Constant magnetic field and $2d$ non-commutative inverted oscillator

Stefano Bellucci

*INFN, Laboratori Nazionali di Frascati, P.O. Box 13, I-00044 Frascati, Italy*

*e-mail: bellucci@lnf.infn.it*

**Abstract**

We consider a two-dimensional non-commutative inverted oscillator in the presence of a constant magnetic field, coupled to the system in a “symplectic” and “Poisson” way. We show that it has a discrete energy spectrum for some value of the magnetic field.

PACS number: 03.65.-w

**Introduction**

Non-commutative quantum field theories have been studied intensively during the last several years, owing to their relationship with M-theory compactifications [1], string theory in nontrivial backgrounds [2] and quantum Hall effect [3] (see e.g. [4] for a recent review). At low energies the one-particle sectors become relevant, which prompted an interest in the study of non-commutative quantum mechanics (NCQM) [5] - [19] (for some earlier studies of NCQM see [20] - [22]). Most of the attention was focused on quantum mechanics on two- and three-dimensional noncommutative spaces. Two-dimensional NCQM in the presence of a constant magnetic field was considered on a plane [8, 11], torus [9], sphere [8] and pseudosphere (Lobachevski plane, or AdS$_2$) [16, 19].

NCQM on a plane has a critical point, specified by the vanishing of the dimensionless parameter

$$\kappa = 1 - B\theta,$$

where the system becomes effectively one-dimensional [8, 11]. Away from the critical point, the rotational properties of the model become qualitatively dependent on the sign of \(\kappa\): for \(\kappa > 0\) the system admits an infinite number of states with a given value of the angular momentum, while for \(\kappa < 0\) the number of such states is finite [11]. From NCQM on a (pseudo)sphere originate, in some sense, the “phases” of planar NCQM [13]: the “monopole number” is defined, in such phases, in a different way. In the planar limit the NCQM on (pseudo)sphere results in the “non-conventional”, or the so-called “exotic” NCQM [10], where the magnetic field is introduced via “minimal”, or symplectic coupling.

Notice that the only two-dimensional NCQM with a non-zero potential term that has been solved explicitly corresponds to a noncommutative circular oscillator in the presence of a magnetic field. Although this system has been solved for both conventional and exotic coupling of the magnetic field, its rotational properties have been analyzed only for a conventional coupling of the magnetic field. The present interest in NCQM was initiated by Chaichian, Sheikh-Jabbari and Tureanu [6], who calculated the corrections to the hydrogen atom spectrum, which arise in non-commutative QED. Later on, the three-dimensional noncommutative oscillator with a conventional coupling of the magnetic field has also been considered and explicitly solved [23]. Hence, the recent studies of NCQM concentrated mainly on the systems with either an attractive potential, or without potential.

On the other hand, Ho and Kao [18], accurately considering the multi-particle non-relativistic limit of the noncommutative field theory, found that particles of opposite charges have opposite non-commutativity. Hence, in the center-of-mass frame, the “particle-antiparticle” system has no noncommutative correction, in contrast to the system of two identical particles. Therefore, in the context of a two-particle interpretation, NCQM with a repulsive potential becomes especially important.\(^1\)

We recall that, at the present time, the only example of an exactly solvable NCQM system with a nonzero potential term is the oscillator, which adds on to the distinguished role played by the harmonic

---

\(^1\)In [18] it was claimed, that “since proton and electron have opposite charges, the hydrogen atom has no noncommutative corrections, in spite of the statement of [6]”. However, the authors of the latter paper argued in their reply [24] that this prediction does not work in the case of the hydrogen atom, owing to the composite nature of the proton.
oscillator in both theoretical physics and mathematics.\(^2\) Owing to this reason, in the present work we decided to examine the effects of non-commutativity in systems with a repulsive potential in the presence of a constant magnetic field (coupled in a conventional and an exotic way), considering the simple model of the two-dimensional inverted oscillator. For completeness, we shall present also the known facts about the non-commutative harmonic oscillator.

**Basic properties**

As stated above, we will determine the effects of introducing a constant magnetic field in the following two cases: the “conventional” picture and the “exotic” one. We will also consider, as a by-product, the rotational properties of the noncommutative oscillator with exotic coupling of the constant magnetic field. Thus, we consider NCQM with the potential

\[
V = \varepsilon \frac{\omega^2 q^2}{2},
\]

where \(\varepsilon = +1\) corresponds to the harmonic oscillator and \(\varepsilon = -1\) to the inverted one.

The two-dimensional non-commutative quantum mechanical system with an arbitrary central potential in the presence of a constant magnetic field \(B\) is given by the Hamiltonian \([8]\)

\[
\mathcal{H} = \frac{p^2}{2} + V(q^2),
\]

with the operators \(p, q\) which obey the commutation relations

\[
[q_1, q_2] = i\theta, \quad [q_\alpha, p_\beta] = i\delta_{\alpha\beta}, \quad [p_1, p_2] = iB \quad \text{in the “conventional” picture}
\]

\[
[q_1, q_2] = i\theta/\kappa, \quad [q_\alpha, p_\beta] = i\delta_{\alpha\beta}/\kappa, \quad [p_1, p_2] = iB/\kappa \quad \text{in the “exotic” picture},
\]

where \(\alpha, \beta = 1, 2\) and the non-commutativity parameter \(\theta > 0\) has the dimension of length\(^2\).

The system can be also represented as follows:

\[
\mathcal{H} = \frac{(\pi + q/\theta)^2}{2} + V(q^2),
\]

where the operators \(\pi\) and \(q\) form the algebra

\[
[\pi_\alpha, q_\beta] = 0, \quad [\pi_1, \pi_2] = -i\kappa/\theta, \quad [q_1, q_2] = i\theta \quad \text{“conventional”}
\]

\[
[\pi_1, \pi_2] = -i/\theta, \quad [q_1, q_2] = i\theta/\kappa \quad \text{“exotic”}.
\]

At the “critical point”, i.e. for \(\kappa = 0\), the system becomes effectively one-dimensional \([11, 10]\)

\[
[q_1, q_2] = i\theta, \quad \mathcal{H}^{\text{plane}} = \left\{ \begin{array}{ll}
q^2/2\theta^2 + V(q^2), & \text{“conventional”} \\
V(q^2), & \text{“exotic”}.
\end{array} \right.
\]

Away from the point \(\kappa = 0\) the angular momentum of the systems is defined by the operator

\[
L = \left\{ \begin{array}{ll}
q^2/2\theta - \theta q^2/2\kappa, & \text{“conventional”} \\
\kappa q^2/2\theta - \theta q^2/2 & \text{“exotic”}.
\end{array} \right.
\]

The eigenvalues of the angular momentum operator are given by the expression

\[
l = \pm \left( (n_1 + 1/2) - \text{sgn} \kappa \left( n_2 + 1/2 \right) \right), \quad n_1, n_2 = 0, 1, ...
\]

where \((n_1, n_2)\) denote, respectively, the eigenvalues of the operators \((q^2, \pi^2)\) for the “conventional” NCQM and those of the operators \((\pi^2, q^2)\) for the “exotic” one. In (9) the upper sign corresponds to the\(^2\)

A "maximally integrable" isotropic oscillator on the complex projective space has been recently introduced in \([25]\), where the behaviour of the system in a constant magnetic field and its supersymmetric generalization on the Kähler space based upon the N=4 mechanics formulated in \([26]\) have been discussed.
“conventional” system and the lower sign to the “exotic” one. Hence, the rotational properties of NCQM qualitatively depend on the sign of $\kappa$.

Let us remind [10], that for a non-constant $B$ the Jacobi identities fail in the “conventional” model, while in the “exotic” model the Jacobi identities hold for any $B = A_{[1,2]}$, by definition. This reflects the different origin of the magnetic field $B$ appearing in the two models. In the “conventional” model, $B$ appears as the strength of a non-commutative magnetic field, while in the “exotic” model, $B$ appears as a commutative magnetic field, obtained by means of the Seiberg-Witten map from the non-commutative one.

**Conventional coupling**

For non-vanishing values of $\kappa$, it is convenient to introduce the operators

$$a^\pm = \frac{q_1 \mp i q_2}{\sqrt{2\theta}}, \quad b^\pm = \frac{\sqrt{\theta} \pi_1 \mp i \pi_2}{\sqrt{2|\kappa|}},$$

with the following non-zero commutators:

$$[a^-, a^+] = 1, \quad [b^-, b^+] = -\text{sgn} \kappa.$$  

In terms of the operators in (10), the angular momentum reads

$$L = (a^+ a^- + a^- a^+)/2 - \text{sgn} \kappa (b^+ b^- + b^- b^+)/2,$$

and the Hamiltonian (5) takes the form

$$H = \frac{1}{2\theta} \left( |\kappa|(b^+ b^- + b^- b^+) - 2i\sqrt{|\kappa|}(b^+ a^- - a^+ b^-) + \mathcal{E}(a^+ a^- + a^- a^+) \right),$$

where we have used the notation

$$\mathcal{E} = 1 + \varepsilon(\omega\theta)^2.$$  

Let us introduce the orthonormal basis in the Hilbert space consisting of the states

$$|n_1,n_2\rangle = \frac{(a^+)^{n_1} (b^+ \text{sgn} \kappa)^{n_2}}{\sqrt{n_1!n_2!}} |0,0\rangle, \quad a^- |0,n_1\rangle = b^- \text{sgn} \kappa |n_2,0\rangle = 0,$$

where $b^- \text{sgn} \kappa = b^-$ for $\kappa > 0$, and $b^- \text{sgn} \kappa = b^+$ for $\kappa < 0$.

Recall that the eigenvalues of the angular momentum are given by (9). One can see that the spectrum essentially depends on the sign of $\kappa$. Indeed, the angular momentum $l$ and the occupation number $n_1$ take the values

$$n_1 = 0,1,\ldots, \quad l = n_1,n_1+1,\ldots \quad \text{for } \kappa < 0,$$

$$n_1 = 0,1,\ldots, \quad l = -\infty,\ldots,-1,0,\ldots,n_1, \quad \text{for } \kappa > 0.$$  

At the critical point $\kappa = 0$, the system reduces to the one-dimensional oscillator with the energy spectrum

$$E_{\text{osc}}^{(0)}(n) = \frac{\mathcal{E}}{\theta}(n + 1/2), \quad n = 0,1,2,\ldots$$

Even though we are dealing with an inverted oscillator, the latter possesses a discrete spectrum, and gets a ground state when $\mathcal{E} > 0$.

For $\kappa \neq 0$, let us try to diagonalize the Hamiltonian, by performing an appropriate (pseudo)unitary transformation

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow U \cdot \begin{pmatrix} a \\ b \end{pmatrix},$$

We remind that $n_1$ defines the eigenvalue of the operator $|q|^2/2\theta$ and has the meaning of the quantized radius of the system $r_2^2 = \theta(2n_1 + 1)$. 

3
where the matrix $U$ belongs to SU(1,1) for $\kappa > 0$, and to SU(2) for $\kappa < 0$,
\[
U = \begin{cases}
  \begin{pmatrix}
    \cos \chi e^{i\phi} & \sin \chi e^{i\phi} \\
    \sinh \chi e^{-i\phi} & \cosh \chi e^{-i\phi}
  \end{pmatrix}, & \text{for } \kappa > 0 \\
  \begin{pmatrix}
    \cos \chi e^{i\phi} & \sin \chi e^{i\phi} \\
    -\sinh \chi e^{-i\phi} & \cosh \chi e^{-i\phi}
  \end{pmatrix}, & \text{for } \kappa < 0
\end{cases}
\] (19)

The Hamiltonian becomes diagonal, when $\phi, \psi, \chi$ obey the conditions
\[
\begin{align*}
\cos(\phi + \psi) & = 0 \\
(\mathcal{E} + \kappa) \sinh 2\chi - 2\sqrt{\kappa} \cosh 2\chi \sin(\phi + \psi) & = 0, \quad \text{for } \kappa > 0 \\
(\mathcal{E} + \kappa) \sin 2\chi + 2\sqrt{-\kappa} \cos 2\chi \sin(\phi + \psi) & = 0, \quad \text{for } \kappa < 0 
\end{align*}
\] (20)

The diagonalized Hamiltonian should take the form
\[
\mathcal{H}_{\text{osc}} = \frac{1}{2} \omega_{-}(b^{+}b^{-} + b^{-}b^{+}) + \frac{1}{2} \omega_{+}(a^{+}a^{-} + a^{-}a^{+}),
\] (21)

where
\[
2\theta \omega_{\pm} = \begin{cases}
  \pm(\kappa - \mathcal{E}) + (\mathcal{E} + \kappa) \cosh 2\chi - 2\sqrt{\kappa} \sinh 2\chi \sin(\phi + \psi), & \text{for } \kappa > 0 \\
  (\mathcal{E} - \kappa) \pm \left[(\mathcal{E} + \kappa) \cos 2\chi - 2\sqrt{-\kappa} \cosh 2\chi \sin(\phi + \psi)\right], & \text{for } \kappa < 0
\end{cases}
\] (22)

“Conventional” harmonic oscillator. In the case of the harmonic oscillator the above equations have a solution for any $\kappa$. After some work one gets
\[
2\theta \omega_{\pm} = \begin{cases}
  \pm(\mathcal{E} - \kappa) + \sqrt{(\mathcal{E} + \kappa)^{2} - 4\kappa}, & \text{for } \kappa > 0 \\
  (\mathcal{E} - \kappa) \pm \sqrt{(\mathcal{E} + \kappa)^{2} - 4\kappa}, & \text{for } \kappa < 0
\end{cases}
\] (23)

Hence, the energy spectrum of the “conventional” oscillator takes the form
\[
E_{\text{osc},n_{a},n_{b}} = \omega_{+}(n_{a} + 1/2) + \omega_{-}(n_{b} + 1/2) = \left[\sqrt{\mathcal{E} + \kappa} (n_{1} + 1/2) - \sqrt{(\mathcal{E} + \kappa)^{2} - 4\kappa} \right] \right] / \theta.
\] (24)

Since the transformation (19) belongs to the symmetry group of the rotational momentum $L$, the magnetic number is given by the same equation as above, i.e. (9). It can be seen that the expressions (16) arise, in the case of the harmonic oscillator, from the requirement of the positivity of the energy spectrum.

There exists an “isotropic point”, $\mathcal{E} = \kappa > 1$, where the frequencies become equal to each other
\[
\omega_{\pm}^{\text{isotr}} = \sqrt{1 + (\omega \theta)^{2}},
\]

and the system has the symmetry of the ordinary circular oscillator.

In the commutative limit, i.e. for $\theta \to 0$, the effective frequencies read
\[
\omega_{\pm}^{0} = \pm B/2 + \sqrt{\omega^{2} + B^{2}/4}.
\] (25)

In the case of the Landau problem, $\mathcal{E} = 1$ (or, equivalently, $\omega = 0$), one of the frequencies vanishes and the spectrum reads
\[
E_{n} = |B|(n + 1/2), \quad l = n_{1} - \text{sgn} \kappa n_{2}, \quad n = \begin{cases}
  n_{2} = 0, 1, \ldots \quad & \text{for } \kappa > 0 \\
  n_{1} = 0, 1, \ldots \quad & \text{for } \kappa < 0
\end{cases}
\]

Hence, although the energy spectrum of the Landau problem is independent from the non-commutativity parameter, its expression in terms of the angular momentum essentially depends on $\text{sgn} \kappa$.

“Conventional” inverted oscillator. When we deal with the “conventional” inverted oscillator, $\varepsilon = -1$, the equations (22) have a solution for any $\kappa < 0$, as well as for the values $\kappa > 0$ which satisfy the condition
\[
(\mathcal{E} + \kappa)^{2} - 4\kappa \geq 0
\] (26)

Hence, the non-commutativity of the plane, together with the presence of a magnetic field, yield different “regimes” in the “conventional” inverted oscillator:
• $\kappa = 0$: the spectrum of the inverted oscillator is given by the expression (17), where $E < 1$. Hence, when $E > 0$, the inverted oscillator transmutes into a one-dimensional harmonic one, and the system possesses a ground state. For $E = 0$ the energy of the system vanishes.

• $\kappa < 0$: the Hamiltonian of the inverted oscillator can be diagonalized by a $SU(2)$ transformation. The resulting system is again given by (21), where $\omega^+ > 0$, $\omega^- < 0$.

• $\kappa > 0$, $(\kappa + E)^2 > 4\kappa$ (or, equivalently, $\kappa \in \left[ 0, (1 - \omega\theta)^2 \right] \cup \left[ (1 + \omega\theta)^2, \infty \right]$).

In this case, we can diagonalize the Hamiltonian by an appropriate $SU(1,1)$ transformation. The energy spectrum is defined by (21), where

$$
\omega^+ > 0, \quad \omega^- < 0 \quad \text{for } E > \kappa,
\omega^+ < 0, \quad \omega^- > 0 \quad \text{for } E < \kappa.
$$

Hence, although the spectrum is discrete, the system has no ground state. This regime also appears in the absence of a magnetic field, i.e. for $B = 0$, when $E < -3$. In this case the “frequencies” are $\omega_{\pm} = 0, 2(\omega\theta)^2$.

• $\kappa > 0$, $(\kappa + E)^2 \leq 4\kappa$ (or, equivalently, $\kappa \in \left[ (1 - \omega\theta)^2, (1 + \omega\theta)^2 \right]$).

In this case we cannot diagonalize the Hamiltonian, neither by $SU(1,1)$, nor by $SU(2)$ transformations. This indicates that in the given regime the system possesses a smooth energy spectrum. Notice that this is the only regime which has commutative limit.

So, considering the simplest system with a repulsive potential, we find that the non-commutativity of the coordinates, together with the presence of a non-vanishing magnetic field, essentially change its initial properties.

### “Exotic” coupling

Here we must distinguish, once more, between two different cases.

**“Exotic” harmonic oscillator.** The spectrum of the “exotic” oscillator, away from the point $\kappa = 0$, can be obtained in a way similar to that described above. For this purpose we should make, in the above derivation, the following replacements:

$$
\pi \to -q/\theta, \quad q/\theta \to -\pi, \quad \kappa \to 1/\kappa.
$$

(27)

Thus, away from the point $\kappa = 0$, the spectrum of the “exotic” oscillator reads

$$
E_{n_1,n_2}^{osc} = \omega_-(n_1 + 1/2) + \omega_+(n_2 + 1/2),
$$

(28)

where

$$
2\theta\omega_{\pm} = \left\{ \begin{array}{l}
\pm(E - 1/\kappa) + \sqrt{(E + 1/\kappa)^2 - 4/\kappa}, \quad \text{for } \kappa > 0 \\
(E - 1/\kappa) \pm \sqrt{(E + 1/\kappa)^2 - 4/\kappa}, \quad \text{for } \kappa < 0 .
\end{array} \right.
$$

(29)

Since the transformation (19) belongs to the symmetry group of the angular momentum $L$, the eigenvalues of the latter operator are given by (9). As in the case of the “conventional” oscillator, the expressions (16) arise, also in the case of the “exotic” oscillator, from the requirement of the positivity of the energy spectrum. The “isotropy point” of the “exotic” oscillator is defined by the expression

$$
E = 1/\kappa, \quad \kappa > 0.
$$

(30)

**“Exotic” inverted oscillator.** The regimes of the “exotic” inverted oscillator can be found in the same way, as the regimes of the inverted “conventional” oscillator. One has

• $\kappa < 0$: the Hamiltonian of the inverted oscillator can be diagonalized by a $SU(2)$ transformation.
\( \kappa > 0, (1/\kappa + \mathcal{E})^2 > 4/\kappa \) (or, equivalently, \( \kappa \in [0, (1 + \omega \theta)^{-2}] \cup [(1 - \omega \theta)^{-2}, \infty] \)).

In this case we can diagonalize the Hamiltonian by an appropriate \( SU(1,1) \) transformation. The energy spectrum is discrete, however the system has no ground state.

\( \kappa > 0, (1/\kappa + \mathcal{E})^2 \leq 4/\kappa \) (or, equivalently, \( \kappa \in [(1 + \omega \theta)^{-2}, (1 - \omega \theta)^{-2}] \)).

In this case we cannot diagonalize the Hamiltonian, neither by \( SU(1,1) \), nor by \( SU(2) \) transformations. The system has a smooth energy spectrum.

**Discussion**

We considered quantum mechanics with a repulsive oscillator potential on a non-commutative plane, interacting with a constant magnetic field. This system could be interpreted as a two-particle system in a center-of-mass frame. We found that non-commutativity, combined with the presence of a constant magnetic field, generates a special regime, where the system gets a discrete spectrum. The qualitative behaviour of the system is the same for both types of coupling of the magnetic field, i.e. symplectic and Poisson ones (which we called, respectively, “conventional” and “exotic”).

It should be quite interesting to take into consideration similar non-commutative systems on two- and higher-dimensional spheres, as well as to study non-commutativity effects on supersymmetric quantum mechanics. The appropriate *commutative* candidate, namely a (super)oscillator, in a constant magnetic field, on complex projective spaces, is constructed in [25].

It can be seen from eq. (7) that, at the critical point, the “conventional” NCQM gets a ground state not only for attractive potentials, but for repulsive potentials as well, when the latter are long-range distance ones, namely

\[
2\theta^2 \frac{dV(q^2)}{dq^2} > -1.
\]

This class includes the physically important cases of the logarithmic potential \( V = -\gamma^2 \log q^2/2\theta \) and the Coulomb \( V = \gamma^2/|q| \) one, as well as a simplest repulsive potential, i.e. that describing an inverted oscillator.

**Acknowledgments**

I would like to thank Armen Nersessian for useful discussions. This work was supported in part by the European Community’s Human Potential Programme under the contract HPRN-CT-2000-00131 Quantum Spacetime, the INTAS-00-0254 grant, the NATO Collaborative Linkage Grant PST.CLG.979389 and the Iniziativa Specifica MI12 of the INFN Commissione Nazionale IV.

**References**