Monotonic decrease of the quantum nonadditive divergence by projective measurements

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Nonadditive (nonextensive) generalization of the quantum Kullback-Leibler divergence, termed the quantum $q$-divergence, is shown not to increase by projective measurements in an elementary manner.

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In recent papers [1-3], we have developed a nonadditive generalization of information theory and have discussed its distinguished roles in the study of quantum entanglement extensively (see also, [4-8]). These works have primarily been concerned with the Tsallis nonadditive (nonextensive) entropy [9] and the associated generalized conditional entropy [1]. On the other hand, quite recently, the role of the generalized Kullback-Leibler divergence, termed the quantum $q$-divergence, has been examined as a measure of the degree of state purification [10]. There, an advantageous point of the quantum $q$-divergence over the ordinary quantum Kullback-Leibler divergence has been clarified.

In this article, we study the behavior of the quantum $q$-divergence under measurements, i.e., quantum operations. In particular, we present an elementary proof that the quantum $q$-divergence does not increase by projective measurements.

The quantum $q$-divergence is the relative entropy associated with the Tsallis entropy. The Tsallis entropy reads

$$S_q[\rho] = -\frac{1}{1-q} \text{Tr}(\rho^q \ln_q \rho).$$

(1)

Here, $\rho$ is the normalized density matrix of the quantum system under consideration and $q$ is the entropic index which can be an arbitrary positive number at this level. $\ln_q x$ stands for the $q$-logarithmic function [11] defined by $\ln_q x = (x^{1-q} - 1) / (1-q)$.
which tends to the ordinary logarithmic function, $\ln x$, in the limit $q \to 1$. Then, the quantum $q$-divergence of $\rho$ with respect to the reference density matrix $\sigma$ is given by

$$K_q[\rho \| \sigma] = \text{Tr} \left[ \rho^q \left( \ln_q \rho - \ln_q \sigma \right) \right].$$

(The classical counterpart of this quantity has been introduced independently and almost simultaneously in [12-14].) Using the definition of the $q$-logarithmic function, Eq. (2) can also be written in the following compact form:

$$K_q[\rho \| \sigma] = \frac{1}{1-q} \left[ 1 - \text{Tr} \left( \rho^q \sigma^{1-q} \right) \right].$$

Since this quantity should not be too sensitive to small eigenvalues of the density matrices, the range of $q$ is taken to be

$$0 < q < 1.$$  \hspace{1cm} (4)

Let $s_{\rho}$ and $s_{\sigma}$ be the supports of $\rho$ and $\sigma$, respectively. In the case when $s_{\rho} \leq s_{\sigma}$, $K_q[\rho \| \sigma]$ has the well-defined limit $q \to 1-0$, which yields the ordinary quantum Kullback-Leibler divergence introduced by Umegaki [15]
\[ K[\rho \| \sigma] = \text{Tr}[\rho (\ln \rho - \ln \sigma)]. \] (5)

Here, the condition, \( s_\rho \leq s_\sigma \), is crucial. In fact, \( K[\rho \| \sigma] \) becomes singular when \( s_\rho > s_\sigma \). Therefore, \( K[\rho \| \sigma] \) cannot be defined if \( \sigma \) is a pure state (i.e., an idempotent operator), for example. In marked contrast to this, \( K_q[\rho \| \sigma] \) with \( q \in (0, 1) \) remains well-defined even in such a case [10].

In Ref. [10], it has been shown that (i) \( K_q[\rho \| \sigma] \geq 0 \) and \( K_q[\rho \| \sigma] = 0 \) if and only if \( \rho = \sigma \), (ii) for product states, \( \rho(A, B) = \rho_1(A) \otimes \rho_2(B) \) and \( \sigma(A, B) = \sigma_1(A) \otimes \sigma_2(B) \), of a bipartite system \((A, B)\), \( K_q[\rho \| \sigma] \) satisfies pseudoadditivity:

\[ K_q[\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2] = K_q[\rho_1 \| \sigma_1] + K_q[\rho_2 \| \sigma_2] + (q-1)K_q[\rho_1 \| \sigma_1]K_q[\rho_2 \| \sigma_2] \] and

(iii) \( K_q \) can be observed as the \( q \)-analog (i.e., \( q \)-deformation) of \( K \) in the sense in [16].

In addition to the properties (i)-(iii), we wish to notice another important one anew here. That is, \( K_q[\rho \| \sigma] \) is jointly convex

\[ K_q\left[\sum_i \lambda_i \rho^{(i)} \| \sum_i \lambda_i \sigma^{(i)}\right] \leq \sum_i \lambda_i K_q[\rho^{(i)} \| \sigma^{(i)}], \] (6)

where \( \lambda_i > 0 \) and \( \sum_i \lambda_i = 1 \). This directly follows from the expression in Eq. (3) as well as Lieb’s theorem [17] stating that \( \text{Tr}(L^{1-x} M^x) \) with \( x \in (0, 1) \) is jointly concave in any positive operators, \( L \) and \( M \). Eq. (6) generalizes joint convexity of the ordinary quantum divergence (see [18], for example).
Now, let us discuss the behavior of $K_q[\rho\|\sigma]$ under projective measurement of $\rho$ and $\sigma$. This measurement can be regarded as a particular kind of positive trace-preserving quantum operation, but is quite common from the experimental viewpoint [19]. Let $Q$ be an observable with eigenspaces defined by orthogonal projections $P_k$ and $\{q_k\}$ be its measured values. Then, $Q = \sum_k q_k P_k$, $P_k P_k' = \delta_{kk'} P_k$, and $\sum_k P_k = I$. The finite probability $p_k$ of obtaining the value $q_k$ of $Q$ in a state $\rho$ of the system through the projective measurement is $p_k = \text{Tr}(\rho P_k)$. From this, $\rho$ is transformed to $\rho_k = p_k^{-1} P_k \rho P_k$. Averaging over all possible outcomes, we have

$$\Pi(\rho) = \sum_k p_k \rho_k = \sum_k P_k \rho P_k.$$  \hspace{1cm} (7)

Clearly, $\Pi$ is a positive trace-preserving operation.

Let us employ the diagonal representations of $\rho$ and $\sigma$:

$$\rho = \sum_a r(a) |a\rangle\langle a|, \quad \sigma = \sum_b s(b) |b\rangle\langle b|,$$  \hspace{1cm} (8)

where $r(a) \geq 0$, $\sum_a r(a) = 1$, $\langle a|a\rangle = \delta_{aa}$, $\sum_a |a\rangle\langle a| = I$ and so on. Under the operation of a projective measurement, they are replaced by
\[ \Pi(\rho) = \sum_a r(a) \Pi(|a\rangle\langle a|), \quad \Pi(\sigma) = \sum_b s(b) \Pi(|b\rangle\langle b|), \]  

(9)

respectively. Let us further use the diagonal representations

\[ \Pi(|a\rangle\langle a|) = \sum_{\alpha} \mu(\alpha, a) |\alpha\rangle\langle \alpha|, \quad \Pi(|b\rangle\langle b|) = \sum_{\beta} \nu(\beta, b) |\beta\rangle\langle \beta|, \]  

(10)

where \( \mu(\alpha, a) = \sum_k |\langle \alpha| P_k |a\rangle|^2 \geq 0, \quad \sum_a \mu(\alpha, a) = \sum_{\alpha} \mu(\alpha, a) = 1, \quad \langle \alpha|\alpha'\rangle = \delta_{\alpha\alpha'}, \) 

\[ \sum_{\alpha} |\alpha\rangle\langle \alpha| = I \text{ and so on. Accordingly, we have} \]

\[ \left[ \Pi(\rho) \right]^q = \sum_{a, \alpha} [r(a) \mu(\alpha, a)]^q |\alpha\rangle\langle \alpha|, \quad \left[ \Pi(\sigma) \right]^{1-q} = \sum_{b, \beta} [s(b) \nu(\beta, b)]^{1-q} |\beta\rangle\langle \beta|, \]  

(11)

which leads to

\[ \text{Tr}\left[ \left[ \Pi(\rho) \right]^q \left[ \Pi(\sigma) \right]^{1-q} \right] = \sum_{a, \alpha} \sum_{b, \beta} [r(a) \mu(\alpha, a)]^q [s(b) \nu(\beta, b)]^{1-q} |\langle \alpha|\beta\rangle|^2. \]  

(12)

Since \( 0 \leq \mu, \nu \leq 1 \) and \( 0 < q < 1 \), we see that \( \mu^q > \mu, \quad \nu^q > \nu \). Therefore, we have

\[ \text{Tr}\left[ \left[ \Pi(\rho) \right]^q \left[ \Pi(\sigma) \right]^{1-q} \right] \geq \sum_{a, \alpha} \sum_{b, \beta} [r(a)]^q [s(b)]^{1-q} \text{Tr}\left[ \Pi(|a\rangle\langle a|) \Pi(|b\rangle\langle b|) \right]. \]  

(13)
Noting that

\[
\text{Tr}\left[\Pi(a\langle a\rangle) \Pi(b\langle b\rangle)\right] = \sum_{k,k'} |\langle a|P_k P_{k'}|b\rangle|^2
\]

\[
\geq \left|\sum_{k,k'} \langle a|P_k P_{k'}|b\rangle\right|^2 = |\langle a|b\rangle|^2,
\]  

we find

\[
\text{Tr}\left[\left[\Pi(\rho)^q [\Pi(\sigma)]^{1-q}\right]\right] \geq \text{Tr}\left(\rho^q \sigma^{1-q}\right),
\]

leading to

\[
K_q [\Pi(\rho)\|\Pi(\sigma)] \leq K_q [\rho\|\sigma].
\]

Therefore, we obtain the main result that the quantum \(q\)-divergence does not increase by projective measurements.

In conclusion, we have shown that the quantum \(q\)-divergence is jointly convex and does not increase by projective measurements.
References


[11] For the various properties of the $q$-functions, see, S. Abe, Y. Okamoto (Eds.),


[Erratum, 40 (1999) 2196].


