Can the Two-Higgs-Doublet Model Survive the Constraint from the Muon Anomalous Magnetic Moment as Suggested?

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Abstract

Requiring the two-Higgs-doublet model II to accommodate the $3\sigma$ deviation in the muon anomalous magnetic moment imposes specific constraints on the Higgs spectrum. We analyze the combination of all the relevant, available, constraints on the model parameter space. The use of constraints from $b \rightarrow s\gamma$, the precision electroweak measurements of $R_b$, and the $\rho$ parameter, together with exclusions from direct searches at LEP, give extremely severe restrictions on the model parameters. That is “almost enough” to kill the model altogether. The exclusion would be even stronger if the direct searches can be optimized to complement the other constraints, as will be discussed in details in this work.

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I. INTRODUCTION

Physicists have been delighted by the extraordinary success of the standard model (SM), while many got frustrated by the lack of experimental clues for the construction of the theory beyond. The most recent measurement on the muon anomalous magnetic moment $a_{\mu}$ [1] revealed a plausible deviation as large as $3\sigma$ from the SM prediction, as suggested by the papers in Ref.[2]. While such a scenario is only a favorable, rather than unquestionably established, conclusion from the analyses, it has been taken by some physicists as a strong suggestion for physics beyond the SM. ¹ Since many extensions of the SM are capable of giving rise to such a deviation, theorists, typically, would like to check the constraints imposed on the parameter space of a specific model. Such constraints are particularly interesting, because they are likely to give information not only on excluded regions but also on the predictions for where else should other evidences on the model be expected. For example, one expects to find a lower bound on the mass of a new particle playing the role on generating the extra contribution to $a_{\mu}$. The information has especially strong implications on models that have a small number of parameters which are already stringently constrained by various precision electroweak data. We present here such a case study, illustrating how far such a $3\sigma$ deviation can take us.

One of the simplest extensions of the SM is the two-Higgs-doublet model (2HDM) [3], which adds one Higgs doublet in addition to the one required in the SM. A generic 2HDM allows flavor-changing neutral currents (FCNC), which can be avoided by restricting the couplings of the doublets, say, by imposing an ad hoc discrete symmetry[4]. The most popular version, known as model II, has one Higgs doublet coupled to the down-type quarks (and charged leptons) and the second doublet to the up-type quarks. The physical content of the Higgs sector (assuming no CP violation) includes a pair of CP-even neutral Higgs bosons $H$ and $h$, a CP-odd neutral boson $A$, and a pair of charged-Higgs bosons $H^\pm$. The model fits in well with the criteria mentioned above for the $a_{\mu}$ result to have a very strong impact. We focus on the 2HDM II in this paper, answering the question of to what extent the model can survive a requirement of generating the $3\sigma$ deviation in $a_{\mu}$, as suggested.

The 2HDM II has been extensively studied in literature and tested experimentally. One of the most stringent tests is the radiative decay of $B$ mesons, specifically, the inclusive decay rate of $b \to s \gamma$, which has the least hadronic uncertainties. In the 2HDM, the rate of $b \to s \gamma$ can be enhanced substantially for large regions in the parameter space of the mass $m_{H^\pm}$ of the charged-Higgs boson and $\tan\beta(= v_2/v_1$, where $v_1$ and $v_2$ are the vacuum expectation values of the down- and up-sector Higgs doublets, respectively). An earlier analysis has already put a constraint on the charged-Higgs boson mass at $m_{H^\pm} > 380$ GeV[5] (see also Ref.[6, 7]). Updating the constraint while asking for the model to give rise to the $a_{\mu}$ deviation

¹ This $3\sigma$ deviation was derived using the $e^+e^-$ data. If the $\tau$ decay data are used, the deviation would be reduced to about $0.9\sigma$ [2], which, however, has model dependence and thus less reliable.
as suggested already imposes a strong and specific mass hierarchy between the pseudoscalar and the charged scalar. Electroweak precision data also have strong implications on the model.

The 2HDM can explain the muon anomalous magnetic moment deviation with a light pseudoscalar boson $A$ contributing via a 2-loop Barr-Zee-type diagram \cite{8, 9}. Our interest here is in updating a previous analysis\cite{9} and extending it to a comprehensive treatment of all the relevant constraints. The OPAL Collaboration \cite{10} has recently published an update on their search for the Higgs bosons within the 2HDM framework. Since their result is more stringent than before, by combining the updated constraints from $a_\mu$ and other precision measurements, such as the $\rho$ parameter, $R_b$, and the $b \to s \gamma$ rate, together with this new OPAL result and a study on Yukawa processes from DELPHI \cite{11}, we are able to limit the 2HDM to a tiny window of parameter space with an, perhaps, uncomfortably large value of $\tan\beta$. All in all, we will piece together a story of the very stringently constrained 2HDM, almost to the extent of killing it altogether. We note that there have been previous analyses on the 2HDM using the electroweak precision data \cite{12–14}, to which we are partially in debt. Our study here can be considered an update, with a presentation along a different line.

We are here talking about the intricate interplay of a few stringent constraints on the overall parameter space of the model. It is not a simple matter to illustrate results on the space of a large number (six, as discussed in the next section) of parameters. In our opinion, the best way to do it may depend on how each of the constraints really works. Our presentation of the constraints follows what we consider the most efficient way to appreciate the overall results. It may be not very conventional, but is considered particularly illustrative. Wherever explicit plots are shown, we are typically plotting two-parameter fits of one or more constraints, based on the usual $\chi^2$-analysis. We will show $\chi^2 < 4$ regions, corresponding to $2\sigma$ deviation limits, which we consider as "solution" regions — where the 2HDM survives the particular constraints. We will also show regions where the model fits better than the SM, wherever appropriate.

The organization is as follows. In the next section we briefly describe the 2HDM (II) and the relevant parameters used in our analysis. In Sec. III, we look at the $b \to s \gamma$ rate and the $B^0 - \bar{B}^0$ mixing, which require a heavy charged Higgs boson. In Sec. IV, we discuss the Higgs-sector contributions to $a_\mu$ and $R_b$, and show the strongly complementary character of the two data and how that plays off in the 2HDM. We give our fits to the data, with the exclusions from DELPHI and OPAL further imposed. In Sec. V, we add the consideration of the $\rho$ parameter. We conclude in Sec. VI.

\section*{II. PARAMETER SPACE OF THE TWO HIGGS DOUBLET MODEL II}

We take the parameter space of the model as given by a set of six Higgs-sector parameters

$$m_h, \ m_H, \ m_A, \ m_{H^\pm}, \ \tan\beta, \ and \ \alpha.$$
Here, the first four are masses of the physical Higgs states. The last one, $\alpha$, is the real scalar Higgs mixing angle as defined in Ref.[3]. We would like to emphasize that we are taking this set of six parameters as mutually independent experimental parameters. From the theoretical point of view, one has parameters in the scalar potential from which the above can be derived. However, without supersymmetry or anything else to avoid the hierarchy problem, the masses for the Higgs states suffer from quadratic divergences. The tree-level relations among parameters are modified substantially by loop corrections which depend on the renormalization approach and the cut-off imposed. To stay away from such uncertainties, we do not discuss the scalar potential here, except noting that there are enough degrees of freedom in the model in general, and especially with the loop corrections taken into account, to allow us to take the above six parameters as mutually independent. Here we only consider the 2HDM-II without CP-violation, and are not interested in couplings among the Higgs states. It is then obvious that we do not have to consider more than the six parameters. It should be noted that any substantial CP violation in the Higgs sector that may largely invalidate our analysis here is ruled out by the electron electric dipole moment constraint.

The four physical masses are direct experimentally measurable quantities. The other two parameters, $\tan\beta$ and the angle $\alpha$, come into the game as effective couplings. The Yukawa couplings of $h$, $H$, and $A$ to up- and down-type quarks are given by, with a common factor of $-igm_f/2M_W$,

\[
\begin{align*}
  h^t & : \cos\alpha/\sin\beta & h^b & : -\sin\alpha/\cos\beta \\
  H & : \sin\alpha/\sin\beta & H & : \cos\alpha/\cos\beta \\
  A & : -i \cot\beta \gamma_5 & A & : -i \tan\beta \gamma_5
\end{align*}
\]

while the charged Higgs $H^-$ couples to $t$ and $\bar{b}$ via

\[
\bar{b}tH^- : \frac{ig}{2\sqrt{2}M_W} [m_t \cot\beta (1 + \gamma_5) + m_b \tan\beta (1 - \gamma_5)]
\]

From the perspective of our study here, the couplings given above may be considered as defining implicitly the two parameters $\tan\beta$ and $\alpha$. Other relevant couplings in our study are those to gauge bosons, as given by,

\[
\begin{align*}
  hZZ & : ig \frac{M_Z \sin(\beta - \alpha)}{\cos\theta_W} g^{\mu\nu} \\
  HZZ & : ig \frac{M_Z \cos(\beta - \alpha)}{\cos\theta_W} g^{\mu\nu} \\
  hAZ & : g \frac{\cos(\beta - \alpha)}{2 \cos\theta_W} (p - p')^\mu \\
  HAZ & : -g \frac{\sin(\beta - \alpha)}{2 \cos\theta_W} (p - p')^\mu \\
  H^+H^-Z & : -ig \frac{\cos2\theta_W}{2 \cos\theta_W} (p - p')^\mu,
\end{align*}
\]

where $p(h, H, H^+)$ and $p'(A, H^-)$ are the 4-momenta going into the vertex.
III. REQUIREMENT OF A HEAVY CHARGED HIGGS BOSON

It has been well appreciated that $B$ physics bars the 2HDM from admitting a relatively light charged-Higgs state. We first review the constraints here. The first important constraint comes from the inclusive $B \to X_s\gamma$ result, and the second one comes from the $B^0 - \bar{B}^0$ mixing. The essential point here is that without a direct source of FCNC, the charged Higgs mediates the only significant contributions to flavor-changing processes, in addition to the $W^\pm$-mediated SM process. The experimental data then allows us to bound the charged Higgs mass independent of the other Higgs states.

The detail description of the effective Hamiltonian approach can be found in Refs. [15, 16]. Here we present the highlights that are relevant to our discussions. The effective Hamiltonian for $B \to X_s\gamma$ at a factorization scale of order $O(m_b)$ is given by

$$H_{\text{eff}} = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[ \sum_{i=1}^{6} C_i(\mu) Q_i(\mu) + C_{7\gamma}(\mu) Q_{7\gamma}(\mu) + C_{SG}(\mu) Q_{SG}(\mu) \right]. \quad (1)$$

The operators $Q_i$ can be found in Ref.[15], of which the $Q_1$ and $Q_2$ are the current-current operators and $Q_3 - Q_6$ are QCD penguin operators. $Q_{7\gamma}$ and $Q_{SG}$ are, respectively, the magnetic penguin operators specific for $b \to s \gamma$ and $b \to s g$. Here we also neglect the mass of the external strange quark compared to the external bottom-quark mass. There have been more recent analyses [7, 17, 18] on $b \to s \gamma$ involving the NLO and other corrections, but the LO treatment here is sufficient for our purpose to put a lower bound on the charged Higgs mass $m_{H^\pm}$, which is then used in the central part of our analysis.

The decay rate of $B \to X_s\gamma$ normalized to the experimental semileptonic decay rate is given by

$$\frac{\Gamma(B \to X_s\gamma)}{\Gamma(B \to X_c e\nu_e)} = \frac{|V_{ts}^* V_{tb}|^2}{|V_{cb}|^2} \frac{6 \alpha_{\text{em}}}{\pi f(m_c/m_b)} |C_{7\gamma}(m_b)|^2,$$

where $f(z) = 1 - 8z^2 + 8z^6 - z^8 - 24z^4 \ln z$. The Wilson coefficient $C_{7\gamma}(m_b)$ is given by

$$C_{7\gamma}(\mu) = \eta^{\frac{16}{3}} C_{7\gamma}(M_W) + \frac{8}{3} \left( \eta^{\frac{14}{3}} - \eta^{\frac{16}{3}} \right) C_{SG}(M_W) + C_2(M_W) \sum_{i=1}^{8} h_i \eta^{a_i}, \quad (3)$$

where $\eta = \alpha_s(M_W)/\alpha_s(\mu)$. The $a_i$’s and $h_i$’s can be found in Ref. [15]. The coefficients $C_i(M_W)$ at the leading order in 2HDM II are given by

$$C_j(M_W) = 0 \quad (j = 1, 3, 4, 5, 6), \quad (4)$$

$$C_2(M_W) = 1, \quad (5)$$

$$C_{7\gamma}(M_W) = -\frac{A(x_t)}{2} - \frac{A(y_t)}{6} \cot^2 \beta - B(y_t), \quad (6)$$

$$C_{SG}(M_W) = -\frac{D(x_t)}{2} - \frac{D(y_t)}{6} \cot^2 \beta - E(y_t), \quad (7)$$

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where \( x_t = m_t^2/M_w^2 \), and \( y_t = m_t^2/m_{h^+}^2 \). The Inami-Lim functions\(^{[19]}\) are given by

\[
A(x) = x \left[ \frac{8x^2 + 5x - 7}{12(x - 1)^3} - \frac{(3x^2 - 2x) \ln x}{2(x - 1)^4} \right],
\]

\[
B(y) = y \left[ \frac{5y - 3}{12(y - 1)^2} - \frac{(3y - 2) \ln y}{6(y - 1)^3} \right],
\]

\[
D(x) = x \left[ \frac{x^2 - 5x - 2}{4(x - 1)^3} + \frac{3 \ln x}{2(x - 1)^4} \right],
\]

\[
E(y) = y \left[ \frac{y - 3}{4(y - 1)^2} + \frac{\ln y}{2(y - 1)^3} \right].
\]

The most recent experimental data on \( b \to s \, \gamma \) rate has been reported\(^{[20]}\), giving

\[
B(b \to s \, \gamma)|_{\text{exp}} = 3.88 \pm 0.36(\text{stat}) \pm 0.37(\text{sys})^{+0.43}_{-0.28}(\text{theory}).
\]

The most updated SM prediction is\(^{[21]}\)

\[
B(b \to s \, \gamma)|_{\text{SM}} = (3.64 \pm 0.31) \times 10^{-4},
\]

which agrees very well the data. Both the experimental data and the SM prediction have been extrapolated to the total branching ratio. Therefore, there is only a little room for new physics contributions. The constraint on new physics contribution is, explicitly,

\[
\Delta B(b \to s \, \gamma) \equiv B(b \to s \, \gamma)|_{\text{exp}} - B(b \to s \, \gamma)|_{\text{SM}} = (0.24^{+0.67}_{-0.59}) \times 10^{-4},
\]

where we have added the various errors of the experimental data in quadrature. (Note that the theory error quoted in the experimental data is larger than the one quoted by the SM prediction. We take the more conservative value.)

The quantity that parameterizes the \( B^0 - \bar{B}^0 \) mixing is

\[
x_d \equiv \frac{\Delta m_B}{\Gamma_B} = \frac{G_F^2}{6\pi^2} |V_{td}|^2 |V_{tb}|^2 f_B^2 m_B \eta_B \tau_B M_w^2 (I_{ww} + I_{wh} + I_{hh}),
\]

where\(^{[22]}\)

\[
I_{ww} = \frac{x}{4} \left[ 1 + \frac{3 - 9x}{(x - 1)^2} + \frac{6x^2 \log x}{(x - 1)^3} \right],
\]

\[
I_{wh} = xy \cot^2 \beta \left[ \frac{(4z - 1) \log y}{2(1 - y)^2(1 - z)} - \frac{3 \log x}{2(1 - x)^2(1 - z)} + \frac{x - 4}{2(1 - x)(1 - y)} \right],
\]

\[
I_{hh} = \frac{xy \cot^4 \beta}{4} \left[ \frac{1 + y}{(1 - y)^2} + \frac{2y \log y}{(1 - y)^3} \right],
\]

with \( x = m_t^2/M_w^2 \), \( y = m_t^2/m_{h^+}^2 \), \( z = M_w^2/m_{h^+}^2 \), and the running top mass \( m_t = m_t(m_t) = 166 \pm 5 \text{ GeV} \). The experimental value is\(^{[23]}\)

\[
x_d = 0.755 \pm 0.015.
\]
We use the following input parameters [23]: 
\[ |V_{tb}V_{td}^*| = 0.0079 \pm 0.0015, \quad f_B^2 B_B = (198 \pm 30 \text{ GeV})^2(1.30 \pm 0.12), \quad m_B = 5279.3 \pm 0.7 \text{ MeV}, \quad \eta_B = 0.55, \quad \text{and} \quad \tau_B = 1.542 \pm 0.016 \text{ ps}. \]
Note that the value of \( |V_{tb}V_{td}^*| \) is in fact determined by the measurement of \( x_d \). Now we can use the data to constrain the new contribution from the charged-Higgs boson.

The two constraints discussed above are quite stringent, giving a lower bound on \( m_{H^\pm} \) close to 500 GeV at 95% C.L. for intermediate and large values of \( \tan \beta \). The result is illustrated in Fig. 1. The heavy charged Higgs mass means that the state pretty much decouples, playing a little role in the contributions of the 2HDM to quantities like \( a_\mu \) and \( R_b \), which we turn to in the next section.

IV. \( a_\mu \) VS \( R_b \)

The most recent data on the \( a_\mu \) indicates [2] (see also the footnote \# 1)
\[ \Delta a_\mu \equiv a_\mu^{\exp} - a_\mu^{\text{SM}} = (33.9 \pm 11.2) \times 10^{-10}. \] (15)
The result shows a 3\( \sigma \) deviation to be explained by new physics. Adopting the view that the \( a_\mu \) problem is real and demands new physics contributions, we will see that it has a strong and definite implication on the Higgs spectrum of the 2HDM. In fact, we had performed an analysis [9] along the line for the earlier data. The major point is that a light pseudoscalar, together with a large \( \tan \beta \) value, is required to explain the positive \( \Delta a_\mu \) contribution via a two-loop Barr-Zee diagram [24]. A real scalar contributes in the negative direction. To avoid a cancellation, the real scalar mass has to be heavy. We will show here that such a mass splitting is strongly disfavored by the allowed contribution to \( R_b \), for which the experimental data agrees well with the SM prediction. The two constraints are hence strongly complementary.

It has been emphasized in Ref.[8, 9] that for the Higgs boson mass larger than about 3 GeV, the dominant Higgs contributions to \( a_\mu \) actually come from the two-loop Barr-Zee diagram with a heavy fermion \( (f) \) running in the upper loop. A \( m_f^2/m_\mu^2 \) factor could easily overcome the \( \alpha/4\pi \) loop factor. In our calculation here, we include all one-loop contributions and all two-loop Barr-Zee-type contributions with an internal photon and a third-family fermion running in the loop. The latter diagrams with the bottom and tau loops are strongly enhanced by \( \tan \beta \). If the internal photon was replaced by a \( W^\pm \) or a \( Z^0 \), the contributions will be much suppressed (see [25], for examples). The \( W^\pm \) case is in particular strongly suppressed, partly as a result of the fact that the Higgs boson has to be \( H^\pm \), the mass of which we have shown above to be heavier than 500 GeV. The only other important diagrams of the Barr-Zee type are the SM diagrams, such as the one with the \( W^\pm \) replacing the heavy fermion. We neglect the small “extra” contributions from such diagrams[26] because of the small difference between the Higgs boson mass used here and that used in the Ref. [27].
Explicitly, we first write the fermion couplings of a neutral Higgs mass eigenstate \( \phi^0 \) as

\[
\mathcal{L}^{\phi^0 f} = -\lambda_f \frac{m_f}{v} \bar{f} \phi^0 f + i\gamma_5 A_f \frac{m_f}{v} \bar{f} \phi^0 f ,
\]

where \( \lambda_f \frac{m_f}{v} \) and \( A_f \frac{m_f}{v} \) are the effective scalar and pseudoscalar couplings explicitly given in Sec. II, and \( v = 246 \text{ GeV} \). The two-loop photon Barr-Zee diagram contribution from \( \phi^0 \), with a heavy fermion \( f \) running in the second loop, is given by

\[
\Delta a^\phi_{\mu} = \frac{N_f}{4\pi^2} \frac{\alpha_{em}}{v^2} m^2_\mu Q_f \left[ A_\mu A_f g\left(\frac{m^2_f}{m^2_\phi}\right) - \lambda_\mu \lambda_f \left(\frac{m^2_f}{m^2_\phi}\right) \right],
\]

where

\[
\begin{align*}
 f(z) &= \frac{1}{2} z \int_0^1 dx \frac{1-2x(1-x)}{x(1-x)-z} \ln \frac{x(1-x)}{z}, \\
g(z) &= \frac{1}{2} z \int_0^1 dx \frac{1}{x(1-x)-z} \ln \frac{x(1-x)}{z};
\end{align*}
\]

\( N_f \) represents the number of color degrees of freedom in \( f \), and \( Q_f \) its electric charge. Here, we have three scalars and no CP violating mixing is assumed. The real scalars \( h \) and \( H \) give negative contributions only (from the second part) while a pseudoscalar \( A \) gives a positive contribution only (from the first part). The diagrams with the \( b \) and \( \tau \) loops are \( \tan \beta \) enhanced.

We plot in Fig. 2 the 2\( \sigma \) range of solution to \( a_\mu \) on the plane of the pseudoscalar mass \( m_A \) versus \( \tan \beta \), considering only the pseudoscalar contribution. Note that while the region below the solution band is excluded, the solution in the region above the band may be admissible when the \( a_\mu \) is compensated by some negative contributions from the real scalar(s). We also superimpose on the plot the excluded region from the DELPHI study on Higgs Yukawa processes in the \( 4b \) and \( 2b2\tau \) final states [11]. We can see that one obtains lower bounds on \( m_A \) and \( \tan \beta \) as 26 GeV and 30 respectively.\(^2\)

\( \Delta a^\phi_{\mu} \) is given in Ref. [9] plots of the solution regions on the \( m_h-m_A \) plane for the old \( a_\mu \) data with specific values of \( \tan \beta \) and the scalar mixing angle \( \alpha \). We will present similar results here with, however, the complementary \( R_b \) constraint included. We will see that not much area of the parameter space can survive the combination of both \( a_\mu \) and \( R_b \) data.

The current \( R_b \) measurement is given by [29]

\[
R_b^{\text{exp}} = 0.21646 \pm 0.00065.
\]

\(^2\) The plot is similar to the one given in Ref. [14], in which some more constraints are superimposed. We include here only the important ones. Note that the Tevatron exclusion region claimed in the paper is not used here. The exclusion result was from Ref. [28], which is an analysis based on the minimal supersymmetric standard model. The result should not be directly applicable to the present case of 2HDM. We do not find a similar study on the Tevatron data based on the 2HDM.
With $R_{b}^{\text{SM}} = 0.215768$, we have

$$\Delta R_{b} \equiv R_{b}^{\exp} - R_{b}^{\text{SM}} = 0.000692 \pm 0.00065.$$  

(19)

The $\Delta R_{b}$ contributions in the 2HDM are given, for example, by formulas in Ref.[30]. The charged Higgs contribution is always negative. On the other hand, there is a window of parameter space for the neutral Higgs contributions to be positive. From our discussions above, we need a light pseudoscalar and have to live with a heavy charged Higgs. We are therefore more interested in the neutral Higgs contributions.

Let us first focus on the contributions from the lighter Higgs bosons, the pseudoscalar $A$ and the real scalar $h$, pushing the other scalar $H$ to the heavy-mass limit together with $H^{\pm}$. The scenario will actually be well justified by our discussion on the constraint of the $\rho$ parameter in the next section. The smallness of $\Delta R_{b}$ contributions admitted here generally disfavors a large mass splitting between $m_{h}$ and $m_{A}$. This is in contrast to the requirement of a positive contribution to $a_{\mu}$. Since the SM result (with $M_{H_{SM}}$ at 115 GeV) now represents a $3\sigma$ deviation in $a_{\mu}$, better fits to the combined $a_{\mu}$-$R_{b}$ data are possible from the 2HDM. We illustrate some such fits in Fig. 3. In the figure, we take the case of $\tan\beta = 58$ and check various values of the Higgs mixing angle $\alpha$. Here, and in the discussion below unless specifically stated otherwise, we stick to $m_{H^{\pm}} = 500$ GeV and $m_{H} = 1$ TeV. The exact value of $m_{H}$ does not matter at all here, one, however, should note that the charged Higgs boson still gives a contribution of $-2.32 \times 10^{-4}$ to $R_{b}$, which is about $1.3\sigma$ in strength. Bearing this in mind, it is easy to estimate from our plots the slight shift in each of the admissible region as $m_{H^{\pm}}$ is being pushed towards the decoupling limit.

Each of the plots in Fig. 3 gives ±2σ limits for $a_{\mu}$ and $R_{b}$ fits, with darker shaded regions indicating the solution of interest defined by a total $\chi^{2}$ of 4 or less. Also marked in the plots are regions with a total $\chi^{2}$ less than the SM value of 10.3 (the sum of $a_{\mu}$ and $R_{b}$). The purpose of showing the area with a total $\chi^{2} < \chi^{2}(\text{SM})$ is to indicate the region of parameter space that can fit the $a_{\mu}$ and $R_{b}$ better than the SM, other than the decoupling limits. From now on, we concentrate on the dark area of a total $\chi^{2} < 4$ as a valid solution to the $a_{\mu}$ and $R_{b}$ data. We can see that there are no solutions for $-\pi/8 < \alpha < \pi/4$. While a larger magnitude of the $\alpha$ looks more favorable, the best solution, with higher Higgs masses ($m_{h}$ especially), stay close to $|\sin\alpha| = 1$, inclining more towards the negative sign. The most favorable range is around $-\pi/2 < \alpha < -3\pi/8$ [cf. plots (c) and (d)]. The plots in Fig. 3 illustrate well the trend of the changes in the $a_{\mu}$-$R_{b}$ solution regions with variations in $\alpha$, which is quite generic for $\tan\beta$ around and larger than 50. A higher $m_{h}$ solution is preferred as the light mass solutions are easily killed by searches at LEP, including the DELPHI exclusion used in Fig. 2 and a particularly focused 2HDM analysis from OPAL[10], to be discussed below. In fact, as we will illustrate below, if any part of the solutions to the $a_{\mu}$-$R_{b}$ fits survive the exclusions from LEP, it is more or less the part with a high enough $m_{h}$ value. 

\[3\] In these plots, we used the standard $R_{b}$ formulas for 2HDM as available in the reference [30] quoted. The
For instance, the DELPHI exclusion we used in Fig. 2 obviously kills quite a part of the above solutions. At \( \tan\beta = 58 \), the lower bound on \( m_A \) is actually about 40 GeV. It is particularly interesting to check the case of a relatively small \( \tan\beta \). We show in Fig. 4 the case of \( \tan\beta = 40 \), in which the DELPHI almost kills all solutions. In the figure, only a tiny window survives at \( \sin\alpha = \frac{3\pi}{8} \). This is, however, at a \( m_h \) (as well as \( m_A \)) value too small to survive the OPAL exclusion discussed below. From the same study by DELPHI, the Yukawa processes were also used to impose bounds on the \( m_h \)[11]. Note that the plots in the Ref. [11] give bounds on Higgs masses versus the \( b - \tau \) coupling enhancement factor, which is simply \( \tan\beta \sqrt{B(A \rightarrow b\bar{b}, \tau^+\tau^-)} \) for the pseudoscalar case, but has an added \( |\sin\alpha| \) dependence for the case of the scalar \( h \). In the range of \( m_A \) values of interest, however, the \( m_h \) bound is typically superseded by the OPAL exclusion. Note that the \( a_\mu \) solution with \( m_h \) at a few GeV is also inadmissible, from the consideration of \( T \)-decay[9] and otherwise. Figure 4 also illustrates that a positive value of \( \alpha \), around \( \frac{3\pi}{8} \), tends to give \( a_\mu - R_b \) solutions with the largest \( m_A \). This is mainly a result of the \( a_\mu \) constraint. At a fixed \( m_h \), the contribution of the scalar through the 2-loop Barr-Zee diagram to \( a_\mu \) is suppressed by \( |\sin\alpha| \), and thus allows a larger \( m_A \). A slight asymmetry in the cases of positive and negative \( \alpha \) values comes in as a result of the different way the 1-loop contributions go.

We should also point out that we find no solutions to the \( a_\mu - R_b \) fit at all for much lower \( \tan\beta \). The \( a_\mu \) solutions shown in Fig. 2 simply produce a \( m_h - m_A \) splitting too large to accommodate the \( R_b \) constraint. Therefore, the solutions to the \( a_\mu - R_b \) fits start to emerge as \( \tan\beta \) approaches a large enough value, not much below 40. To get solutions with high enough masses for the Higgs bosons so as to survive the exclusion limits from the LEP searches, one will have to get to a higher and higher \( \tan\beta \) value. To check the details, we first turn to the powerful exclusions from OPAL.

Based also on the LEP data, OPAL has been publishing Higgs-search analyses specifically focused on the 2HDM. Here, we use the most recent results available [10, 31]. The results exclude a region at the lower left corner of the \( m_h - m_A \) plane for each specific value of the mixing angle \( \alpha \) (four explicitly shown). As presented in Ref.[10], however, the results are not \( \tan\beta \) specific. We show in Fig. 5 the solution regions from \( a_\mu - R_b \) with the OPAL and the above mentioned DELPHI excluded regions superimposed. Here, we show the cases of \( \tan\beta = 50 \) and 58 for two \( \alpha \) values, \( -\frac{\pi}{4} \) and \( \pm\frac{\pi}{2} \). The latter are chosen as they are among the \( \alpha \) values for which the OPAL paper gives the explicit exclusion. The exclusion for non-specific \( \alpha \) values only presents a substantially weakened result to be of interest here.

formulas have not taken into consideration the possible tree-level decays of the \( Z^0 \) boson into a pair of Higgs states. Such very light Higgs regions are typically not of interest to theorists, as they would be easily excluded by direct searches. By the same token, we use the formulas as they are and show the \( a_\mu - R_b \) fits without bothering about the tree-level modifications of \( R_b \) from such Higgs channels, leaving such fake solutions to be taken care of by the exclusions from LEP searches to be discussed below. We thank A. Sanda for alerting us to explicitly address the issue here.
The two $\alpha$ values are also close to or within the range of value for an optimal $a_\mu - R_b$ fit, as illustrated in Fig. 3 above.

In fact, the OPAL exclusions are based on the $\tan\beta$ value in the range $1 - 58$, given only at four values of $\alpha$, which we adopted here for the plots. An excluded point is one excluded at all value of $\tan\beta$ within the range. The excluded ranges may hence be extended at each specific value of $\tan\beta$, with more detailed analysis of the data[32]. In fact, one would expect the enhanced couplings at a larger $\tan\beta$ generally push the exclusion regions towards higher masses. One can see in the plots that the $a_\mu - R_b$ solution regions are largely cut off by the presented OPAL exclusions in general. Actually, nothing survives in the illustrated plots for $\tan\beta = 50$ [cf. plots (a) and (b) of Fig. 5]; and it is also quite obvious that the same is true for the case of $\tan\beta = 40$. Figures 4 and 5 together clearly illustrate the general trend of how the increase in $\tan\beta$ gives better results. While it does not do much for the case of $\alpha = -\frac{\pi}{4}$, at $\alpha = \pm\frac{\pi}{2}$, the admissible $m_h$ and $m_A$ values are pushed to be high enough to escape the OPAL exclusion at $\tan\beta > 50$ [cf. plot (c) of Fig. 5]. However, we are not bold enough to say for sure if no solution survives at $\tan\beta = 50$ though. The optimal case giving the largest $m_h$ value for the $a_\mu - R_b$ is likely to be around $\alpha = -\frac{3\pi}{8}$, at which the present result of OPAL did not show its greatest strength.

We pick $\tan\beta = 58$ for the above detailed illustration of the $a_\mu - R_b$ fits because it gives the most favorable case within the limit of a direct application of the OPAL exclusion. There is apparently a surviving window in the parameter space. Combining our present study with a refined version of the OPAL analysis to focus on our $a_\mu - R_b$ solution regions, especially the upper blots as shown in plots (a) and (c) of Fig. 5, will certainly be very interesting. Exclusions have to be checked for each specific value of $\tan\beta$ and that of $\alpha$. Such a study will further narrow down the survival parameter space regions of the 2HDM, especially with further improved exclusion limits. We are told that the LEP data actually allows such an improvement[32]. A very tantalizing question is if the current constraints are actually strong enough to kill the model altogether!

The OPAL analysis is limited to $\tan\beta$ value at or below 58, while the DELPHI analysis stops at 100. The very large $\tan\beta$ region is theoretically unfavorable and may provide practical problems due to the much enhanced $b$-quark Yukawa coupling, which signals a breakdown of the perturbative treatment. Nevertheless, we will include some results from such uncomfortably large $\tan\beta$ values, and urge the OPAL group to push on a bit further in their analysis.

For $\tan\beta > 58$, we again illustrate some results for $\alpha = -\frac{\pi}{4}$ and $\pm\frac{\pi}{2}$ in Fig. 6, in which we still put in the (no longer exactly valid) OPAL exclusion from the $\tan\beta = 1 - 58$ scan for reference. For the $\alpha = -\frac{\pi}{4}$ case, the $a_\mu - R_b$ solution regions never rise above $m_h = 60$ GeV (as also for the smaller $\tan\beta$ cases), and partially excluded by DELPHI. There seem to be good enough reasons to believe that such low Higgs mass regions should be excluded by the available LEP data if an analysis along the OPAL line is performed. The $\alpha = \pm\frac{\pi}{2}$ case is better. The surviving region seen at $\tan\beta = 58$ moved further to the right, towards larger
$m_A$, and further up a bit for larger $m_h$. The rise in $m_h$ actually more or less saturates at \(\tan \beta = 100\), and falls for even larger values of \(\tan \beta\). Such a region still lives at the boundary of the DELPHI exclusion, but would still have a surviving part even if the OPAL exclusion can still be imposed. The region is, however, shrinking with increases in \(\tan \beta\), due to a more fine-tuned $a_\mu$-$R_b$ solution. One should also bear in mind that the (upward) shifting, and hence slight enlargement, of this solution region with a further increase in the $m_{h\pm}$ value. For this purpose, we give the charged-Higgs contribution to $R_b$ in Fig. 7.

Let us summarize our results so far. We have seen that combining the suggested requirement of producing a definite positive contribution to $a_\mu$ while keeping a limited deviation from the SM $R_b$ result is an extremely stringent constraint on the parameter space of the 2HDM. When the available direct experimental search results from the LEP experiments are further implemented, there is at most a tiny window of parameter space that can survive. The apparently surviving region from the above discussions is restricted to very large values of \(\tan \beta\). In fact, it may be already uncomfortably large, inviting the problem of the perturbativity of the Yukawa coupling of the $b$ quark. As for the mixing angle $\alpha$, it is being pushed close to the \(\frac{-\pi}{2} < \alpha < -\frac{3\pi}{8}\) region. All these are based on a strong mass splitting between the charged Higgs and the light scalars, with the pseudoscalar $A$ lying typically below 80 GeV and the scalar $h$ below 140 GeV.

Recall that we have essentially decoupled the heavy Higgs boson $H$ in the above analysis. We promised to justify this as a physics requirement. We first noted that a not too heavy $H$ would more or less add to the effect of the other real scalar $h$ in the contributions to $a_\mu$ and $R_b$. So, we expect it to ask for a larger mass splitting when fitting $a_\mu$ is concerned, but a smaller mass splitting to fit $R_b$. In another word, further tightening of the apparent solution window. In the section below, we will show that fitting another precision EW parameter, the $\rho$ parameter, actually does require sending $m_{h\pm}$ to a very large value, indeed well beyond $m_{h\pm}$.

V. THE $\rho$ PARAMETER CONSTRAINT

The parameter $\rho$ was introduced to measure the relation between the masses of $W^\pm$ and $Z^0$ bosons. In the SM $\rho \equiv M_W^2/M_Z^2 \cos^2 \theta_W = 1$ at tree-level. However, the $\rho$ parameter receives contributions from the SM corrections and from new physics. The deviation from the SM prediction is usually described by the parameter $\rho_0$ defined by

\[
\rho_0 \equiv \frac{M_W^2}{\rho M_Z^2 \cos^2 \theta_W},
\]

(20)

where the $\rho$ in the denominator absorbs all the SM corrections, including the corrections from the top quark and the SM Higgs boson. By definition, $\rho_0 = 1$ in the SM. Sources of new physics that contribute to $\rho_0$ can be written as

\[
\rho_0 = 1 + \Delta \rho_0^{\text{new}},
\]

(21)
where $\Delta \rho_0^{\text{new}} = \Delta \rho^{2\text{HDM}} - \Delta \rho^{\text{SM-Higgs}}$ in our case. Note that since the two-doublet Higgs sector (in the 2HDM) is employed here to replace the SM Higgs, the latter contribution to $\Delta \rho$ has to be subtracted out.

The most recent reported value of $\rho_0$ is [23]

$$\rho_0 = 1.0004 \pm 0.0006, \quad (\text{with } M_{\text{h}}^{\text{SM}} \text{ fixed at 115 GeV}) \, .$$

In terms of new physics the constraint becomes:

$$\Delta \rho_0^{\text{new}} = 0.0004 \pm 0.0006 \, .$$

In 2HDM $\Delta \rho$ receives contributions from all Higgs bosons given by [3, 12]

\[
\Delta \rho^{2\text{HDM}} = \frac{\alpha_{\text{em}}}{4\pi \sin^2 \theta_W M_W^2} \left[ F(m_A, m_{h^+}) + \cos^2(\beta - \alpha) [F(m_{h^+}, m_h) - F(m_A, m_h)] \\
+ \sin^2(\beta - \alpha) [F(m_{h^+}, m_h) - F(m_A, m_h)] \\
+ \cos^2(\beta - \alpha) \Delta \rho^{\text{SM}}(m_h) + \sin^2(\beta - \alpha) \Delta \rho^{\text{SM}}(m_h) \right],
\]

where

\[
F(x, y) = \frac{1}{8} x^2 + \frac{1}{8} y^2 - \frac{1}{4} x^2 y^2 \log \left( \frac{x^2}{y^2} \right) = F(y, x),
\]

\[
\Delta \rho^{\text{SM}}(M) = -\frac{\alpha_{\text{em}}}{4\pi \sin^2 \theta_W M_W^2} \left[ 3F(M, M_W) - 3F(M, M_Z) + \frac{1}{2}(M_Z^2 - M_W^2) \right].
\]

Let us take a closer look into the implication of the formulas above. First of all, we note that $\Delta \rho^{\text{SM}}(M)$ has a negative value with magnitude increasing with $M$. As to be expected from above, the value is about $-0.0004$ at $M = 115$ GeV. It has a relatively mild variation, and does not go beyond $-0.005$ even as $M$ gets to $10$ TeV. In direct contrast, the other contributions to $\Delta \rho^{2\text{HDM}}$ in the above formula are very sensitive to the masses involved. The $F(x, y)$ function is always positive, vanishes only at $x = y$, and increases with a faster and faster rate with the splitting between $x$ and $y$. In a typical scenario that is of interest here, we expect the pseudoscalar to be the lightest Higgs state with a quite heavy charged Higgs. That makes the contribution from the first term [involving $F(m_A, m_{h^+})$] large; indeed of order 0.01 for $m_{h^+}$ satisfying the lower bound from $b \to s \gamma$ and $B \to \overline{B}$ mixing. To get the required almost zero value of $\Delta \rho^{2\text{HDM}}$, we need some negative contributions from the terms involving $[F(m_{h^+}, m_h) - F(m_A, m_h)]$ and $[F(m_{h^+}, m_h) - F(m_A, m_h)]$. And the solution is obviously a fine-tuned one. Consider a case of degenerated Higgs real scalars, $m_t = m_h = M$. The $(\beta - \alpha)$ dependence in the formula is removed as the sine-square and cosine-square parts are combined. We need $[F(m_{h^+}, M) - F(m_A, M)]$ to have a negative value close in magnitude to that of $F(m_A, m_{h^+})$. The former is obviously positive roughly when $M$ is closer to $m_A$ than $m_{h^+}$, and negative when it is the other way round; and it can be larger than $F(m_A, m_{h^+})$ only for $M > m_{h^+}$. The $(\beta - \alpha)$ dependence comes back in the generic situation, with a
mass splitting between the two real scalars. Obviously, at least one of them, by definition $H$, has to be heavy. However, they cannot be both heavy, because the $R_b$ constraint does not allow only a light pseudoscalar giving a substantial contribution. Hence, this additional requirement of a limited splitting between $m_A$ and $m_h$ clearly suggests a large $m_H$, typically larger than $m_{H^+}$. The larger the $m_H$ value, the smaller the $\sin^2(\beta - \alpha)$ is required. Admissible solutions, though fine-tuned, can be obtained so long as the required $\sin^2(\beta - \alpha)$ falls into the legitimate interval. A large splitting between $m_H$ and $m_h$ also resulted in a very narrow range of admissible $(\beta - \alpha)$ values, and hence the $\alpha$ values at a fixed $\tan\beta$ of interest.

Our numerical results corroborate well with the above analytical discussions. We illustrative our discussion with a plot in Fig. 8. Here, we take a “surviving” solution point to the $a_\mu$-$R_b$ fits and perform a further fitting together with the $\rho$ parameter by varying $m_H$ and $m_{H^+}$. The extremely fine-tuned nature of the solution is well-illustrated by the very narrow $\chi^2$ bands. Moreover, $m_H$ is always more than twice of $m_{H^+}$ for an $|\sin\alpha|$ larger than 0.8. Note that $\sin\alpha$ of $-0.8$ and $-0.92$ roughly correspond to an $\alpha$ value of $\frac{-3\pi}{10}$ and $\frac{-3\pi}{8}$, respectively. The basic features remain if other solution points are taken, hence, we refrain from showing more plots.

VI. CONCLUSIONS

The requirement for the 2HDM II to give rise to the suggested $+3\sigma$ deviation in the muon anomalous magnetic moment demands a light pseudoscalar, preferably around 40 GeV for $\tan\beta \leq 58$. The FCNC constraints, in particular the $b \to s\gamma$, require a heavy charged Higgs beyond 400 GeV. This large mass splitting in the Higgs mass spectrum is difficult to be accommodated by the precision EW measurements. In particular, the $\rho$ parameter constraint then admits only very fine-tuned solutions, with cancellation from opposite contributions good to one in a hundred, favoring heavy real scalars. The fine-tuned nature of the solutions, while making many physicists uncomfortable with the model, is not in itself a good enough reason to pronounce the death of the model. If one has reasons to be confident about the correctness of the model, one would say that the available constraints are just strong enough to pin down for us the values of the unknown model parameter. A good example of such a situation is given by the pinning down of the top mass value from precision EW data prior to the experimental discovery of the top quark.

Nevertheless, we have not had much of a reason to believe in the correctness of the 2HDM. In fact, in our analysis here, we focus more on the simultaneous fits to $\Delta a_\mu$ and $R_b$. Having only the pseudoscalar contribution dominated the corrections to $R_b$ is fatal. Hence, we require a relatively light $m_h$, while pushing $m_H$ to way beyond $m_{H^+}$ in order to satisfy the $\rho$ parameter. Even then, not much of a solution to the $a_\mu$-$R_b$ fits survives the direct search exclusion. Here, we require a total $\chi^2$ of 4 or smaller for the $a_\mu$-$R_b$ fits to claim a good solution. For roughly $\tan\beta < 40$, no solution to $a_\mu$-$R_b$ fits survives the DELPHI
exclusion. In fact, there is no solution to the $a_\mu-R_b$ fit for a $\tan\beta$ value quite a bit smaller than 40. For $\tan\beta$ value from around 40 to a bit beyond 50, solutions surviving the DELPHI exclusions exist, but only to be killed by the OPAL exclusions. For even larger $\tan\beta$, the $a_\mu-R_b$ solutions surviving both the DELPHI and OPAL exclusions started to emerge. This is very much restricted to an $\alpha$ value in the range $-\frac{\pi}{2} < \alpha < -\frac{3\pi}{8}$. Solution regions shrink fast outside the range as the $m_h$ values given by the $a_\mu-R_b$ fits drop towards OPAL exclusion bound.

In summary, under the strong restriction of the available constraints, we show that only a very tiny window of apparent solutions exist close to the limit of $\tan\beta \leq 58$. If the OPAL group could tailor their analysis of the LEP data to focus on the apparent solution window as shown here, we would be able to have a more definite conclusion. It looks to us that the solution window will be shut down quite substantially. So, we have “almost enough” constraints to kill the model altogether. For $\tan\beta$ beyond the 58 limit, our hands are tied at the moment by the unavailability of the strong LEP exclusion results as presented by OPAL. The very large $\tan\beta$ values certainly make many of us uncomfortable though. It is theoretical undesirable as limited by the blowing up of the bottom Yukawa couplings.

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[26] There are complicated gauge dependence notion when some of the class of diagrams are calculated separately. The photon Barr-Zee diagrams with fermions running in the second loop form a gauge invariant set on their own. We thank Wai-Yee Keung for comments on the issue.


[31] Further update or improved results may soon be available — P. Ferrari, private communications.


FIG. 1: 95% C.L. lower limit on the charged Higgs mass vs $\tan \beta$ due to the constraints on $b \rightarrow s \gamma$ and/or $B - \overline{B}$ mixing.
FIG. 2: The 2σ allowed region in the $(m_A, \tan\beta)$ plane due to the $a_\mu$ data. Here only the pseudoscalar contribution is considered. The DELPHI excluded region is represented by the thick line.
FIG. 3: The 2σ allowed regions in the \((m_A, m_h)\) plane due to the constraints of \(a_\mu\) and \(R_b\) for \(\tan\beta = 58\). The smaller dark region is where the total \(\chi^2\) is less than 4 while the lighter region is where the total \(\chi^2\) is less than the \(\chi^2\text{(SM)} = 10.3\).
FIG. 4: The 2σ allowed regions in the \((m_A, m_h)\) plane due to the constraints of \(a_\mu\) and \(R_b\) for \(\tan\beta = 40\). The smaller dark region is where the total \(\chi^2\) is less than 4 while the lighter region is where the total \(\chi^2\) is less than the \(\chi^2_{(SM)} = 10.3\). Here the lower mass limits on \(m_A\) and \(m_h\) from DELPHI are shown.
FIG. 5: The 2σ allowed regions in the $(m_A, m_h)$ plane due to the constraints of $a_\mu$ and $R_b$ for $\tan\beta = 50$ (a,b) and for $\tan\beta = 58$ (c,d). The smaller dark region is where the total $\chi^2$ is less than 4 while the lighter region is where the total $\chi^2$ is less than the $\chi^2(\text{SM}) = 10.3$. Here the lower mass limits on $m_A$ and $m_h$ from DELPHI, and the excluded region from OPAL are shown.
FIG. 6: Same as Fig. 5, but for $\tan \beta = 80$ (a,b) and for $\tan \beta = 100$ (c,d). Note that the OPAL exclusion regions put in is actually for $\tan \beta \leq 58$, hence no longer directly applicable here. In lack of the corresponding applicable results, they are kept here for reference.
FIG. 7: The charged-Higgs contribution to $R_b$ in units of $10^{-3}$ for $\tan\beta = 40 - 100$ (from the top to the bottom).
FIG. 8: The allowed region in the plane of $(m_H, m_{H^+})$ due to constraints of $a_\mu$, $R_b$, and the $\rho$ parameter. Here the $m_A = 40$ GeV and $m_h = 100$ GeV for $\tan \beta = 58$ are chosen from the allowed darker region in Fig. 5(c).