Critical Behavior of Non Order-Parameter Fields

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We show that all of the relevant features of a phase transition can be determined using a non order parameter field which is a physical state of the theory. This fact allows us to understand the deconfining transition of the pure Yang-Mills theory via the physical excitations rather than using the Polyakov loop.

INTRODUCTION

Some of the relevant intrinsic properties of the phase transition are best investigated using the order parameter of the theory. Sometimes the order parameter can be a non physical object or not experimentally accessible. The pure Yang-Mills deconfinement phase transition constitutes a time honored example. Here the order parameter is the Polyakov loop which is not a physical state of the theory while the physical field is the glueball field. We argue that it is always possible to extract all of the relevant information about a specific phase transition using a non order parameter field which is a physical state of the underlying theory.

We predict the universal behavior of this field near and at the phase transition. More specifically we predict a finite drop in the mass of the singlet field at the phase transition. Our predictions seem to reproduce surprisingly well the general features of some lattice data and can be put to further and more stringent tests via new first principle lattice simulations. Finally, we define new physical quantities able to characterize the phase transition.

A RENORMALIZABLE THEORY AND CRITICAL BEHAVIOR

We consider a temperature regime close to the phase transition. In order for our results to be as universal as possible we study a renormalizable Lagrangian containing a field neutral under the global symmetries and the order parameter field of theory as well as their interactions. The protagonists of our theory are two canonically normalized fields $h$ and $\chi$. The field $h$ is a scalar of $Z_N$ while $\chi$ transforms according to $\chi \rightarrow z\chi$ with $z \in Z_N$.

Here we consider the case $N = 2$. This theory is suitable for understanding the deconfining phase transition of 2 color Yang-Mills which has been heavily studied via lattice simulations. A general, renormalizable, potential is:

$$V(h, \chi) = \frac{m^2}{2} h^2 + \frac{m^2}{2} \chi^2 + \frac{\lambda}{4!} \chi^4 + g_0 h \chi + \frac{g_1}{2} h^2 \chi^2 + \frac{g_2}{4} h^4 + \frac{g_3}{3!} h^3 + \frac{g_4}{4!} h^4.$$

The coefficients are all real with $g_0 = 0$. Stability requires $\lambda \geq 0$ and $g_1 \geq 0$. It is straightforward to generalize this potential to $Z_N$. Since we will relate $\chi$ to the Polyakov loop, we take it to depend only on space coordinates. This leads to a mixed situation in which the physical states depend also on time. We will also briefly comment on what happens when the order parameter is taken to be four dimensional. The Lagrangian we use to define our Feynman rules is:

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} h \partial^\mu h - \frac{1}{2} \nabla \chi \cdot \nabla \chi - V(h, \chi).$$

We start our investigation at temperatures below $T_c$. Since the phase transition is of second order, $m_\chi$ vanishes at the transition point. All of the coupling constants are such that $h$ and $\chi$ have zero vacuum expectation value in this regime.

We focus our attention on the $h$ self energy at the one loop level at a given non zero temperature:

$$= 12 + 2 + 2 + 18$$

The double lines indicate $\chi$ fields while single lines stand for $h$ ones. We are not considering tadpole diagrams which we require to vanish at each given loop order by adding a counterterm linear in $h$. The number in front of each diagram is the associated combinatorial factor.

We consider the limit in which the $h$ mass is much larger than the temperatures in play, hence all of the loops containing only $h$ fields are then infrared finite at any temperature. The second diagram in eq. is, however, linearly ultraviolet divergent but this divergence is absorbed in the respective mass counterterm. After all the ultraviolet divergences have been taken into account, we can show that the finite temperature corrections due to $h$ are Boltzman suppressed. We are left with the following infrared divergent graph for $T \rightarrow T_c$,

$$= T \frac{g_1^2}{16\pi m_\chi},$$

and this diagram constitutes the relevant one loop correction to the $h$ mass which reads:

$$m^2(T) = m^2 - T \frac{g_1^2}{16\pi m_\chi}.$$
This is the way the nearby phase transition is directly felt by the non order parameter field. It shows that the mass of the singlet field must decrease fast close to the phase transition. An unpleasant consequence is that this one loop result breaks down at the transition point due to the infrared singularity.

Besides, since $h$ is not the order parameter, its correlation length (i.e. $1/m$) is not expected to diverge at the phase transition. To cure the infrared behavior we need to go beyond the one-loop approximation and consistently resum the following set of diagrams:

\[
\begin{align*}
\raisebox{-1em}{\includegraphics[width=1cm]{diagram1}} & + \raisebox{-1em}{\includegraphics[width=1cm]{diagram2}} + \raisebox{-1em}{\includegraphics[width=1cm]{diagram3}} + \cdots.
\end{align*}
\]

Each diagram is infrared divergent but when resummed $m$ is finite at $T$ and reads:

\[
m^2(T) = m^2 - T \frac{\lambda T}{16\pi m^2 + \lambda T}.
\]

At $T = T_c$ we have

\[
m^2(T_c) = m^2 - \frac{\lambda^2}{\lambda}.
\]

Hence we predict that close to the phase transition the singlet state has a decreasing mass. The drop at the phase transition point is the ratio between the square of the coupling constant governing the interaction of the singlet state with the order parameter ($g_1$) and the order parameter self interaction coupling constant ($\lambda$).

The slope is:

\[
D^- \equiv \lim_{T \to T_c^-} \frac{1}{\Delta m^2} \frac{d m^2(T)}{dT} = \frac{16\pi}{\lambda T_c} \lim_{T \to T_c^-} \frac{d m^2(T)}{dT},
\]

with $\Delta m^2 = m^2(T_c) - m^2 = g_1^2/\lambda$. This slope encodes the critical behavior of the theory even though it is constructed using the non order parameter field. For example if $m_x^2$ vanishes as $(T_c - T)^{\nu}$ close to the phase transition, then $D^-$ scales with the exponent $(\nu/2 - 1)$. When considering an $O(N)$ symmetry group rather than $Z_2$ the resummation procedure becomes exact in the large $N$ limit.

When $T > T_c$ $\chi$ develops a vacuum expectation value which will induce one also for $h$:

\[
\langle \chi^2 \rangle \equiv \nu^2 = \frac{M_x^2}{\lambda - \frac{\lambda^2}{\lambda}} \quad \text{and} \quad \langle h \rangle = -\frac{g_1}{2 m^2} \langle \chi^2 \rangle,
\]

where we denoted the mass of $\chi$ in the broken phase by $M_x^2 = 2 |m_\chi|^2$ and considered small $h$ fluctuations (i.e. we kept only the $g_1$ interaction term for $h$). This is in agreement with the results found in [4]. The fields $\chi$ and $h$ mix in this phase and the mixing angle $\theta$ is proportional to $g_1^2/m^2$. The mixing can be neglected within the present approximations. Like for $T < T_c$, we consider also now only the effects due to $\chi$ loops for the $h$ propagator. Due to symmetry breaking we now have the trilinear $\chi$ couplings:

\[
-\frac{\lambda}{3!} \theta^3 \chi^3,
\]

which substantially affect the analysis and the results in this phase. At the one loop level the diagram to compute is again the one in eq. (10) with $m^2$ replaced by $M_x$. So we predict a drop on the right hand side of $T_c$ while this diagram is clearly infrared divergent. Curing the divergence now is more involved due to the appearance of the trilinear $\chi$ coupling. A typical diagram in the consistent set of diagrams we consider is of the form:

\[
\begin{align*}
\raisebox{-1em}{\includegraphics[width=1cm]{diagram4}} & \cdots \raisebox{-1em}{\includegraphics[width=1cm]{diagram5}} \cdots \raisebox{-1em}{\includegraphics[width=1cm]{diagram6}}
\end{align*}
\]

We choose this subset of diagrams since the associate infinite sum can be performed exactly [3] and it has knowledge of the onset of chiral symmetry breaking via the trilinear vertices.

\[
m^2(T) = m^2 - \frac{g_1^2 T}{2} \frac{1 + \frac{\lambda^2}{3 T} + \frac{\lambda^2}{6 T^2}}{1 + \frac{\lambda^2}{2 T} + \frac{\lambda^2}{6 T^2}}
\]

\[
T = \frac{1}{8\pi M_x}.
\]

The infrared divergence has been cured for $T \geq T_c$. We also see that $m(T_c)$ from the right hand side of the transition equals exactly the one from the unbroken side of the transition. This remarkable result does not hold order by order in the loop expansion but only when an infinite sum of the diagrams is performed. The $h$ mass square is a continuous function throughout the phase transition and the associated correlation length remains finite. However the slope of the $h$ mass near the phase transition from the right hand side does not mirror the one on the left. This fact is due to the onset of spontaneous symmetry breaking communicated to $h$ via the trilinear $\chi$ interaction term. Indeed:

\[
D^+ \equiv \lim_{T \to T_c^+} \frac{1}{\Delta m^2} \frac{d m^2(T)}{dT} = \frac{3}{2} \left(\frac{16\pi}{\lambda T_c}\right)^2 \lim_{T \to T_c^+} \frac{d |m_\chi|^2}{dT}
\]

\[
= \frac{3}{2} \left(\frac{16\pi}{\lambda T_c}\right)^2 \lim_{T \to T_c^+} \frac{d |m_\chi|^2}{dT},
\]

and we deduce the relation:

\[
D^+ = -6 \frac{16\pi |m_\chi|^2}{\lambda T_c} D^-.
\]

The $h$ mass drops on the left hand side of the phase transition and rises on the right one. The slope on the
DECONFINEMENT AND CONCLUSIONS

The $SU(N)$ deconfinement phase transition is a hard problem. Importance sampling lattice simulations provide crucial information about the nature of the temperature driven phase transition for 2 and 3 colors Yang-Mills theories with and without matter fields (see \ref{fig:mass} for 3 colors). Different approaches \cite{4, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19} are used in literature to tackle the features of this phase transition. At zero temperature $SU(N)$ Yang-Mills theory is asymptotically free and the physical spectrum of the theory consists of a tower of hadronic states referred to as glueballs and pseudo-scalar glueballs. The theory develops a mass gap and the lightest glueball has a mass of the order of few times the confining scale.

At nonzero temperature the $Z_N$ center of $SU(N)$ is a relevant global symmetry \cite{20}, and it is possible to construct a number of gauge invariant operators charged under $Z_N$ among which the most notable one is the Polyakov loop:

$$\ell(x) = \frac{1}{N} \text{Tr}[L] = \frac{1}{N} \text{Tr} \left[ \mathcal{P} \exp \left[ i g \int_0^{1/T} A_0(x, \tau) d\tau \right] \right] \quad (16)$$

$\mathcal{P}$ denotes path ordering, $g$ is the $SU(N)$ coupling constant and $x$ is the coordinate for the three spatial dimensions while $\tau$ is the euclidean time. The $\ell$ field is real for $N = 2$ while otherwise complex. This object is charged with respect to the center $Z_N$ of the $SU(N)$ gauge group \cite{20} under which it transforms as $\ell \rightarrow z \ell$ with $z \in Z_N$. A relevant feature of the Polyakov loop is that its expectation value vanishes in the low temperature regime while is non zero in the high temperature phase. The Polyakov loop is a suitable order parameter for the Yang-Mills temperature driven phase transition \cite{20}.

Pisarski used this feature \cite{12} to model the Yang-Mills phase transition via a mean field theory of Polyakov loops. The model is referred to as the Polyakov Loop Model (PLM). Recently some interesting phenomenological PLM inspired models aimed to understand RHIC physics were constructed \cite{21, 22}.

Here we consider pure gluon dynamics. This allows us to have a well defined framework where the $Z_N$ symmetry is exact. The hadronic states of the Yang-Mills theory are the glueballs. At zero temperature the Yang-Mills trace anomaly has been used to constrain the potential of the lightest glueball state $H$ \cite{23}:

$$V[H] = \frac{H}{2} \ln \left[ \frac{H}{\Lambda^4} \right]. \quad (17)$$

$\Lambda$ is chosen to be the confining scale of the theory and $H$ is a mass dimension four field. This potential correctly saturates the trace anomaly when $H$ is assumed to be proportional to $\text{Tr}[G_{\mu \nu} G^{\mu \nu}]$ and $G_{\mu \nu}$ is the standard Yang-Mills field strength.

In \cite{4} a concrete model was proposed able to transfer the information about the Yang-Mills phase transition encoded in the $Z_N$ global symmetry to the hadronic states of the theory. This model is constructed using trace anomaly and the $Z_N$ symmetry. The model in \cite{4} is supported by recent theoretical investigations \cite{24}. Once
the relation between the fields \( H \) and \( h \), and \( \ell \) and \( \chi \) is made the relations between the couplings of the Lagrangian for \( H \) and \( \ell \) constructed in \( \mathbb{R}^4 \) and the renormalizable one presented here can be obtained. More specifically, our renormalizable theory is a truncated (up to fourth order in the fields) version of the full glueball theory. For example one can take the following relation between \( H \) and the glueball field \( h \):

\[
H = \langle H \rangle \left(1 + \frac{h}{\sqrt{c}(H)^{1/4}}\right).
\]

(18)

Here \( \langle H \rangle = \Lambda^4/e \) is the vacuum expectation value of the glueball field below the critical temperature and \( c \) is a positive dimensionless constant fixed by the mass of the glueball. For the \( \ell \) field we have \( \chi = \sqrt{\kappa} \ell \) with \( \kappa \) a mass dimension two constant (at high temperature proportional to \( T^2 \)).

Our present results are simply the higher loop corrections to the glueball model presented in \( \mathbb{R}^4 \) and can be immediately applied to the two color Yang-Mills phase transition. We confronted already our theoretical results close to the phase transition studied in \( \mathbb{R}^4 \) for three colors. Our analysis not only is in agreement with the numerical analysis but allows us to provide a better understanding of the physics in play. More recent lattice results should provide a still more detailed test of our predictions.

Finally, using a renormalizable theory containing a singlet heavy field uncharged under a global symmetry and the associated order parameter with their possible interactions we demonstrated that the information about the phase transition can be extracted via the singlet field.

The transfer of information from the order parameter to the heavy field is in fact complete. Moreover, if we consider the large \( N \) limit for the \( O(N) \) theory the analysis in the unbroken phase becomes exact. Thanks to the result established here and envisioned in \( \mathbb{R}^4 \) we can use directly the knowledge about \( h \), or more specifically of some of its properties defined in the text, to characterize the phase transition. We considered a \( \mathbb{Z}_2 \) symmetry group appropriate for example to discuss the deconfining phase transition for 2 color Yang-Mills theory. The generalization to a general group of symmetries is straightforward.

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