Security of Quantum Key Distribution with entangled quNits.

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Abstract

We consider a generalisation of Ekert’s entanglement-based quantum cryptographic protocol where qubits are replaced by quNits (i.e., $N$-dimensional systems). In order to study its robustness against optimal incoherent attacks, we derive the information gained by a potential eavesdropper during a cloning-based individual attack. In doing so, we generalize Cerf’s formalism for cloning machines and establish the form of the most general cloning machine that respects all the symmetries of the problem. We obtain an upper bound on the error rate that guarantees the confidentiality of quNit generalisations of the Ekert’s protocol for qubits.

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1 Introduction

In quantum cryptographic protocols, the presence of an eavesdropper in the communication channel can be revealed through disturbances in the transmission of the message. To realize such protocols, it is necessary to encode the signal into quantum states that belong to non-compatible bases, as in the original protocol of Bennett and Brassard\cite{1}. In 1991, Ekert suggested\cite{2} a scheme in which the security of quantum cryptography is based on entanglement. In this scheme, one encrypts the key into the non-compatible qubit bases that maximize the violation of local realism.

Recently it was shown that this violation is more pronounced for the case of entangled quNits\cite{3, 4, 5} for $N > 2$. Moreover, the qutrit generalisation of Ekert’s protocol is more robust...
and safer than its qubit counterpart[6, 7]. This naturally serves as a significant motivation for studying the generalisation of Ekert’s protocol where qubits are replaced by quNits.

There are several ways of realizing quNits experimentally. One realization of quNits, possibly the most straightforward one, exploits time-bins[8]. This approach has already been demonstrated for entangled photons up to eleven dimensions[9]. Another possibility is to utilize multiport-beamsplitters, and more specifically those that split the incoming single light beam into $N[10]$ outputs. Thus, an entanglement-based quantum cryptographic protocol based on quNits instead of qubits can be realized with the current state-of-the-art quantum optical techniques.

In this paper, we establish the generality of a class of eavesdropping attacks that are based on (state-dependent) quantum cloning machines[11, 12, 13, 14, 15, 16]. We shall show that such attacks cannot be thwarted by Alice and Bob, the authorized users of the quantum cryptographic channel, because the disturbance due to the presence of the eavesdropper (Eve) perfectly mimics the correlations of an unbiased noise, at least for what concerns correlations between the encryption and decryption bases that characterize the $N$ dimensional Ekert protocol. The security is shown to be higher for higher dimensional systems, a property that was already noted for several qutrit-based protocols in comparison to their qubit-based counterparts[6, 7, 17, 18, 19, 20].

2 The four quNit bases that maximize the violation of local realism

In the Ekert91 protocol[2], the four qubit bases chosen by Alice and Bob are the bases that maximize the violation of CHSH inequalities[21]. There exists a natural generalisation of this set of bases in the case of quNits[4]. In analogy with the Ekert91 bases that belong to a great circle on the Bloch sphere, these bases belong to a set of bases parametrized by a phase $\phi$, that we shall from now on call the $\phi$ bases. These bases are related to the computational basis \{\ket{0}, \ket{1}, \ldots, \ket{N-1}\}:

$$ |l_\phi \rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{ik(\frac{2\pi}{N}l+\phi)} |k \rangle, l = 0, \ldots, N - 1 $$

(1)

It has been shown that when local observers measure the correlations exhibited by the maximally entangled state $|\phi^+_N \rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N-1} |i \rangle \otimes |i \rangle$ in the four $\phi$ bases that we obtain when $\phi_i = \frac{2\pi}{iN} \cdot i(i = 0, 1, 2, 3)$, the degree of nonclassicality or violation of local realism that characterizes the correlations increases with the dimension $N[3, 4, 5]$. It is also higher than the degree of nonclassicality allowed by Cirelson’s theorem[22] for qubits, and higher than for a large class of other quNit bases. Indeed, this can be shown by estimating the resistance of non-locality against noise[3, 4], or by considering generalisations of the CHSH inequality to bipartite entangled quNit states[5]. From now on, we call the four quNit bases that maximize the violation of local realism the optimal bases.
3 The $N$-dimensional Entanglement based ($N$-DEB) protocol

Let us now assume that the source emits the maximally entangled quNIt state $|\phi_N^{\pm}\rangle$ and that Alice and Bob share this entangled pair and perform measurements along one of the optimal bases described in the previous section.

In order to show that these bases are pairwise perfectly correlated, it is useful to present an interesting property of the state $|\phi_N^{\pm}\rangle$. Let us consider two bases: the $\psi$ basis chosen arbitrarily (with $\langle i | \psi_j \rangle = U_{ij}$), and its conjugate basis, the $\psi^*$ basis (with $\langle i | \psi_j^* \rangle = U_{ij}^*$). When Alice and Bob share the maximally entangled state $|\phi_N^{\pm}\rangle$ and that Alice measures it in the $\psi^*$ basis and Bob in the $\psi$ basis, their results are perfectly correlated. To see this, we note that by virtue of the unitarity of the matrix $U_{ij}$, $|\phi_N^{\pm}\rangle = \frac{1}{\sqrt{N}} \sum_{k,l,m=0}^{N-1} |\psi^*_i \rangle \langle \psi^*_k | \otimes |\psi^*_m \rangle \langle \psi_l |$ whenever Alice projects the state $|\phi_N^{\pm}\rangle$ into the conjugate basis $\psi^*$, she projects Bob’s component into the $\psi$ basis and reciprocally.

Moreover, the state $|\phi_N^{\pm}\rangle$ can be rewritten as $|\phi_N^{\pm}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N-1} (|i^*_\phi \rangle \otimes |i_\phi \rangle)$ where

$$|i^*_\phi \rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-ik(\frac{2\pi}{N}i+\phi)} |k\rangle (l : 0, ..., N - 1) \quad (2)$$

Therefore, when Bob performs a measurement in the $\phi$ basis ($|k^*_\phi\rangle$) and Alice in its conjugate basis ($|k_\phi\rangle$), their results are perfectly correlated. Now, the four optimal bases are pairwise conjugate and hence perfectly correlated as well. To show the perfect correlation, we note that for the phase that appears in Eq. (2), $-k(\frac{2\pi}{N}l+\phi) = k(\frac{2\pi}{N}(N - l - j) - \phi + j\frac{2\pi}{N})$ mod $2\pi$ where $j$ is an arbitrary integer number. Now, $N - l - j$ varies from 0 to $N - 1$ (mod $N$) when $l$ varies from 0 to $N - 1$ which shows that the $\phi^*$ basis is the same as the $\phi'$ basis (with $\phi' = -\phi + j\frac{2\pi}{N}$). It is easy to check that, thanks to an appropriate choice of $j$, the bases associated to even values of $i$ ($i = 0, 2, \phi_i = \frac{2\pi}{4N} \cdot i$) are preserved under phase conjugation, while the bases associated to odd values of $i$ ($i = 1, 3$) are interchanged. The four optimal bases are thus pairwise maximally correlated.

In a natural generalisation of the Ekert91 protocol for quNits, denoted as the $N$-DEB protocol (i.e. the $N$-dimensional entanglement based protocol) in analogy with the notation adopted in Ref.[7] for the case $N = 3$, Alice and Bob share the entangled state $|\phi_N^{\pm}\rangle$ and choose their measurement basis at random among one of the four optimal quNIt bases (according to the statistical distribution that they consider to be optimal). Because of the existence of perfect correlations between the conjugate bases, a fraction of the measurement results can be used in order to establish a deterministic cryptographic key. The rest of the data, for which Alice’s and Bob’s bases are not perfectly correlated, can be used in principle in order to detect the presence of an eavesdropper for example with the help of generalized CHSH inequalities[5]. Let us now study the safety of this protocol against optimal incoherent attacks.
4 Individual attacks and optimal cloning machines

We assume from now on that the noise that characterizes the transmission (this includes dark counts in the detectors, misalignments of Alice and Bob’s bases, transmission losses and so on) is unbiased (this is a very general assumption). Under such conditions, the state shared by Alice and Bob must mimic the correlations that is observed when they share a state that has the following form:

\[ \rho_{R,A}(F_N) = (1 - F_N)\langle \phi_N^+ | \phi_N^+ \rangle_{R,A} + F_N \rho_{R,A}^{\text{noise}}, \]

where \( \rho_{R,A}^{\text{noise}} = \frac{1}{N^2} \hat{I}_{R,A} \), and the positive parameter \( F_N \leq 1 \) determines the “noise fraction” within the full state. In the following, we make the conservative hypothesis that all the errors of transmission could be due to the presence of an eavesdropper who possesses a perfect (noiseless) technology, controls the line of transmission and lets her state(s) \( |\phi_N^+\rangle_{R,A} \) interact with the one originally shared by Alice and Bob and an auxiliary system (or probe). In principle, Eve is free to use any interaction between one or several quNit pairs (originally prepared in the state \( |\phi_N^+\rangle_{R,A} \)) and an auxiliary system of her choice. Eve could keep this auxiliary system isolated and unperturbed during an arbitrary long period of time. On hearing the public discussion between Alice and Bob, she performs the measurement of her choice on her system. In principle she could let her auxiliary system interact with a series of successive signals and/or carry out her measurement on a series of auxiliary systems. We limit ourselves to the situation where she couples each signal individually to an auxiliary system (individual or incoherent attacks). It could happen that coherent attacks (so-called joint or collective attacks, for a review, see e.g. [23]) are more dangerous but presently nobody knows whether this is the case.

Let us assume that initially Alice and Bob share the state \( |\phi_N^+\rangle \) and that during an individual attack, Eve lets this state interact with her probe (which also belongs to a \( N^2 \)-dimensional Hilbert space[23] which can be seen as the “mirror space” of Alice and Bob’s \( N^2 \)-dimensional Hilbert space). Then, the most general cloning state \( |\Psi\rangle_{R,A,B,C} \) where the labels \( R,A,B,C \) are respectively associated to the reference \( R \) (Alice), the two output clones (\( A \) for Bob and \( B \) for Eve), and to the \( (N \)-dimensional) cloning machine (\( C \)) is the element of a \( N^4 \)-dimensional Hilbert space. Our task is to optimize this state in such a way that Eve maximizes her information and minimizes the disturbance on the information exchanged between Alice and Bob. Moreover, as the real disturbance along a transmission line is assumed to be unbiased, the disturbance induced by the presence of Eve must mimic an isotropic disturbance so to say, it may not depend (1) on the state that Alice and Bob measure when they measure in perfectly correlated (conjugate) bases, and also (2) on which pair of such bases is selected; finally (3) it must also mimic the correlations between the non-conjugate bases that would be observed in the presence of real (unbiased) noise.

4.1 Invariance under relabelings of the detectors and Cerf states

Clearly the complexity of the problem to solve increases as the fourth power of the dimension \( N \). It is thus necessary to simplify the treatment as much as possible by taking account of the intrinsic symmetries of the problem. According to the notation introduced in the previous
section, let us consider an arbitrary basis, the $\phi$ basis associated to Bob, and its conjugate basis, the $\phi^*$ basis, associated to Alice. A fundamental symmetry characterizes such bases: the states of the $\phi$ ($\phi^*$) basis are permuted if we vary the phase $\phi$ by any multiple of $\frac{2\pi}{N}$. It is easy to show that any such permutation is generated by $C$, the generator of the cyclic permutations that shifts each label of the states of the $\phi$ ($\phi^*$) basis by unity ($l \rightarrow l + 1 \ (\text{mod} \ N)$). When the condition (1) is fulfilled, so to say when the error rate does not depend on the label $l$, it is natural to impose this invariance at the level of the cloning state. In order to do so, it is useful to introduce the so-called Bell states. The $N^2$ generalized Bell states are defined as follows[7]:

$$|B_{m^*,n}^{\psi}\rangle_{R,A} = N^{-1/2} \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |\psi_{k}^{*}\rangle_{R} |\psi_{k+m}\rangle_{A}$$

(4)

with $m$ and $n \ (0 \leq m, n \leq N - 1)$ labeling these Bell states. Obviously they are eigenstates under the generator of cyclic permutations $l \rightarrow l + 1$ for the eigenvalue $e^{-2\pi i (n/N)}$. Moreover they form an orthonormal basis of the $N^2$ dimensional space spanned by the product states $|\psi_{k}^{*}\rangle_{R} |\psi_{l}\rangle_{A} \ (l, j = 0, ..., N - 1)$.

Note that all the Bell states are maximally entangled and that $|\phi_{k}^{+}\rangle = |B_{0,0}^{\psi}\rangle$. Let us consider the fraction of the signal that is measured by Alice in the $\psi$ basis and by Bob in its conjugate basis (the $\psi^*$ basis). As these are perfectly correlated bases, this signal is not discarded and it will be used afterwards, during the reconciliation protocol, to establish a fresh cryptographic key. At this level, when Alice and Bob reveal publicly their choices of bases, Eve will measure her ancilla in the basis that she deems to be optimal[23]. Eve is free to redefine this basis thanks to an arbitrary unitary transformation according to her convenience. Taking account of the dual nature of the “mirror” space assigned to Eve, it is natural that Eve fixes this unitary transformation in such a way that she measures her copy in the same basis as Bob (the $\psi$ basis) and the cloning state in the conjugate basis (the $\psi^*$ basis). Expressed in these bases, the most general state $|\Psi\rangle_{R,A,B,C}$ that is invariant under cyclic permutations of the labels assigned to the detectors has then (up to an arbitrary redefinition of Eve’s basis) the following form:

$$|\Psi\rangle_{R,A,B,C} = \sum_{m,m',n=1} a_{m,m',n} |B_{m^*,n}^{\psi}\rangle_{R,A} |B_{m',-n^*}^{\psi}\rangle_{B,C}$$

(5)

where by definition

$$|B_{m',-n^*}^{\psi}\rangle_{B,C} = N^{-1/2} \sum_{k=0}^{N-1} e^{-2\pi i (kn/N)} |\psi_{k}^{*}\rangle_{B} |\psi_{k+m}\rangle_{C}$$

(6)

Note that we introduced at this level two apparently different definitions of the Bell states. Both can be re-expressed according to the synthetic expression:

$$|B_{m^{(*)},n^{(*)}}^{\psi}\rangle = N^{-1/2} \sum_{k=0}^{N-1} e^{2\pi i (kn/N)} |\psi_{k}^{(*)}\rangle_{R} |\psi_{k+m}\rangle_{C}$$

(7)

In the computational basis (where $|k\rangle = |k^*\rangle$), these definitions coincide with the usual definition[13]. Indeed, our approach constitutes a covariant generalisation of Cerf’s formalism for cloning machines[13]. At this level, the complexity of the problem to solve is only in $N^3$, because we projected the state $|\Psi\rangle_{R,A,B,C}$ onto the set of states that remains invariant under a shift unity of the labels assigned to the basis states. We shall now show that it is still possible to reduce
the complexity of the problem if we consider the question of optimality. Prior to this, let us introduce the following definitions:

**Definition 1:** The pure state $|Ψ\rangle_{R,A,B,C}$ is a Cerf state iff, for a given basis (say the $ψ$ basis),

$$|Ψ\rangle_{R,A,B,C} = \sum_{m,n=1}^{N-1} a_{m,n} |B^{ψ}_{m,n}⟩_{R,A} |B^{ψ}_{m,-n}⟩_{B,C}$$  \hspace{1cm} (8)

Note that then $Tr_{B,C}|Ψ\rangle_{R,A,B,C}⟨Ψ|_{R,A,B,C}$ is diagonal in the Bell basis $|B^{ψ}_{m,n}⟩_{R,A}$ and $Tr_{R,A}|Ψ\rangle_{R,A,B,C}⟨Ψ|_{R,A,B,C}$ is diagonal in the Bell basis $|B^{ψ}_{m,n,*}⟩_{B,C}$. Beside, normalization imposes that $\sum_{m,n} |a_{m,n}|^2 = 1$.

**Definition 2:**

A (pure) Cerf state is optimal among the Cerf states (for a given quantum cryptographic protocol) iff, for the same mutual information between Alice and Bob, the mutual information between Alice and Eve corresponding to this state is superior or equal to the one corresponding to any other (pure) Cerf state or to any mixture of them.

**Theorem:**

- Let us assume that an attack is characterized by a state $|Ψ\rangle_{R,A,B,C}$ that is invariant under cyclic permutations of the labels of Alice and Bob’s basis states (in the $ψ^*$ and $ψ$ bases).

- If the optimal Cerf state exists, then it is also optimal among all possible states $|Ψ\rangle_{R,A,B,C}$.

**Proof:**

We have shown that the most general state $|Ψ\rangle_{R,A,B,C}$ that is invariant under cyclic permutations of the labels of the optimal bases must necessarily fulfill Eq. (5) which does not necessarily imply that it is a Cerf state. Nevertheless, by virtue of Eq. (7), we have that

$$|Ψ\rangle_{R,A,B,C}⟨Ψ|_{R,A,B,C} = N^{-2} \sum_{m,m',n,m',n,k,l,k,l=0}^{N-1} a_{m,m',n} a^{*}_{m,m',n} e^{i(2\pi/N)((k-l).n-(k-\bar{l}).\bar{n})},$$  \hspace{1cm} (9)

$$|ψ^*_{k+m}⟩_B|ψ_{l+m'}⟩_C ⟨ψ^*_{k-l}⟩_B⟨ψ_{l-m'}⟩_C$$

The mutual informations must be estimated on the non-discarded signal that is measured in the $ψ^*$ basis by Alice and in the $ψ$ basis by Bob, while Eve measures product states of the type $|ψ_l⟩_B|ψ^*_C⟩_C$. Therefore only the diagonal coefficients that appear in the expression of the density matrix (Eq. (9)) are relevant. It is easy to check that for such coefficients $m - m' = \bar{m} - \bar{m}'$, so that $|Ψ\rangle_{R,A,B,C}⟨Ψ|_{R,A,B,C}$ is equivalent to a reduced density matrix $ρ^{\text{red}}_{R,A,B,C}$ defined as follows:

$$ρ^{\text{red}}_{R,A,B,C} = \sum_{m,n,\bar{m},\bar{n}=0}^{N-1} a_{m,m',n} a^{*}_{m',m,n} |B^{ψ}_{m,n}⟩_{R,A} |B^{ψ}_{m,n,*}⟩_{B,C}$$

which in turn corresponds to the following mixture:

$$ρ^{\text{red}}_{R,A,B,C} = \sum_{i=0}^{N-1} P_i ρ^{i\text{red}}_{R,A,B,C},$$

$$P_i ρ^{i\text{red}}_{R,A,B,C} = \sum_{m,n=0}^{N-1} a_{m,m'=m+i,n} |B^{ψ}_{m,n}⟩_{R,A} |B^{ψ}_{m+i,-n}⟩_{B,C}$$

$$\sum_{\bar{m},\bar{n}=0}^{N-1} a^{*}_{\bar{m},\bar{m}'=\bar{m}+i,\bar{n}} ⟨B^{ψ}_{\bar{m},\bar{n}}⟩_{R,A} ⟨B^{ψ}_{\bar{m}+i,-\bar{n}}⟩_{B,C}$$  \hspace{1cm} (10)
and \( P_i = \sum_{m,n=0}^{N-1} |a_{m,m'}=m+i,n|^2 \). \( \rho_{R,A,B,C}^{i, \text{red}} \) is the projector onto the state \( |\Psi\rangle_{R,A,B,C}^i \), with

\[
|\Psi\rangle_{R,A,B,C}^i = \frac{1}{\sqrt{P_i}} \sum_{m,n=0}^{N-1} a_{m,m'=m+i,n} |B_m^\psi, n\rangle_{R,A} |B_{m+i,n}^\psi\rangle_{B,C}.
\]

(11)

Everything happens as if the above state was chosen with probability \( P_i \) without that Eve is able to control this choice or even to get informed about it. Her information is thus certainly less than the information that she would get if she was informed about this choice. Beside, once the choice of a particular \( |\Psi\rangle_{R,A,B,C}^i \) is realized, the mutual information between Eve and Alice is invariant when Eve chooses to re-label her detectors, in particular if she re-labels them according to the rule \( |\psi\rangle_B |\psi_{i,m+i}^*\rangle_C \rightarrow |\psi\rangle_B |\psi_{i+m}^*\rangle_C \) which sends \( |B_{m+i,n}^\psi\rangle_{B,C} \) onto \( |B_{m,-n}^\psi\rangle_{B,C} \). Note that this re-labeling does not influence at all the statistical distribution of Alice and Bob’s results and their mutual information. In conclusion, Eve’s information is in the best case equivalent to the information that she would get by realizing the state \( |\Psi\rangle_{R,A,B,C}^i \), and being informed about the nature of this choice. This corresponds to a mixture of Cerf states, which ends the proof: when optimal Cerf state exist(s), then, if a state \( |\Psi\rangle_{R,A,B,C} \) belongs to the class of states defined in Eq. (5) and is optimal it is necessarily equal to this (one of these) Cerf state(s).

This theorem shows that it is sufficient to optimize the cloning machines described by a Cerf state which again reduces the complexity of the problem: such a state is now described, according to Eq. (8) by \( N^2 \) parameters \( a_{m,n} \) instead of the \( N^3 \) parameters \( a_{m,m',n} \). Note that in Cerf’s approach, Eq. (8) was considered to be an ansatz[13]; the previous theorem shows that its generality can be established on the basis of more general assumptions. The property that was encapsulated in the previous theorem expresses a deep property of the Bell states. Actually, from Eve’s perspective, everything happens as if different families of Bell states were separated by a classical super-selection rule. It helps to understand why, when the optimal state is pure it is sufficient to limit oneself to the quest of the optimal Cerf state.

4.2 Invariance under the choice of the optimal basis

At this level we did not exploit all the symmetries of the problem, we only made use of the fact that inside a given pair of perfectly correlated (conjugate) bases, the labeling of the detectors is defined up to a cyclic permutation. Another symmetry of the problem that we did not exploit yet is the following: according to the condition (2), all pairs of perfectly correlated bases must also be treated on equal footing. Therefore it is natural to impose that the optimal Cerf state associated to the \( N\text{-DEB} \) protocol fulfills the following constraints

\[
|\Psi\rangle_{R,A,B,C} = \sum_{m,n=1}^{N-1} a_{m,n} |B_{m,n}^{\phi=0}\rangle_{R,A} |B_{m,n}^{\phi=0}\rangle_{B,C} = \sum_{m,n=1}^{N-1} a_{m,n} |B_{m,n}^{\phi=\frac{2\pi}{N}}\rangle_{R,A} |B_{m,n}^{\phi=\frac{2\pi}{N}}\rangle_{B,C} = \sum_{m,n=1}^{N-1} a_{m,n} |B_{m,n}^{\phi=\frac{4\pi}{N}}\rangle_{R,A} |B_{m,n}^{\phi=\frac{4\pi}{N}}\rangle_{B,C} = \sum_{m,n=1}^{N-1} a_{m,n} |B_{m,n}^{\phi=\frac{6\pi}{N}}\rangle_{R,A} |B_{m,n}^{\phi=\frac{6\pi}{N}}\rangle_{B,C}
\]

(12)
The treatment of this type of constraint is developed in appendix. The result is extremely simple: whenever \( \langle B_{k^*}^{\phi_{-j}} | B_{k^*}^{\phi_{-j}} \rangle \neq 0 \) (where \( p, q = 0, 1, 2, 3 \)), then \( a_{i,j} = a_{k,l} \). Actually these results were already used in [7] for the treatment of the qutrit case but have not been published yet. These constraints express the necessary and sufficient conditions for which the Cerf state (Eq. (8)) that characterizes the cloning machine possesses biorthogonal Schmidt decompositions in the Bell bases (Eq. (7)) associated to the four optimal bases simultaneously. By a straightforward computation we get that

\[
\langle B_{k^*}^{\phi_{-j}} | B_{k^*}^{\phi_{-j}} \rangle = \langle B_{k^*}^{0} | B_{k^*}^{0} \rangle = \frac{1}{N} \delta_{j,l} \sum_{p,q=0}^{N-1} \delta_{p-q,j(\text{mod} N)} e^{i(-p(\phi_2-\phi_1) + q((\phi_2-\phi_1) + \frac{2\pi}{N}(k-i)))}
\]

(14)

In particular, we have that when \( j = l = 0 \), \( \langle B_{k^*}^{\phi_1} | B_{k^*}^{\phi_2} \rangle = \delta_{i,k} \), which means that \( B_{k^*}^{\phi=0} = B_{k^*}^{\phi=0} \forall \phi \). When \( j = l \neq 0 \), \( \langle B_{k^*}^{\phi_{-j}} | B_{k^*}^{\phi_{-j}} \rangle \) reaches the extremal values 1 or 0 only when \( \phi_1 - \phi_2 \) is an integer multiple of \( \frac{2\pi}{N} \). This is due to the fact that for such values the basis states \( \{|\phi_i\rangle\} \) are equivalent to the states \( \{|\phi_{i,l}\rangle\}, \) up to a cyclic permutation of the labels of the basis states and we showed in a previous section that the Bell states are eigenstates under such permutations. Otherwise, for intermediate values of \( \phi \) these in-products are never equal to zero. Therefore the Cerf state \( |\Psi\rangle_{R,A,B,C} \) that is invariant for at least two distinct values of \( \phi \mod \frac{2\pi}{N} \) is characterized by the NxN matrix \( a_{m,n} \) that obeys the following equations:

\[
(a_{m,n}) = \begin{pmatrix}
    u & y_1 & y_2 & \ldots & y_{N-1} \\
x_1 & y_1 & y_2 & \ldots & y_{N-1} \\
x_2 & y_1 & y_2 & \ldots & y_{N-1} \\
    \vdots & \vdots & \vdots & \ldots & \vdots \\
x_{N-1} & y_1 & y_2 & \ldots & y_{N-1}
\end{pmatrix}
\]

(15)

The (normalized) matrix \( a_{m,n} \) still contains \( 2(N-1) \) independent parameters. This shows that it is not enough to impose the invariance of the state \( |\Psi\rangle_{R,A,B,C} \) under cyclic permutations of the basis states or under changes of bases in order to fix all the parameters of the cloning state. We shall do this by optimizing the information gained by Eve. We shall impose that, in virtue of the “mirror” property of the cloning transformation, when the detector associated to the projector onto the state \( |k_{\phi^*}\rangle_E |l_{\phi}\rangle_{E'} \) fires, the probability of the inference that Alice and Bob’s state is \( |k_{\phi^*}\rangle_A |l_{\phi}\rangle_B \) is maximal. It is easy to check that the probability that Alice and Bob’s state is \( |k_{\phi^*}\rangle_A |l_{\phi}\rangle_B \) conditioned on the observation by Eve of the state \( |k_{\phi^*}\rangle_E |l_{\phi}\rangle_{E'} \) is equal to \( \delta_{k^*-k,l-k} e^{i\pi \cdot \frac{2\pi}{N} (k'-k) n)} \).

Let us first assume that the fidelity and the disturbances are fixed, (including the coefficients \( a_{i0} \) \( i = 0, \ldots N-1 \)). These parameters will be varied later. Using the method of Lagrange’s multipliers with the constraint that \( \sum_{j=0}^{N-1} |a_{ij}|^2 \) is constant, and maximizing the function \( \sum_{j=0}^{N-1} |a_{ij}|^2 \) under the variations of \( a_{i1}, a_{i2}, \ldots a_{i,N-1} \), we get that the \( N - 1 \) dimensional vector \( (\sum_{i=0}^{N-1} a_{i1}, \sum_{j=0}^{N-1} a_{i2}, \ldots, \sum_{j=0}^{N-1} a_{i1} ) \) is parallel to the \( N - 1 \) dimensional vector \( (a_{i1}, a_{i2}, \ldots, a_{i,N-1} ) \).

The solution of these constraints that corresponds to a maximum is the following: \( a_{i1} = a_{i2} = \ldots = a_{i,N-1} \) and the phase
of these complex numbers is the same as the phase of \(a_{i,0}\). Now, as \((a_{i,1}, a_{i,2}, ..., a_{i,N-1}) = (a_{0,1}, a_{0,2}, ..., a_{0,N-1}) = (y_1, y_2, ..., y_{N-1})\) and \((a_{0,0}, a_{1,0}, ..., a_{N-1,0}) = (v, x_1, ..., x_{N-1})\) in virtue of Eq. (15), we get that \(y_1 = y_2 = ... = y_{N-1}\), and all the coefficients \(x_i\) and \(y_i\) have the same phase as \(v\). We can without loss of generality assume that this phase is zero. Moreover, we must impose that all disturbances are equal; otherwise, Eve’s presence could be detected easily by Alice and Bob. Indeed, at this level all the states of a same \(\phi\) basis are not treated on the same footing. This can be checked for instance by estimating the disturbances. There are \(N-1\) possible errors when copying the basis state \(|k\phi\rangle\) Depending on it being transformed into \(|(k+i)\phi\rangle \mod N, with i = 1, ..., N - 1\). Therefore, we define \(N-1\) disturbances \(D_1, D_2\) and \(D_{N-1}\) corresponding to these \(N-1\) errors. By a straightforward but lengthy computation, we get that the \(i\)th disturbance is equal to \(|x_i|^2 + \sum_{j=0}^{N-1} |y_j|^2\) \((j = 1, ..., N - 1)\) which, in general, is not independent on the label \(i\). If we impose that the disturbances are independent of the label \(i\), we get \(x_1 = x_2 = ... = x_{N-1}\) and the matrix \(a_{mn}\) contains only real positive coefficients.

Taking account of Eq. (15), we obtain the final form of the matrix \(a_{m,n}\):

\[
(a_{m,n}) = \begin{pmatrix}
    v & y & y & ... & y \\
    x & y & y & ... & y \\
    x & y & y & ... & y \\
    . & . & . & ... & . \\
    . & . & . & ... & . \\
    x & y & y & ... & y \\
\end{pmatrix}
\] (16)

Note that it is sufficient that the cloning state \(|\Psi\rangle_{R,A,B,C}\) is invariant for two distinct values of \(\phi\) in order that it is invariant for all values of \(\phi\), so to say that it acts identically on each state of the \(\phi\) bases. Such a cloner is thus phase-covariant, in analogy with the qubit[24] and qutrit cases[7].

We determined numerically the values of \(v, x\) and \(y\) for which Eve’s information is maximal while the fidelity of Bob’s clone is fixed. Letting vary this fidelity, we determined the error rate that corresponds to the crossing point of Bob and Eve’s mutual information relative to Alice’s data \((I_{AB} = I_{AE})\). According to Csiszár and Körner theorem [25] Alice and Bob can distill a secure cryptographic key if the mutual information between Alice and Bob \(I_{AB}\) is larger than the mutual information between Alice and Eve \(I_{AE}\), i.e., \(I_{AB} > I_{AE}\). If we restrict ourselves to one-way communication on the classical channel, this actually is also a necessary condition. Consequently, the quantum cryptographic protocol above ceases to generate secret key bits precisely at the point where Eve’s information matches Bob’s information.

The threshold fidelities below which the security of the protocol is no longer guaranteed are listed in function of the dimension \(N\) in the table 1. These values are the exactly the same values as those obtained from a higher dimensional generalization of Ref. [26]. Note that in the qubit \((N=2)\) and qutrit \((N=3)\) cases, we recover the properties (optimal fidelity, upper bound on the error rate and so on) derived in the literature following Cerf’s approach[17, 27, 7] or more general approaches[24, 28, 6]. The threshold fidelities that we obtained are lower than the corresponding values in the case of symmetric cloners, which were derived in Ref.[29]. This is due to the fact that in our approach Eve considers the full information contained in the clone (B) and in the ancilla (C). In the large \(N\) limit, it is easy to show that \(I_{AE} = 1 - F_A\) and \(I_{AB} = F_A\).
Thus, in this limit, we find that the cloner converges to the universal (isotropic) cloner while the fidelity goes to fifty percent. The tolerable error rate has also been shown recently Ref. [30] to tend to fifty percent with the dimension of quVits going to infinity (with N being a prime number) in another prepare-and-measure scheme. It seems that this asymptotic behaviour is quite general.

4.3 Correlations between non-conjugate bases

The unbiased noise that appears in Eq. (3) is characterized by a density matrix that is proportional to the identity $\hat{I}_{R,A}$. Now, $\hat{I}_{R,A} = \hat{I}_A \hat{I}_R = \sum_{i=0}^{N-1} |\psi_i^r\rangle_R \langle \psi_i^r|_R \sum_{j=0}^{N-1} |\tilde{\psi}_j\rangle_A \langle \tilde{\psi}_j|_A$ where the $\psi$ and the $\tilde{\psi}$ bases can be chosen arbitrarily. This arbitrariness suggests some invariance of the noise under local changes of basis. In particular, when Eve replaces the signal by a clone, this invariance must be respected. According to the condition (3), this clone must mimic the noise under local changes of basis. In particular, when Eve replaces the signal by a clone, this invariance must be respected. According to the condition (3), this clone must mimic the correlations between the non-fully correlated (non-conjugate) bases that would be observed in the presence of real, unbiased, noise. We shall now show that this is well the case. In order to do so, let us consider the reduced cloning state $\rho_{R,A} = Tr_{B,C} |\Psi\rangle_{R,A,B,C} \langle \Psi|_{R,A,B,C}$ obtained after averaging over Eve’s degrees of freedom; according to Eqs. (1,2,4,8,16), and thanks to the identities

$$|B_{0^*,0}\rangle = |\phi_N^+\rangle, \sum_{m=0,n=0}^{N-1} |B_{m^*,n}\rangle_{R,A} \langle B_{m^*,n}|_{R,A} = \hat{I}_{R,A}, and \sum_{n=0}^{N-1} e^{-2\pi i (k-l)n/N} = N \delta_{k,l},$$

we get:

$$\rho_{R,A} = v^2 |B_{0^*,0}\rangle_{R,A} \langle B_{0^*,0}|_{R,A} + x^2 \sum_{m=1}^{N-1} |B_{m^*,0}\rangle_{R,A} \langle B_{m^*,0}|_{R,A} + y^2 \sum_{m=1,n=0}^{N-1} |B_{m^*,n}\rangle_{R,A} \langle B_{m^*,n}|_{R,A}$$

$$= (v^2 - x^2) |\phi_N^+\rangle_{R,A} \langle \phi_N^+|_{R,A} + (x^2 - y^2) \sum_{m=0}^{N-1} |B_{m^*,0}\rangle_{R,A} \langle B_{m^*,0}|_{R,A} + y^2 \sum_{m=0,n=0}^{N-1} |B_{m^*,n}\rangle_{R,A} \langle B_{m^*,n}|_{R,A}$$

$$= (v^2 - x^2) |\phi_N^+\rangle_{R,A} \langle \phi_N^+|_{R,A} + N(x^2 - y^2) \frac{1}{N} \sum_{n=0}^{N-1} |n\rangle_R \langle n|_A \langle n|_R \langle n|_A + N^2 y^2 \rho_{R,A}^{\text{noise}}.$$  

In comparison to Eq. (3) a new factor weighted by $N(x^2 - y^2)$ appears in the previous expression. It is a mixture of projectors on products of the states of the computational basis $|n\rangle_R |n\rangle_A$.  

<table>
<thead>
<tr>
<th>$N$</th>
<th>$F_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.853553</td>
</tr>
<tr>
<td>3</td>
<td>0.775276</td>
</tr>
<tr>
<td>4</td>
<td>0.734178</td>
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<tr>
<td>5</td>
<td>0.708043</td>
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<tr>
<td>6</td>
<td>0.689788</td>
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<tr>
<td>7</td>
<td>0.676230</td>
</tr>
<tr>
<td>8</td>
<td>0.665708</td>
</tr>
<tr>
<td>9</td>
<td>0.657267</td>
</tr>
<tr>
<td>10</td>
<td>0.650319</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 1: Fidelity for dimension $2 \leq N \leq 10$
For what concerns measurements performed by Alice and Bob in the optimal bases (that they are conjugate or not), any such product has the same statistical properties as the unbiased noise $\rho_{\text{noise}}^{R,A} = \frac{1}{N^2} I_{R,A}$. The deep reason for this property is that the in-product between any state of the computational basis and any $\phi$ state is in modulus squared equal to $\frac{1}{N}$, in other words, both bases are mutually unbiased. Henceforth, the modulus squared of $\langle n|_R \langle n|_A |k^*_\phi_1\rangle |l\phi_2\rangle$ is equal to $\frac{1}{N^2}$, whatever the values of the labels $k$ and $l$ could be.

5 Conclusions

The Ekert91 protocol[2] and its quNit extension, the N-DEB protocol which is analyzed in the present paper, involve encryption bases for which the violation of local realism is maximal. If Alice and Bob measure their member of a maximally-entangled quNit pair in two “conjugate” bases, this gives rise to perfect correlations. After measurement is performed on each member of a sequence of maximally-entangled quNit pairs, Alice and Bob can reveal on a public channel what were their respective choices of basis and identify which quNit was correctly distributed, from which they will make the key. They can use the rest of the data in order to check that it does not admit a local realistic simulation. For instance they can check that their correlations violate some generalized Bell or CHSH inequalities[5]. More generally, they can check that the correlations between their results (that they are perfectly correlated or not are the same as the results that they expect in the presence of unbiased noise. Note that the optimal bases do not allow them to differentiate a fully unbiased noise (described by a fully incoherent reduced density matrix proportional to the unity matrix) from a “colored” noise that would contain projectors (with arbitrary weights) on product states of the computational basis. As shown at the end of the last section, this is due to the fact that all the optimal bases are mutually unbiased relatively to the computational basis. This explains why the phase-covariant attacks are more dangerous than the universal (state-independent) attacks. For instance, the maximal admissible error rate (when attacks based on state-dependent cloners are considered) was shown to be equal to $E_A = 1 - F_A = 1 - \left(\frac{1}{2} + \frac{1}{\sqrt{8}}\right) \simeq 14.64\%$ for the 2-DEB protocol (or Ekert91 qubit protocol) [24, 28, 27] and to 22.47 % for the 3-DEB protocol[6, 7]. The corresponding rates, if we restrict ourselves to state-independent cloners[13] were shown in Ref.[17] to be respectively equal to 15.64% and 22.67%. These results were derived for a slightly asymmetric state-independent cloner[13, 17] that clones all the states with the same fidelity. Universal attacks correspond to protocols in which Alice and Bob have the physical possibility to measure (distinguish) experimentally any coefficient of the reduced density matrix which is not the case here. This is the price to pay, but, at the same time, as the resistance of the violation of local realism against noise is maximal when the maximally-entangled quNit pair is measured in the optimal quNit bases discussed here[3, 4], the N-DEB protocol is optimal from the point of view of the survival of non-classical correlations in a noisy environment.

Actually, it has been shown[5] that the violation of a Bell inequality extended to quNits is possible, as long as the “visibility” of the two-quNits interference exceeds a threshold value $V_{\text{thr}}$ given by the equation $N^2 V_{\text{thr}}(N) = \sum_{k=0}^{[N/2]-1} (1 - \frac{2k}{N-1}) (\frac{1}{\sin^2(\pi(4k+1)/4N)} - \frac{1}{\sin^2(\pi(4k+3)/4N)})$. The visibility mentioned above is directly related to the threshold fraction of unbiased noise, $(1 - V_{\text{thr}}(N))$, which has to be admixed to the maximally entangled state in order to erase the non-classical character of the correlations, and therefore is a measure of robustness of non-classicality (see
also \([3, 4]\)). This means that the non-existence of a local realistic model of the correlations is guaranteed if the fidelity \(F_{thr}\) that characterizes the communication channel between Alice and Bob, (detectors included, so \(1-F_{thr}\) is the effective error rate in the transmission) is larger than \(\frac{N-1}{N} \times V_{thr}(N) + \frac{1}{N}\). For instance, for \(N = 2, 3, 4, 5\) and \(10\), \(1-F_{thr}\) is equal to 14.64 \%, 20.26 \%, 23.21 \%, 25.03 \% and 28.77 \% respectively. On the other hand, the \(N\)-DEB protocol is secure against a cloning-based individual attack, if \(F = 1 - E_A > F_A = F_{thr}\) (see table). If we compare the previous values of \(1-F_{thr}\) with the corresponding values of \(1-F_A\) (with \(F_A\) given in the table), it is easy to check that when a violation of local realism occurs, the security of the \(N\)-DEB protocol against individual attacks is automatically guaranteed. Therefore, the violation of Bell inequalities is a sufficient condition for security, as it implies that Bob’s fidelity is higher than the security threshold. For qubits the sufficient condition \((F_A > 0.8436)\) is also necessary (\([23]\)).

In addition, the violation of Bell inequalities guarantees that the \(N\)-DEB protocol is secure against so-called Trojan horse attacks during which the eavesdropper would control the whole transmission line and replace the signal by a fake, predetermined local-variable dependent, signal that mimics the quantum correlations (see Ref.[7] and references therein). All the protocols in which no entanglement is present (such as BB84[1], the 6-state qubit protocol[28, 16], or the 12-state qutrit protocol[18]) admit a local realistic model, so that they are not secure against Trojan horse attacks, although, according to the results of Ref.[17] they are slightly more resistant against noise than the \(N\)-DEB protocol.

As we have already noted, our results confirm the results that can be found in the literature relatively to the security of the 2-DEB and 3-DEB protocols, but at the same time our results are confirmed by the corresponding results in the case that they were derived under constraints more general than the ones that we postulated. Note that in order to find an expression for the cloning state that is valid for arbitrary dimension, it is impossible presently to avoid some extra-assumptions, for instance that the optimal cloning state is pure, \(N^4\) dimensional, symmetric under cyclic permutations of the labels of the optimal bases and so on. These assumptions are very reasonable anyhow. If we would try to avoid any extra-constraint, the complexity of the problem would increase with the dimension \(N\) and we could not find a solution for all values of \(N\). Note also that the security of quantum cryptographic protocols against incoherent attacks was never clearly established, simply because the problem is too complicate to tackle.

In summary, we have established the generality of Cerf’s approach of quantum state-dependent cloning machines under fairly general assumptions. We have shown that the acceptable error rate of the \(N\)-DEB protocol turns out to increase with the dimension \(N\). Our analysis confirms a seemingly general property that the robustness against noise of quNit schemes increases with the dimensionality \(N\).

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References


6 Appendix: Invariance of the cloning state.

Let us consider a reference basis, the \( \psi \) basis, its conjugate basis the \( \psi^* \) basis, another basis, the \( \tilde{\psi} \) basis and its conjugate basis the \( \tilde{\psi}^* \) basis (with \( \langle i | \psi_j \rangle = U_{ij} \) and \( \langle i | \tilde{\psi}_j \rangle = \tilde{U}_{ij} \)). Let us assume that during the realisation of a quantum cryptographic protocol, Alice and Bob share the maximally entangled state \( |\phi^+_N\rangle (|B_{0,0}\rangle) \) and that Alice measures it either in the \( \psi^* \) basis or in the \( \tilde{\psi}^* \) basis (with \( \langle i | \psi^*_j \rangle = U^*_{ij} \) and \( \langle i | \tilde{\psi}^*_j \rangle = \tilde{U}^*_{ij} \)). According to the proof given in the section 2, she projects Bob’s component onto either the \( \psi \) or the \( \tilde{\psi} \) basis. If we require that the cloning machine is invariant in both bases, the joint (Cerf) state of the reference states of the (N-dimensional) cloning machine \( C \) (Eq. (8)) must fulfill the following condition:

\[
\sum_{m,n=0}^{N-1} a_{m,n} |B_{m^*,n}\rangle_R |B_{m,-n^*}\rangle_B = \sum_{m,n=0}^{N-1} a_{m,n} |\tilde{B}_{m^*,n}\rangle_R |\tilde{B}_{m,-n^*}\rangle_B \tag{18}
\]

As the \( N^2 \) states form an orthonormal basis, we can project the righthand side of the previous equality onto them, which gives:

\[
\sum_{m,n,m',n',m'',n''=0}^{N-1} a_{m,n} |B_{m^*,n}\rangle_R |B_{m,-n^*}\rangle_B = \sum_{m,n,m',n',m'',n''=0}^{N-1} a_{m,n} |\tilde{B}_{m^*,n}\rangle_R |\tilde{B}_{m,-n^*}\rangle_B
\]

Denoting \( V_{i,j,k,l} \) the in-product \( \langle B_{i^*,j} | \tilde{B}_{k^*,l} \rangle \), we get:

\[
\sum_{m,n,k,l=0}^{N-1} a_{m,n} \delta_{(m,n),(k,l)} |B_{m^*,n}\rangle_R |B_{k,-l^*}\rangle_B = \sum_{m,n,m',n',m'',n''=0}^{N-1} a_{m,n} \delta_{(m,n),(m',n')} |B_{m^*,n}\rangle_R |B_{m,-n^*}\rangle_B
\]
∑_{m,n,m',n'=0}^{N-1} a_{m,n} |B_{m',n'}⟩_{R,A} V_{m',n',m,n} |B_{m'',-n''}⟩_{B,C} V_{m'',n'',m,n}' = ∑_{m,n,i,j,k,l=0}^{N-1} a_{i,j} δ(i,j),(i',j') V_{m,n,i,j} V_{k,l,i',j'}^{*} |B_{m',n'}⟩_{R,A} |B_{k,-l'}⟩_{B,C}. Thanks to the orthonormality of the Bell bases, this constraint can be expressed as a matrix relation of the form \( V\mathcal{A} = \mathcal{A} V \) where \( V \) and \( \mathcal{A} \) are \( N^2 \times N^2 \) matrices defined as follows: \( V_{i,j,k,l} = \langle B_{i',j} | \tilde{B}_{k,l} \rangle \) and \( A_{i,j,k,l} = a_{i,j} δ(i,j),(k,l) \). Such a system of linear equations is extremely simple to solve: if \( V_{i,j,k,l} \neq 0 \), then \( a_{i,j} = a_{k,l} \). The procedure to follow in order to build a cloning machine that is invariant in the \( \psi \) basis and the \( \tilde{\psi} \) basis is thus straightforward: compute the \( N^4 \) in-products \( V_{i,j,k,l} = \langle B_{i',j} | \tilde{B}_{k,l} \rangle \) (\( i, j, k, l = 0, ..., N - 1 \)); if \( V_{i,j,k,l} \neq 0 \), then \( a_{i,j} = a_{k,l} \). The solutions \( a_{m,n} \) of this set of equations define the most general Cerf state (Eq. (8)) that is invariant in the two bases.