function of states to be cloned. Ref. [17] obtained the lower bound on the relative error to mixed-state cloning and related physical operations, in which the ancilla contains some a priori information about the input state. The authors of Ref. [18] modified the method of Ref. [15] and considered the state-dependent cloning of set which contains \( n > 2 \) pure states. They deduced the upper bound on the global fidelity for the case of equal a priori probabilities of states.

In this paper, we extend the known bounds on the global fidelity of state-dependent cloning to mixed quantum states. This is attained by natural development of the method of Ref. [16]. We consider the case, in which the ancilla contains some a priori information about the state to be cloned. The relation between obtained limits and the stronger no-cloning theorem is discussed.
Let us start with a precise formulation of the problem. A register $A$ is composed of $N$ systems, each having an $d$-dimensional Hilbert space $H = \mathbb{C}^d$ ($d > 1$). Initially, every of these $N$ systems is with probability $p_j$ prepared in one and the same state $\rho_j$ from a set $\mathcal{A} = \{\rho_1, \ldots, \rho_n\}$. We want making $L > N$ copies of the given $N$ systems. In order to attain this we use the ancilla which contains some a priori information about the input state $\rho_j$. That is, the ancilla is initially prepared in state $\Upsilon_j$ from a set $\mathcal{S} = \{\Upsilon_1, \ldots, \Upsilon_n\}$ indexed by the same labels. We will mean that the ancilla $BE$ is combined of extra register $B$ and environment $E$. Of course, the extra register $B$ contains $M = L - N$ additional systems, each is to receive the clone of $\rho_j$. Thus, the final state of two registers is described by

$$\tilde{\rho}_j = \text{Tr}_E \left( U(\rho_j^{\otimes N} \otimes \Upsilon_j) U^\dagger \right),$$

(2.1)

which is partial trace over environment space.

In order to estimate an accuracy of cloning we must of course use of some quantitative measure of distinguishability for mixed states. This need is met by the fidelity function [20]. There are several ways to the definition of fidelity. The general expressions is given by

$$F(\chi, \omega) = \left\{ \text{Tr} \left[ (\sqrt{\chi} \omega \sqrt{\chi})^{1/2} \right] \right\}^2,$$

(2.2)

where the trace is taken over the same space on which the density operators $\chi$ and $\omega$ are considered. The right-hand side of Eq. (2.2) is the transition probability between mixed states $\chi$ and $\omega$ introduced by Uhlmann [19]. Jozsa proposed the definition in terms of purifications [20]; it provides a kind of physical interpretation of Eq. (2.2). Note that the concept of purifications of mixed states is a natural development of the "decoherence" point of view. According to this viewpoint [21], any mixed states is really describing the reduced states of a subsystem entangled with a larger system. The total system is always being in a pure state described by vector in a Hilbert space. As it is shown in Ref. [20], the fidelity function is an analogue of the squared modulus of the inner product for pure states. The fidelity can also be defined by means of generalized measurements [3]. (Note that Refs. [3,22] define fidelity to be the square root of the right-hand side of Eq. (2.2).) In Ref. [16] we parametrized the fidelity function by

$$F(\chi, \omega) = \cos^2 \Delta(\chi, \omega),$$

(2.3)

where the angle $\Delta(\chi, \omega)$ ranges between 0 and $\pi/2$. Using the concept of purifications, we extended the triangle inequality to the case of mixed states. That is [16],

$$\Delta(\chi, \omega) \leq \Delta(\chi, \rho) + \Delta(\omega, \rho).$$

(2.4)

Other properties of the angle between two mixed states are listed in Ref. [16]. Note that $\sin \Delta(\chi, \omega)$ provides a reasonable measure of closeness for mixed states $\chi$ and $\omega$. For any (generalized) measurement [17],

$$|p(a|\chi) - p(a|\omega)| \leq \sin \Delta(\chi, \omega),$$

(2.5)

where $p(a|\chi)$ and $p(a|\omega)$ are the probabilities of outcome $a$ generated by $\chi$ and $\omega$ respectively.

Thus, the fidelity function generalizes the squared modulus of the inner product. Therefore, it is natural to define the global fidelity of mixed-state cloning as

$$F_G = \sum_{1 \leq j \leq n} p_j F(\tilde{\rho}_j, \rho_j^{\otimes L}).$$

(2.6)

The definition by Eq. (2.6) uses the state $\rho_j^{\otimes L}$ as ideal output. So we consider the cloning just. Recall that the cloning is a special strong form of broadcasting [3]. The broadcasting is most general type of quantum copying. By broadcasting Ref. [3] means that the marginal density operator of each of the separate system is the same as the input state to be broadcast. The cloning case is specified by choice of state $\rho_j^{\otimes L}$ as perfect. Replacing the squared modulus of the inner product by the fidelity function, the present definition extends the definition of Ref. [9] to mixed quantum states.

We are interested in a nontrivial upper bound on the global fidelity defined by Eq. (2.6). Our approach to obtaining the limits employs triangle inequalities and general properties of the fidelity function. Following the method of Ref.
[16], we shall derive the angle relation from which bound on the global fidelity is simply obtained. In order to be rid of bulky expressions we shall use the notation
\[
\Delta_{jk}^{(L)} = \Delta(\rho_j^L \otimes \rho_k^L) \quad \delta_j = \Delta(\tilde{\rho}_j \otimes \rho_j^L). \tag{2.7}
\]

We also introduce the angle
\[
\alpha_{jk} = \arccos \left[ F(\rho_j^N \otimes N_j, \rho_k^N) F(\Upsilon_j, \Upsilon_k) \right]^{1/2}, \tag{2.8}
\]
which lies in the interval \([0; \pi/2]\).

### III. LIMIT FOR TWO-STATE SET

In this section we establish the limit of state-dependent cloning of two-state set \(\mathcal{A} = \{\rho_1, \rho_2\}\). The initial state of ancilla is \(\Upsilon_1\) or \(\Upsilon_2\) according to the input state which is \(\rho_1\) or \(\rho_2\). We shall restrict our consideration to the case in which
\[
F(\rho_1^M \otimes \rho_2^M) < F(\Upsilon_1, \Upsilon_2). \tag{3.1}
\]

The motivation for this restriction consists in the following. As it is shown in Appendix A, if Eq. (3.1) is not satisfied then there are states sufficient for perfect cloning. That is, there are states \(\Upsilon_1\) and \(\Upsilon_2\) such that
\[
\rho_j^M = \text{Tr}_E \Upsilon_j
\]
for \(j = 1, 2\). Here we can only point to trivial bound \(F_G \leq 1\). So we presuppose that Eq. (3.1) is valid. As result, we have
\[
\alpha_{12} < \Delta_{12}^{(L)}. \tag{3.2}
\]

The desired limit is established by the following theorem.

**Theorem 1** The global fidelity \(F_G\) of \(N \rightarrow L\) cloning for set \(\mathcal{A} = \{\rho_1, \rho_2\}\) is limited above by value
\[
\frac{1}{2} \left\{ 1 + \left[ 1 - 4p_1p_2 \sin^2(\Delta_{12}^{(L)} - \alpha_{12}) \right]^{1/2} \right\}. \tag{3.6}
\]

**Proof** Applying Eq. (2.4) twice, we get
\[
\Delta_{12}^{(L)} \leq \delta_1 + \delta_2 \tag{3.3}
\]
Recall that the fidelity function is multiplicative and preserved by unitary evolution [20]. Therefore,
\[
F(\rho_1^N \otimes N_1, \rho_2^N \otimes N_2) = F\left( U(\rho_1^N \otimes N_1) U^\dagger, U(\rho_2^N \otimes N_2) U^\dagger \right).
\]

Because the fidelity cannot decrease under the operation of partial trace [3],
\[
F(\rho_1^N \otimes N_1, \rho_2^N \otimes N_2) \leq F(\tilde{\rho}_1, \tilde{\rho}_2),
\]
whence we have
\[
\alpha_{12} \geq \Delta(\tilde{\rho}_1, \tilde{\rho}_2). \tag{3.4}
\]

Eqs. (3.3) and (3.4) provide
\[
\delta_1 + \delta_2 \geq \Delta_{12}^{(L)} - \alpha_{12}. \tag{3.5}
\]
By Eq. (3.2), the right-hand side of Eq. (3.5) ranges between 0 and \(\pi/2\). (Note that if our presupposition given by Eq. (3.1) is broken then the right-hand side of Eq. (3.5) is nonpositive and Eq. (3.5) becomes empty.) According to the definition of the global fidelity,
\[
F_G = p_1 \cos^2 \delta_1 + p_2 \cos^2 \delta_2. \tag{3.6}
\]
We want to maximize the right-hand side of Eq. (3.6) with the constraints (3.5), $0 \leq \delta_1 \leq \pi/2$ and $0 \leq \delta_2 \leq \pi/2$. This problem is considered in Appendix B, the result is formulated as Lemma. Performing the relevant substitutions, we obtain the statement of Theorem 1.

If all the states of set $\mathfrak{A}$ are pure, that is $\rho_j = |s_j\rangle\langle s_j|$ for $j = 1, 2$, and the ancilla contains no a priori information, that is $\Upsilon_1 = \Upsilon_2$, then we have

$$\cos \Delta_{12}^{(L)} = \left| \langle s_1^L | s_2^L \rangle \right|,$$
$$\cos \alpha_{12}^{(N)} = \left| \langle s_1^N | s_2^N \rangle \right|.$$

In this special case the limit given by Theorem 1 is reduced to the limit obtained in Ref. [9]. Thus, Theorem 1 provides the extension of the preceding result in two significances. In the first place, it extends the known limit to the case of mixed states. In the second place, it takes into account that the ancilla state can contain a priori information about the state to be cloned. As the proof of Lemma shows, the equality in Eq. (3.5) is necessary to maximize the right-hand side of Eq. (3.6). For pure states, the equality in the spherical triangle inequality is reached only if the state to be cloned. As the proof of Lemma shows, the equality in Eq. (3.5) is necessary to maximize the right-hand side of Eq. (3.6). For pure states, the equality in the spherical triangle inequality is reached only if the state to be cloned.

As is well known, the global fidelity is really maximized when the final states lie in the subspace spanned by exact clones $|s_1^L\rangle$ and $|s_2^L\rangle$. As is well known, the global fidelity is really maximized when the final states lie in the mentioned subspace $[8,9]$. However, we do not know a way to reach the equality in Eq. (3.5) in the general case. So it is unknown whether the above limit is least.

The limit by Theorem 1 is a decreasing function of $p_1p_2$ and increases as the a priori probabilities differ. For pure-state cloning this fact was shown in Ref. [9]. So we are rather interested in dependence of the limit on the parameter $F(\Upsilon_1, \Upsilon_2)$. This parameter marks the top amount of the additional information, which can beforehand be contained in the ancilla. The more $F(\Upsilon_1, \Upsilon_2)$ the less the given amount. Let $F(\Upsilon_1, \Upsilon_2)$ be variable between $F(\rho_1^{\otimes M}, \rho_2^{\otimes M})$ and 1 and let the rest of parameters be fixed. Then we have

$$\Delta_{12}^{(N)} \leq \alpha_{12} \leq \Delta_{12}^{(L)}.$$

In this range the limit by Theorem 1 is an increasing function of $\alpha_{12}$. In line with Eq. (2.9), the angle $\alpha_{12}$ decreases as $F(\Upsilon_1, \Upsilon_2)$ increases. Therefore, the limit by Theorem 1 decreases as $F(\Upsilon_1, \Upsilon_2)$ increases. For $F(\Upsilon_1, \Upsilon_2) = F(\rho_1^{\otimes M}, \rho_2^{\otimes M})$ the perfect cloning can be attained. In harmony with this, the limit by Theorem 1 is equal to 1. For example, the equality $F_G = 1$ is reached by the ancilla state $\Upsilon_j = \rho_j^{\otimes M}$, where $j = 1, 2$. Here the full information needed for the ideal cloning is already provided in the ancilla alone. On the contrary, in the standard cloning there is no a priori information, i.e. $\Upsilon_1 = \Upsilon_2$ and $F(\Upsilon_1, \Upsilon_2) = 1$. Then the limit by Theorem 1 reaches its minimum as a function of $F(\Upsilon_1, \Upsilon_2)$.

For $L$ going to infinity we have $\Delta_{12}^{(L)} = \pi/2$ (except $p_1 = p_2$) and

$$\sin^2(\Delta_{12}^{(L)} - \alpha_{12}) = F(\rho_1^{\otimes N}, \rho_2^{\otimes N}) F(\Upsilon_1, \Upsilon_2).$$

Then the global-fidelity limit becomes

$$F_G \leq \frac{1}{2} \left\{ 1 + \left[ 1 - 4p_1p_2 F(\rho_1^{\otimes N}, \rho_2^{\otimes N}) F(\Upsilon_1, \Upsilon_2) \right]^{1/2} \right\}^{1/2}. \quad (3.7)$$

In the special case of pure states and no a priori information in the ancilla, the right-hand side of Eq. (3.7) is the well-known Helstrom bound [23]. It is the probability of correctly distinguishing between two pure states $|s_1^{\otimes N}\rangle$ and $|s_2^{\otimes N}\rangle$ by optimal strategy.

**IV. LIMIT FOR MULTI-STATE SET**

We are now ready to state the global-fidelity limit when the state-set $\mathfrak{A}$ contains more states than two ($n > 2$). To simplify the exposition, we assume that there is no a priori information about the state to be cloned. Note that in this case we have

$$\alpha_{jk} = \Delta_{jk}^{(N)}.$$

As before, we take that a priori probabilities are arbitrary; ones are constrained only by $p_1 + \cdots + p_n = 1$. 

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Theorem 2 The global fidelity $F_G$ of standard $N \rightarrow L$ cloning for set $\mathcal{A} = \{\rho_1, \ldots, \rho_n\}$ is limited above by value

$$
\frac{1}{n-1} \sum_{1 \leq j < k \leq n} \frac{p_j + p_k}{2} \left\{ 1 + \left[ 1 - \frac{4p_j p_k}{(p_j + p_k)^2} \sin^2 (\Delta_{jk}^{(L)} - \Delta_{jk}^{(N)}) \right]^{1/2} \right\}.
$$

Proof In line with the definition of global fidelity,

$$
F_G = \sum_{1 \leq j \leq n} p_j \cos^2 \delta_j.
$$

Dividing each term of the sum into $n-1$ equal parts and regrouping these, Eq. (4.2) can be rewritten as

$$
F_G = \frac{1}{n-1} \sum_{1 \leq j < k \leq n} \left( p_j \cos^2 \delta_j + p_k \cos^2 \delta_k \right).
$$

Because $p_j/(p_j + p_k) + p_k/(p_j + p_k) = 1$, we can apply Theorem 1 to each term of the sum of Eq. (4.3). By Theorem 1 and Eq. (4.1),

$$
\frac{p_j}{p_j + p_k} \cos^2 \delta_j + \frac{p_k}{p_j + p_k} \cos^2 \delta_k \leq \left\{ \frac{1}{2} \left[ 1 + \left[ 1 - \frac{4p_j p_k}{(p_j + p_k)^2} \sin^2 (\Delta_{jk}^{(L)} - \Delta_{jk}^{(N)}) \right]^{1/2} \right] \right\}.
$$

Eqs. (4.3) and (4.4) provide the statement of Theorem 2. ■

For equal a priori probabilities, that is $p_j = 1/n$, the limit becomes

$$
F_G \leq \frac{1}{2} + \frac{1}{n(n-1)} \sum_{1 \leq j < k \leq n} \cos (\Delta_{jk}^{(L)} - \Delta_{jk}^{(N)}).
$$

In the special case of pure states, Eq. (4.5) gives the limit obtained in Ref. [18]. So we have extended the preceding result to the mixed-state cloning. It is known that the limit given by Theorem 2 is least for cloning of two pure states; the quantum network desired to attain this limit has been presented in Ref. [9]. Probably, it is not the case in general. As the ancilla contains some a priori information, the following modifications must be made in the limit by Theorem 2. If $F(\Upsilon_j, \Upsilon_k) > F(\rho_j^{\otimes M}, \rho_k^{\otimes M})$ for the given pair $[jk]$, then $\Delta_{jk}^{(N)}$ should be replaced by $\alpha_{jk}$. If $F(\Upsilon_j, \Upsilon_k) \leq F(\rho_j^{\otimes M}, \rho_k^{\otimes M})$ for the given pair $[jk]$, then the respective term of the sum should be replaced by $(p_j + p_k)$. However, we know nothing about an attainability of such a limit. Evidently, this open question cannot be clarified without detailing of structure of the ancilla states.

V. CONCLUSION

In this work we have obtained some new results on the limits of state-dependent cloning. The known bounds on the global fidelity of pure-state cloning are extended in two significances. In the first place, the mixed-state cloning is regarded. In the second place, we take into account that the ancilla state can contain a priori information about the state to be cloned. The dependence of limit on the parameter, that roughly describes the amount of a priori information, is considered. The conclusions made look reasonable and contribute to the stronger no-cloning theorem.

Replacing the squared modulus of the inner product by the fidelity function, the known global-fidelity limits of state-dependent cloning of pure states are valid to mixed quantum states. This fact maintains an intuitive belief that use of mixed states hardly adds anything new to our possibilities in the quantum information processing. We study the mixed states rather because all the real devices are inevitably exposed to noise. So the pure states used us will eventually evolve to mixed states. It is in the nature of things.

APPENDIX A: AN EXISTENCE OF SUFFICIENT STATES

Let $\chi$ and $\omega$ be density operators on a finite-dimensional Hilbert space $\mathcal{H}_1$ and let $r$ be real number such that $0 \leq r \leq F(\chi, \omega)$. We shall show an existence of states $\Lambda$ and $\Upsilon$ those have $\chi$ and $\omega$ as the reduced states for subsystem under the constraint $F(\Lambda, \Upsilon) = r$. We briefly recall the definition of the fidelity in terms of purification [20]. A purification of $\chi$ is any pure state $|X\rangle$ in any extended Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with the property that

$$\chi = \text{Tr}_2(|X\rangle\langle X|).$$  \hspace{1cm} (A1)

In general, the dimension of $\mathcal{H}_2$ must not be smaller than the dimension of $\mathcal{H}_1$. The fidelity is defined by

$$F(\chi, \omega) = \max |\langle X|Y \rangle|^2,$$  \hspace{1cm} (A2)

where the maximum is taken over all purifications $|X\rangle$ and $|Y\rangle$ of $\chi$ and $\omega$ respectively [20]. Take the purifications $|X\rangle$ and $|Y\rangle$ those give the maximum in Eq. (A2). Then we have

$$F(\chi, \omega) = |\langle X|Y \rangle|^2.$$  \hspace{1cm} (A3)

Adding the qubit space $\mathcal{H}_3 = \text{span}\{|0\rangle, |1\rangle\}$, we consider the pure states

$$|X0\rangle = |X\rangle \otimes |0\rangle,$$  \hspace{1cm} (A4)

$$|Y\theta\rangle = |Y\rangle \otimes (\cos \theta |0\rangle + \sin \theta |1\rangle),$$  \hspace{1cm} (A5)

which lie in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. It is clear, these states are purifications of $\chi$ and $\omega$ respectively, that is

$$\chi = \text{Tr}_{23}(|X0\rangle\langle X0|),$$

$$\omega = \text{Tr}_{23}(|Y\theta\rangle\langle Y\theta|),$$

In the case of pure states, the fidelity function is equal to the squared modulus of the inner product:

$$F(|X0\rangle\langle X0|, |Y\theta\rangle\langle Y\theta|) = |\langle X0|Y\theta \rangle|^2.$$  \hspace{1cm} (A6)

By Eqs. (A3), (A4) and (A5), we have

$$F(|X0\rangle\langle X0|, |Y\theta\rangle\langle Y\theta|) = F(\chi, \omega) \cos^2 \theta.$$  \hspace{1cm} (A6)

Because $0 \leq r \leq F(\chi, \omega)$, the right-hand side of Eq. (A6) can be set as equal to $r$ by choice of $\theta$. If we now take $\Lambda = |X0\rangle\langle X0|$ and $\Upsilon = |Y\theta\rangle\langle Y\theta|$ then $\chi = \text{Tr}_{23}\Lambda$, $\omega = \text{Tr}_{23}\Upsilon$ and $F(\Lambda, \Upsilon) = r$ too. Thus, if Eq. (3.1) is violated, then there are the ancilla states those are sufficient for the ideal cloning.
Consider the function

$$f(x, y) = p \cos^2 x + q \cos^2 y,$$  \hspace{1cm} (B1)

where $p$ and $q$ are positive numbers such that $p + q = 1$. Let $a \in [0; \pi/2]$ be a fixed parameter. The range of variables is stated by conditions $x + y \geq a$, $0 \leq x \leq \pi/2$ and $0 \leq y \leq \pi/2$. This domain $D$ is a square whose left-lower corner is truncated by line $x + y = a$. We find the maximum of $f(x, y)$ in the domain $D$.

**Lemma** The maximum of the function $f(x, y)$ in the domain $D$ is equal to

$$f_{\text{max}} = \frac{1}{2} \left\{ 1 + \sqrt{1 - 4pq \sin^2 a} \right\}. \hspace{1cm} (B2)$$

**Proof** Interior to the domain $D$, $\partial f/\partial x \neq 0$ and $\partial f/\partial y \neq 0$. Therefore, the maximum is reached on the boundary $\partial D$. Let us consider the boundary segment on which $x + y = a$. Eq. (B1) can be rewritten as

$$f(x, y) = \frac{1}{2} \{ 1 + \cos(x + y) \cos(x - y) + (q - p) \sin(x + y) \sin(x - y) \},$$

that was observed by the authors of Ref. [9]. On the mentioned segment we have

$$f(x, y) = \frac{1}{2} \{ 1 + \cos a \cos(2x - a) + (q - p) \sin a \sin(2x - a) \}. \hspace{1cm} (B3)$$

By the standard procedure, we obtain the requirement

$$\tan(2x - a) = (q - p) \tan a. \hspace{1cm} (B4)$$

Since on the considered segment $x$ ranges between 0 and $a$, the inequality $-a \leq 2x - a \leq a$ holds, where $a \leq \pi/2$. Then $\cos(2x - a)$ is nonnegative and $\sin(2x - a)$ has the same sign as $(q - p)$. We can now reexpress the cosine and the sine in terms of the tangent. By these expressions and Eq. (B4), the maximum is equal to the right-hand side of Eq. (B2). For the rest of boundary segments, the corresponding maximums are trivially obtained. It is easy to check that they do not exceed the right-hand side of Eq. (B2). $\blacksquare$