Doubly Periodic Instantons and their Constituents

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Abstract

Using the Nahm transform we investigate doubly periodic charge one $SU(2)$ instantons with radial symmetry. Two special points where the Nahm zero modes have softer singularities are identified as the locations of instanton core constituents. For a square torus this constituent picture is closely reflected in the action density. In rectangular tori with large aspect ratios the cores merge to form monopole-like objects. For particular values of the parameters the torus can be cut in half yielding two copies of a twisted charge $\frac{1}{2}$ instanton. These findings are illustrated with plots of the action density within a two-dimensional slice containing the constituents.

1 Introduction

Monopoles are at the heart of various kinds of electric-magnetic duality in extended super Yang-Mills theory, which have led to an impressive body of quantitative results. The role of monopoles and other topological constructs in non-supersymmetric Yang-Mills theory is less clear. Using such objects, particularly monopoles and vortices, models of confinement and chiral symmetry breaking have been advanced. A weakness of these models is that classical Yang-Mills does not possess monopole solutions, rather they have instanton solutions. However, in recent years it has become clear that periodic instantons have a substructure that can be identified with monopoles and other extended objects. In particular, charge one $SU(N)$ calorons (finite temperature instantons) can be viewed as bound states of $N$ monopole constituents $[1, 2]$. Provided the ‘size’ of the instanton is larger than the period...
(or inverse temperature) tubes of action density form. These can be identified with the worldlines of the constituent monopoles. It would be interesting to extend these results to higher charge sectors and higher tori, i.e. more than one period so that we are considering instantons on $\mathbb{T}^n \times \mathbb{R}^{4-n}$, $n > 1$. Indeed, lattice based numerical studies indicate that doubly periodic instantons ($n = 2$) have remarkable properties including fractionally charged solutions, vortex-like objects and an exponentially decaying action density [3]. None of these attributes are shared by the calorons. Nevertheless, it would be desirable to have a unified description of calorons and multiply-periodic instantons and their constituents.

The calorons are best understood within a formalism due to Nahm [4, 5]. This is an extension of the ADHM construction. Nahm’s approach was generalised to other four manifolds, notably the four-torus, $\mathbb{T}^4$ [6]. Here it becomes a duality, mapping $U(N)$ charge $k$ instantons on the four torus to $U(k)$ charge $N$ instantons on the dual torus, $\tilde{\mathbb{T}}^4$ (defined by inverting the four periods). The caloron construction and even ADHM can be understood as a limiting case of the four torus duality. One therefore expects that doubly periodic instantons can be treated in a similar fashion. Jardim [7] has used the Nahm transform to discuss the charge one case. In [8] doubly periodic charge one instantons with radial symmetry in the non-compact $\mathbb{R}^2$ directions were considered. Under the Nahm transformation they are mapped to abelian potentials on the dual torus $\tilde{\mathbb{T}}^2$. Starting with these rather simple Nahm potentials the original instantons can be ‘recovered’ via the inverse Nahm transform. Technically, this involves solving certain Weyl-Dirac equations; for each $x \in \mathbb{T}^2 \times \mathbb{R}^2$ one has a different Weyl equation on $\tilde{\mathbb{T}}^2$. The Weyl zero modes were determined explicitly for a two dimensional subspace (corresponding to the origin of $\mathbb{R}^2$).

In a recent letter [9] we used these zero modes to compute the action density, analytically and numerically, within the subspace. We found that there are points in $\mathbb{T}^2 \times \mathbb{R}^2$ where the solution of the Weyl equations have softer singularities. These are obvious candidates for constituent locations. We interpreted these constituents as overlapping instanton cores. For square tori our constituent picture correlates closely to the action density. Some of the charge one solutions we considered can be cut in half yielding two copies of a charge $\frac{1}{2}$ instanton. This procedure is only consistent if the gauge potential has specific asymptotic boundary conditions which are analogous to a four torus twist.

In this paper we obtain further information about doubly periodic instantons. In particular, action density plots for rectangular tori are presented. These clearly show how doubly periodic instantons interpolate between fractionally charged lumps characteristic of four
torus solutions and monopole like tubes of actions density typical of calorons. Furthermore, we present an analytical discussion of the exponential decay of the action density. The paper is organised as follows. In section 2 the basics of the Nahm transform are briefly reviewed, and its application to doubly periodic instantons is discussed. Sections 3 to 5 deal with the Nahm potentials, zero modes and constituents corresponding to charge one instantons with gauge group $SU(2)$. Section 6 describes some technical results regarding Green’s functions that crop up in field strength calculations. Our numerical action density computations are detailed in section 7. Section 8 considers charge $\frac{1}{2}$ solutions and the asymptotics of the gauge field and field strength are discussed in section 9.

2 Nahm Transform for Doubly Periodic Instantons

Before we specialise to $\mathbb{T}^2 \times \mathbb{R}^2$ let us briefly recall how the Nahm transform is formulated on the four torus [6] (for a string-theoretic approach see [10]). Start with a self-dual anti-hermitian $SU(N)$ potential, $A_\mu(x)$, on a euclidean four-torus with topological charge $k$. A gauge field on $\mathbb{T}^4$ is understood to be an $\mathbb{R}^4$ potential which is periodic (modulo gauge transformations) with respect to $x_\mu \rightarrow x_\mu + L_\mu$ ($\mu = 0, 1, 2, 3$), the $L_\mu$ being the four periods of the torus. Self-duality is the requirement that $F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$, and we have taken the convention $\epsilon_{0123} = -1$. The next step is to turn the $SU(N)$ instanton into a $U(N)$ instanton by adding a constant $U(1)$ potential, $A_\mu(x) \rightarrow A_\mu(x) - iz_\mu$, where the $z_\mu$ are real numbers. We can regard the $z_\mu$ as coordinates of the dual torus, $\tilde{\mathbb{T}}^4$, since the shifts $z_\mu \rightarrow z_\mu + 2\pi/L_\mu$ can be effected via periodic $U(1)$ gauge transformations.

Now consider the $U(N)$ Weyl operator

$$ D_z(A) = \sigma_\mu D_\mu^z(A), \quad D_\mu^z(A) = \partial_\mu + A_\mu(x) - iz_\mu \quad (1) $$

with $\sigma_\mu = (1, i\tau_1, i\tau_2, i\tau_3)$ where the $\tau_i$ are Pauli matrices. Provided certain mathematical technicalities are met $D_\mu^z(A) = -\sigma_\mu D_\mu^z(A)$ has $k$ square-integrable zero modes $\psi^i(x;z)$ with $i = 1, 2, ..., k$. The Nahm potential is defined as

$$ \hat{A}^i_\mu(z) = \int_{T^4} d^4x \psi^{i\dagger}(x;z) \frac{\partial}{\partial z_\mu} \psi^j(x;z). \quad (2) $$

Here the zero modes are taken to be orthonormal. Remarkably, $\hat{A}(z)$ is a $U(k)$ instanton on the dual torus with topological charge $N$. The associated field strength can be written as follows

$$ \hat{F}^{ij}_{\mu\nu}(z) = \int_{T^4} d^4x \int_{T^4} d^4x' \psi^{i\dagger}(x;z) \sigma_\mu(D_\nu^z D_\mu^z)^{-1}(x,x') \sigma_\nu^j \psi^j(x;z) - [\mu \leftrightarrow \nu]. \quad (3) $$
The self duality of \( F_{\mu \nu} \) implies \( D^\dagger D = -\sigma_0 D_\mu D_\mu \). Accordingly, \( (D_z^\dagger D_z)^{-1}(x,x') \) commutes with all \( \sigma_\mu \), from which the self duality of \( \hat{F} \) follows. The ‘original’ gauge potential, \( A_\mu(x) \), can be recovered by Nahm transforming \( \hat{A}_\mu(z) \)

\[
A_\mu^p(x) = \int \frac{d^4 z}{T^4} \psi^p(x; z) \frac{\partial}{\partial x^\mu} \psi^q(z; x),
\]

where the \( \psi^p(z; x), p = 1, 2, \ldots, N \) are an orthonormal set of zero modes of \( D_z^\dagger(A) = -\sigma_\mu D_x^\mu(A) \) with \( D_x^\mu = \partial/\partial z_\mu + \hat{A}_\mu(z) - i x^\mu \). In other words the four torus Nahm transform is involutive.

Formally, one can obtain the \( T^2 \times \mathbb{R}^2 \) Nahm transform by taking two of the periods, say \( L_0 \) and \( L_3 \), to be infinite. Given an \( SU(N) \) instanton periodic with respect to \( x_1 \to x_1 + L_1, x_2 \to x_2 + L_2 \) its Nahm transform is

\[
\hat{A}_\mu^{ij}(z) = \int_{T^2 \times \mathbb{R}^2} d^4 x \psi_{ij}^\dagger(x; z) \frac{\partial}{\partial z^\mu} \psi^j(x; z), \quad \mu = 1, 2,
\]

\[
\hat{A}_\mu^{ij}(z) = \int_{T^2 \times \mathbb{R}^2} d^4 x \psi_{ij}^\dagger(x; z) i x_\mu \psi^j(x; z), \quad \mu = 0, 3,
\]

where the \( \psi^i(x; z) \) with \( i = 1, \ldots, k \) are orthonormal zero modes of \( D_z^\dagger(A) \). Note that we may gauge \( z_0 \) and \( z_3 \) to zero. By assumption the zero modes \( \psi_i(x; z) \) are square integrable. This does not guarantee that the integrals in (6) exist. However, the integrals only diverge at special \( z_\mu \) values where the zero modes do not decay exponentially. The number of these points turns out to be \( N \), i.e. it is determined by the gauge group of the instanton. To summarise, the Nahm transform maps doubly periodic \( SU(N) \) instantons with topological charge \( k \) to self dual \( U(k) \) potentials on the dual torus, \( \hat{T}^2 \), with \( N \) singularities.

The dimensionally reduced self-duality equations (or Hitchin equations) take a particularly simple form in complex coordinates

\[
y = z_1 + i z_2, \quad \bar{y} = z_1 - i z_2,
\]

with derivatives \( \partial_y = \frac{1}{2}(\partial_{z_1} - i \partial_{z_2}) \), \( \partial_{\bar{y}} = \frac{1}{2}(\partial_{z_1} + i \partial_{z_2}) \). Combine \( \hat{A}_1(z) \) and \( \hat{A}_2(z) \) into a complex gauge potential, \( \hat{A}_y = \frac{1}{2}(\hat{A}_1(z) - i \hat{A}_2(z)) \) and \( \hat{A}_{\bar{y}} = \frac{1}{2}(\hat{A}_1(z) + i \hat{A}_2(z)) \), and form a ‘Higgs’ field out of the remaining components, \( \Phi(z) = \frac{1}{2}(\hat{A}_0(z) - i \hat{A}_3(z)) \). The self-duality equations read

\[
[D_y, \Phi] = 0, \quad \hat{F}_{y\bar{y}} = [D_y, D_{\bar{y}}] = [\Phi, \Phi^\dagger].
\]

The reader should be aware that we will occasionally use both the cartesian \( z \) and complex \( y \) coordinates in the same equation. The Nahm transform \( \hat{F} \) is a straightforward limit of the
The inverse transform is a different matter. That is, given a (singular) solution of the Hitchin equations on the dual torus, $\tilde{T}^2$, how does one recover the corresponding doubly periodic instanton? To our knowledge there is not a mathematically rigorous treatment of this issue in the literature. Our approach will be to study specific simple solutions of the Hitchin equations and investigate whether the number of normalisable zero modes of $D^\mu_x(\hat{A})$ matches the number of singularities of $\hat{A}(z)$. When this is the case we can then ask whether the natural analogue of (4), namely

$$A_{pq}^\mu(x) = \int_{\tilde{T}^2} d^2z \psi^p(\hat{z}; x) \frac{\partial}{\partial x^\mu} \psi^q(\hat{z}; x),$$

provides doubly periodic instantons.

### 3 Charge One

In the one-instanton case the Nahm potential is abelian and the Hitchin equations reduce to

$$\hat{F}_{y\bar{y}} = 0, \quad \partial_y \Phi = 0,$$

so that $\Phi$ is anti-holomorphic in $y$. As we shall see, $A_y$ can be taken as holomorphic. Since $A_y$ is periodic it is an elliptic function. It is well known that they have at least two singularities. We expect the number of singularities to correspond to $N$. Let us stick to the simplest case $N = 2$. Thus $\hat{A}_y$ will have two simple poles $^1$. Consider the ansatz

$$\hat{A}_y = \partial_y \phi, \quad \hat{A}_\bar{y} = -\partial_{\bar{y}} \phi,$$

which gives $\hat{F}_{y\bar{y}} = -2\partial_y \partial_{\bar{y}} \phi$. It follows from (9) that $\phi$ must be harmonic except at two singularities. A suitable $\phi$ satisfies

$$(\partial^2_{z_1} + \partial^2_{z_2}) \phi(z) = -2\pi \kappa \left[ \delta^2(z - \omega) - \delta^2(z + \omega) \right],$$

where $\kappa$ is a constant and $\pm \omega$ are the positions of the two singularities (we have used translational invariance to shift the `centre of gravity' of the singularities to the origin). The delta functions should be read as periodic (with respect to $z_1 \rightarrow z_1 + 2\pi/L_1$ and $z_2 \rightarrow z_2 + 2\pi/L_2$). Physically, the Nahm potential describes two Aharonov-Bohm fluxes of strength $\kappa$ and $-\kappa$ threading the dual torus. They must have equal and opposite strength to ensure a periodic $\hat{A}$. We may assume that $\kappa$ lies between 0 and 1 since it is possible via a normalisation.

$^1$The Weyl equations for Nahm potentials with higher poles do not admit normalisable zero modes.
(singular) gauge transformation to shift \( \kappa \) by an integer amount (under such a transformation the total flux through \( \mathbb{T}^2 \) remains zero). One can write \( \phi \) explicitly in terms of Jacobi theta functions (see for example [11])

\[
\phi(z) = \frac{\kappa}{2} \left( \log \left| \frac{\theta \left( y + \omega_1 + i \omega_2 \frac{L_1}{2\pi} + \frac{1}{2} + \frac{iL_1}{2L_2}, \frac{iL_1}{2L_2} \right)}{\theta \left( y - \omega_1 - i \omega_2 \frac{L_1}{2\pi} + \frac{1}{2} + \frac{iL_1}{2L_2}, \frac{iL_1}{2L_2} \right)} \right|^2 + \frac{IL_1 L_2 \omega_2}{\pi} (y - \bar{y}) - 2\omega_2 L_1 \right),
\]

where \( y \) and \( \bar{y} \) are the complex coordinates introduced in (6). The theta function is defined as

\[
\theta(w, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau + 2\pi i nw}, \quad \text{Im}\ \tau > 0,
\]

and has the periodicity properties \( \theta(w + 1, \tau) = \theta(w, \tau) \) and \( \theta(w + \tau, \tau) = e^{-i\pi \tau - 2\pi iw} \theta(w, \tau). \) In each cell \( \theta(w, \tau) \) has a single zero located at the centre of the torus \( (w = \frac{1}{2} + \frac{1}{2} \tau) \). We have chosen the constant term in (12) so that the integral of \( \phi \) over the dual torus is zero. This renders \( \phi(z) \) an odd function, \( \phi(-z) = -\phi(z) \). It is easy to see that inserting (12) into (10) yields a holomorphic \( A_y \) with simple poles at the fluxes.

What about the Higgs field? Since we are aiming for an \( SU(2) \) instanton it must, like \( \hat{\lambda}(z) \), have two singularities. The singularities must be at the same two positions as in \( A_y \) or else we have a total of four singularities which means we are considering a (rather special) \( SU(4) \) instanton. The simplest way to arrange for \( A_y \) and the Higgs to have singularities at the same points is to choose them to be proportional

\[
\kappa \Phi = \alpha \partial_y \phi,
\]

where \( \alpha \) is a complex constant. Not all possibilities are exhausted, since while the poles must coincide the zeroes need not. This remaining ambiguity corresponds to the freedom to add to \( \Phi \) a complex constant. However when we insert our Nahm potential into the Weyl equation such a shift is equivalent to a translation of \( x_0 \) and \( x_3 \). This Nahm potential was derived in [8] via the ADHM construction. Here the parameters \( \kappa \) and \( \alpha \) are related to the ‘size’, \( \lambda \) of an instanton centred at \( x_\mu = 0 \),

\[
\sqrt{\kappa^2 + |\alpha|^2} = \frac{\pi \lambda^2}{L_1 L_2}.
\]

Although we do not directly use the ADHM construction in the present paper this relation proves useful in interpreting our results. The Nahm potential we have described has five parameters; \( \omega_1, \omega_2 \) fixing the flux separations, \( \kappa \) the flux strength and the complex parameter
α specifying the Higgs field. Then, with the four translations in \( \mathbb{T}^2 \times \mathbb{R}^2 \), the corresponding doubly periodic instanton has a total of nine parameters.

To reconstruct the charge one \( SU(2) \) instantons corresponding to our simple Nahm potential we require the two zero modes of the Weyl operator

\[
-\frac{i}{2} D^+_x(\hat{A}) = \begin{pmatrix}
\frac{1}{2} \bar{\tau}_1 + i \alpha \kappa^{-1} \partial_y \phi & \partial_y + \partial_y \phi - i \frac{3}{2} \bar{x}_\parallel \\
\partial_y - \partial_y \phi - i \frac{3}{2} x_\parallel & \frac{1}{2} \bar{x}_\perp - i \alpha \kappa^{-1} \partial_y \phi
\end{pmatrix}.
\]  

(16)

In addition to the complex coordinates \( y, \tilde{y} \) on \( \mathbb{T}^2 \) we have introduced two sets of complex coordinates for \( \mathbb{T}^2 \times \mathbb{R}^2 \); in the ‘parallel’ directions \( x_\parallel = x_1 + ix_2, \bar{x}_\parallel = x_1 - ix_2 \), and in the ‘transverse’ non-compact directions \( x_\perp = x_0 + ix_3, \bar{x}_\perp = x_0 - ix_3 \). In this paper we shall concentrate on the special case \( \alpha = 0 \), i.e. zero Higgs. This means that the corresponding instanton will be radially symmetric: the action density depends on \( x_1, x_2 \) and \( r = |x_\perp| \) only. These radially symmetric solutions are a seven parameter subset of the doubly periodic \( SU(2) \) one-instantons.

4 Nahm Zero Modes

From now on we stick to the zero Higgs case (i.e. \( \alpha = 0 \)). Even with this specialisation the Weyl equations are still rather forbidding. However, there is one special case which is immediately tractable: when \( x_\perp = 0 \) the Weyl equation decouples and the two zero modes have a simple form

\[
\psi^1(z; x) = \begin{pmatrix} 0 \\ e^{-\phi(z)} G_+(z - \omega) \end{pmatrix}, \quad \psi^2(z; x) = \begin{pmatrix} e^{\phi(z)} G_-(z + \omega) \\ 0 \end{pmatrix},
\]  

(17)

where \( G_\pm(z) \) are periodic Green’s functions satisfying

\[
(-i\partial_y - \frac{1}{2} \bar{x}_\parallel) G_+(z) = \frac{1}{2} \delta^2(z), \quad (-i\partial_{\tilde{y}} - \frac{1}{2} x_\parallel) G_-(z) = \frac{1}{2} \delta^2(z).
\]  

(18)

Here we have used both the \( z \) and \( y \) coordinates in the same equation; this possibly confusing usage will continue until the end of section. \( G_- \) has a theta function representation

\[
G_-(z) = \frac{i L_1}{4\pi^2} e^{\frac{1}{2} \bar{x}_\parallel} \theta'\left( \frac{1}{2} + \frac{i L_1}{2 L_2}, \frac{i L_1}{2 L_2} \right) \theta\left( \frac{L_1}{2 L_2} y + \frac{1}{2} + \frac{i L_1}{2 L_2}, \frac{L_1}{2 L_2} \right) \theta\left( \frac{L_1}{2 L_2} \bar{x}_\parallel + \frac{1}{2} + \frac{i L_1}{2 L_2}, \frac{L_1}{2 L_2} \right) \theta\left( \frac{L_1}{2 L_2} \bar{x}_\parallel - \frac{1}{2} - \frac{i L_1}{2 L_2}, \frac{L_1}{2 L_2} \right),
\]  

\[
\theta\left( \frac{1}{2} + \frac{i L_1}{2 L_2}, \frac{i L_1}{2 L_2} \right) \theta\left( \frac{L_1}{2 L_2} y + \frac{1}{2} + \frac{i L_1}{2 L_2}, \frac{L_1}{2 L_2} \right) \theta\left( \frac{L_1}{2 L_2} \bar{x}_\parallel + \frac{1}{2} + \frac{i L_1}{2 L_2}, \frac{L_1}{2 L_2} \right) \theta\left( \frac{L_1}{2 L_2} \bar{x}_\parallel - \frac{1}{2} - \frac{i L_1}{2 L_2}, \frac{L_1}{2 L_2} \right)
\]

with \( \theta'(w, \tau) = \partial_w \theta(w, \tau) \). The corresponding result for \( G_+(z) \) can be obtained via \( G_+(z) = G_-^*(z) \). The zero modes have square-integrable singularities at both fluxes: near \( z \sim \omega \).
we have $|\psi^1|^2 \propto |y - \omega_1 - i\omega_2|^{2(\kappa-1)}$ and $|\psi^2|^2 \propto |y - \omega_1 - i\omega_2|^{-2\kappa}$. Near the other flux the singularity profiles of the two modes are exchanged in that $|\psi^1|^2 \propto |y + \omega_1 + i\omega_2|^{-2\kappa}$ and $|\psi^2|^2 \propto |y + \omega_1 + i\omega_2|^{2(\kappa-1)}$. By virtue of the second theta function in the numerator of (19) $\psi^2(z; x)$ has a single zero in $\mathbb{T}^2$ at $y = -\omega_1 - i\omega_2 - 2\pi x_\parallel/(L_1 L_2)$ (see also [12]). Similarly, $\psi^1(z; x)$ has a zero at $y = \omega_1 + i\omega_2 - 2\pi i x_\parallel/(L_1 L_2)$.

5 Soft Zero Modes

When $x_\parallel = 0$ the Green’s functions $G_\pm$ do not exist. This does not mean there are no zero modes, indeed one can see that

$$\psi^1(z; x = 0) = \begin{pmatrix} 0 \\ e^{-\phi(z)} \end{pmatrix}, \quad \psi^2(z; x = 0) = \begin{pmatrix} e^{\phi(z)} \\ 0 \end{pmatrix},$$

(20)

are solutions of the $x_\parallel = 0$ Weyl equation. $\psi^1$ has the expected square-integrable singularity at $z = -\omega$, but for $z = \omega$, $\psi^1(z; x = 0)$ is zero. On the other hand $\psi^2(z, x = 0)$ diverges at $z = \omega$ but not at $z = -\omega$. By choosing $x_\parallel = 0$ we have effectively moved the zeroes of the zero modes to flux locations. There is another way of doing this; choosing $x_\parallel = -iL_1 L_2(\omega_1 + i\omega_2)/\pi$ brings the zero of $\psi^1$ to the other flux (and similarly for $\psi^2$). Thus there are two $x_\parallel$ values where each Nahm zero mode diverges at only one flux. We wish to investigate whether these ‘soft’ points can be interpreted as the locations of some kind of constituent.

The singularity profiles of $\psi^1$ and $\psi^2$ are exchanged under the replacement $\kappa \to 1 - \kappa$ suggesting that the constituents are exchanged under this mapping. That is, if there are indeed lumps at the two points, then $\kappa \to 1 - \kappa$ swaps the two lumps. The following result formalises this idea

$$F_{\mu\nu}(x_\parallel, x_\perp, \kappa) = V^{-1}(x) F_{\mu\nu}(-x_\parallel + \frac{L_1 L_2}{i\pi}(\omega_1 + i\omega_2), -x_\perp, 1 - \kappa) V(x),$$

(21)

where $V(x)$ is some $U(2)$ gauge transformation. The proof of Eq. (21) goes as follows: make the change of variables $z \to -z$ in [8]. The zero mode $\psi^p(-z; x)$ satisfies the same Weyl equation as $\psi^p(z; x)$ except that the signs of $\kappa$ and the $x_\mu$ are flipped. However, $-\kappa$ does not lie between 0 and 1. Under periodic gauge transformations $\hat{A}(z, -\kappa)$ and $\hat{A}(z, 1 - \kappa)$ are equivalent up to a constant potential

$$\hat{A}_\mu(z, -\kappa) = U^{-1}(z) \left( \partial_\mu + \hat{A}_\mu(z, 1 - \kappa) + B_\mu \right) U(z),$$

(22)
where \( \pi B_1 = -iL_1L_2\omega_2 \) and \( \pi B_2 = iL_1L_2\omega_1 \). Explicitly

\[
U(z) = e^{i\frac{1}{2}\omega_1L_1L_2(y-y)/\pi} \times \left[ \frac{\theta \left( \frac{i\omega_2}{2\pi} (\bar{y} - \omega_1 + i\omega_2) + \frac{1}{2} - i\frac{\omega_1}{2L_2}, \frac{i\omega_1}{2L_2}, \frac{i\omega_2}{2L_2} \right) \theta \left( \frac{i\omega_2}{2\pi} y + \omega_1 + i\omega_2 + \frac{1}{2} + i\frac{\omega_1}{2L_2}, \frac{i\omega_1}{2L_2}, \frac{i\omega_2}{2L_2} \right)}{\theta \left( \frac{i\omega_2}{2\pi} (\bar{y} - \omega_1 + i\omega_2) + \frac{1}{2} + i\frac{\omega_1}{2L_2}, \frac{i\omega_1}{2L_2}, \frac{i\omega_2}{2L_2} \right) \theta \left( \frac{i\omega_2}{2\pi} y + \omega_1 - i\omega_2 + \frac{1}{2} - i\frac{\omega_1}{2L_2}, \frac{i\omega_1}{2L_2}, \frac{i\omega_2}{2L_2} \right)} \right]^{\frac{1}{2}}.
\]

In the Weyl equation \( B_\mu \) can be absorbed into \( x_1 \) and \( x_2 \). Thus \( U^{-1}(z)\psi^p(-x|| -iL_1L_2(\omega_1 + i\omega_2)/\pi, -x_|) \) satisfies the same Weyl equation as \( \psi^p(-z; x||, x_|) \).

6 Green’s Function Approach

For the \( T^4 \) transform, the original field strength can be written

\[
F^\mu\nu_{pq}(x) = \int_{T^4} d^4z \int_{T^4} d^4z' \psi^{p\dagger}(z;x)\sigma_\mu (D_\dagger_x D_x)^{-1}(z,z') \sigma_\nu \psi^q(z';x) - [\mu \leftrightarrow \nu],
\]

and as with the corresponding expression for \( \tilde{F}^\mu\nu_{pq}(z) \), equation (3), the self-duality of \( F^\mu\nu_{pq}(x) \) is manifest because the Green’s function \( (D_\dagger_x D_x)^{-1}(z,z') \) commutes with all \( \sigma_\mu \). Starting directly from the inverse Nahm transform \( \Psi \) we have

\[
F^\mu\nu_{pq}(x) = \int_{T^4} d^4z \partial_\mu \psi^{p\dagger}(z;x)\partial_\nu \psi^q(z;x) + \int_{T^4} d^4z \psi^{p\dagger}(z;x)\partial_\mu \psi^q(z;x) \int_{T^4} d^4z' \psi^{p\dagger}(z';x)\partial_\nu \psi^q(z';x) - [\mu \leftrightarrow \nu],
\]

where \( \partial_\mu = \partial/\partial x^\mu \). To get from (25) to (24) we require the derivative of the Nahm zero modes with respect to the coordinates of \( T^4 \)

\[
\frac{\partial}{\partial x^\mu} \psi^q = -i D_x(\hat{A}) \left( D_\dagger_x(\hat{A}) D_x(\hat{A}) \right)^{-1} \sigma_\mu \psi^q + \psi^q R^\mu_{pq}(x),
\]

for some \( N \times N \) matrix valued vector field on the four torus, \( R^\mu_{pq}(x) \). This clearly satisfies the equation obtained by differentiating the Weyl equation, \( D_\dagger_x(\hat{A}) \psi^q(z;x) = 0 \), with respect to \( x^\mu \). Multiplying (26) from the left with \( \psi^{p\dagger} \) and integrating over the dual torus yields

\[
R^{pq}(x) = A^{pq}(x),
\]

which together with equations (25) and (26) give (24).

Our analysis of doubly periodic instantons has been based on the Weyl equation and the inverse Nahm transform (8). These are the exact counterparts of four torus equations.
Therefore one might expect all the results quoted above to carry over directly to $\mathbb{T}^2 \times \mathbb{R}^2$. There are, however, a number of important differences between the two cases. With regard to the $\mathbb{T}^2 \times \mathbb{R}^2$ construction, the Green’s function $(D_x^\dagger(\hat{A})D_x(\hat{A}))^{-1}(z, z')$ does not commute with the $\sigma_\mu$ matrices. Away from the $N$ singularities of $\hat{A}(z)$, $D_x^\dagger(\hat{A})D_x(\hat{A})$ commutes with the $\sigma_\mu$. At the singularities it has source terms which are not proportional to $\sigma_0$. For the $U(1)$ Nahm potential considered in section 3 (as usual we set $\alpha = 0$)

$$D_x^\dagger D_x = -\sigma_0(D_x^\mu)^2 + 2\pi i k \sigma_3 \left[ \delta^2(z - \omega) - \delta^2(z + \omega) \right],$$

(28)

indicating that $(D_x^\dagger D_x)^{-1}$ does not commute with $\sigma_1$ and $\sigma_2$.

We will show that the naive $\mathbb{T}^2 \times \mathbb{R}^2$ limit of (24) does not apply, so that $(D_x^\dagger D_x)^{-1}$ not being proportional to $\sigma_0$ does not signal a breakdown of the inverse Nahm transform. Starting with the inverse transform [3], the field strength can be written in exactly the same way as in (25) except that the integral is over $\mathbb{T}^2$ rather than $\mathbb{T}^4$. The derivative formula (26) also applies without modification to the $\mathbb{T}^2$ zero modes. What does not carry through is equation (27). Much as in the four torus $R_\mu$ can be determined by multiplying the $\mathbb{T}^2 \times \mathbb{R}^2$ analogue of (26) by $\psi^{\dagger \mu}$ and integrating over $\mathbb{T}^2$.

$$A_{\mu}^{pq}(x) = -i \int_{\mathbb{T}^2} \psi^{\dagger \mu} D_x(\hat{A}) \left( D_x^\dagger(\hat{A})D_x(\hat{A}) \right)^{-1} \sigma_\mu^{\dagger \rho} \psi^{\rho} + R_{\mu}^{pq}. \quad (29)$$

Naively one would expect the integral on the right hand side to vanish as $\psi^{\dagger \mu}$ is a (left) zero mode of $D_x(\hat{A})$. However, this property only leads to vanishing $\int_{\mathbb{T}^2} \psi^{\dagger \mu} D_x(\hat{A}) \phi$ for sufficiently smooth functions $\phi$. In (29) $\psi^{\dagger \mu}$ and the Green’s function $(D_x^\dagger D_x)^{-1}$ have coincident singularities.

Retaining the integral yields a modified field strength formula

$$F_{\mu \nu}^{pq}(x) = \int_{\mathbb{T}^2} d^2 z \int_{\mathbb{T}^2} d^2 z' \psi^{\dagger \mu}(z; x) \sigma_\mu f(z, z'; x) \sigma_\nu^{\dagger} \psi^{\rho}(z'; x) - [\mu \leftrightarrow \nu], \quad (30)$$

with

$$f(z, z'; x) = (D_x^\dagger D_x)^{-1}(z, z') \quad (31)$$

This object does commute with the $\sigma_\mu$ consistent with self-duality. The field strength formula (30) is similar to those one can derive for $\mathbb{R}^4$ instantons and calorons via the ADHM formalism.
To make these considerations more concrete we see how they apply to our simple \( U(1) \) Nahm potential. In this case

\[
\frac{1}{4} D^\dagger_x D_x = \begin{pmatrix}
\frac{1}{4}|x_\perp|^2 + (-\partial_y - \partial_y \phi + \frac{i}{2} \bar{x}_\parallel)(\partial_y - \partial_y \phi - \frac{i}{2} x_\parallel) & 0 \\
0 & \frac{1}{4}|x_\perp|^2 + (-\partial_y + \partial_y \phi + \frac{i}{2} \bar{x}_\parallel)(\partial_y + \partial_y \phi - \frac{i}{2} x_\parallel)
\end{pmatrix}.
\]

(32)

When \( x \neq 0 \) this is invertible. The inverse has the form

\[
2 (D^\dagger_x D_x)^{-1} (z, z') = (\sigma_0 + \sigma_3) e^{-\phi(z)} K_+(z, z'; x) e^{-\phi(z')} + (\sigma_0 - \sigma_3) e^{\phi(z)} K_-(z, z'; x) e^{\phi(z')},
\]

(33)

where the \( K_\pm(z, z'; x) \) are finite for all \( z \) and \( z' \) (unless \( x_\parallel = x_\perp = 0 \)). Because of the exponentials \( (D^\dagger_x D_x)^{-1} (z, z') \) is singular if \( z \) or \( z' \) equals \( \pm \omega \). In the \( x_\parallel = 0 \) slice the \( K_\pm \) have simple integral representations

\[
K_\pm(z, z'; x_\parallel = 0) = \int_{\mathbb{T}^2} d^2 s \, G_\pm(z - s) e^{\pm 2\phi(s)} G_\mp(s - z'),
\]

(34)

where \( G_\pm \) are the Green’s functions defined in section 4. The \( x_\parallel = 0 \) zero modes of section 4 can be rewritten as follows

\[
\psi^1(z; x) = -D_x(\hat{A}) \begin{pmatrix} e^{\phi(z)} K_+(z, \omega; x) \\ 0 \end{pmatrix}
\]

(35)

\[
\psi^2(z; x) = -D_x(\hat{A}) \begin{pmatrix} 0 \\ e^{-\phi(z)} K_+(z, -\omega; x) \end{pmatrix}.
\]

In fact, these are also valid for \( x_\parallel \neq 0 \). The modes are orthogonal and can be normalised by dividing them by \( \sqrt{\rho} \), where

\[
\rho(x) = K_+(\omega, -\omega; x) = K_-(\omega, \omega; x).
\]

(36)

Inserting the zero modes into \( \[31] \) enables us to prove that \( f \) commutes with \( \sigma_1, \sigma_2 \). This is detailed in appendix \[A\]. One can also show that \( f(z, z'; x) \), unlike \( (D^\dagger_x D_x)^{-1}(z, z') \), is well behaved at \( x = 0 \). As \( z \) or \( z' \) approaches \( \pm \omega \), \( f \) tends to zero.

We have seen that it is possible to construct the Nahm zero modes out of objects, \( K_\pm \), entering the inverse of \( D^\dagger_x D_x \). The gauge potential can be computed from these zero modes. The final result is very simple

\[
A_{x_\parallel} = -\frac{\tau_3}{2} \partial_{x_\parallel} \log \rho - 2\pi i (\tau_1 - i\tau_2) \kappa \rho \partial_{x_\parallel} \frac{\nu^*}{\rho},
\]

(37)

\[
A_{x_\perp} = -\frac{\tau_3}{2} \partial_{x_\perp} \log \rho + 2\pi i (\tau_1 - i\tau_2) \kappa \rho \partial_{x_\parallel} \frac{\nu^*}{\rho};
\]
where
\[ \nu(x) = K_-(\omega, -\omega; x), \quad \nu^*(x) = K_-(\omega, \omega; x). \] (38)

Note that \( \rho \) is dimensionless, real and periodic (with respect to \( x_1 \to x_1 + L_1, x_2 \to x_2 + L_2 \)), while \( \nu \) is dimensionless, complex and periodic up to constant phases. Eq. (37) can be checked component-wise (see appendix B).

We have given closed forms for the \( K_{\pm} \) in the special case \( x_\perp = 0 \). An explicit construct valid beyond this two-dimensional slice would immediately provide the exact Nahm zero modes and gauge potential. Note that the Green’s functions \( K_{\pm} \) are radially symmetric. Therefore the functions \( \rho \) and \( \nu \) are as well. They can be expressed as power series
\[
\rho = \sum_{n=0}^{\infty} \rho_n |x_\perp|^{2n}, \quad \nu = \sum_{n=0}^{\infty} \nu_n |x_\perp|^{2n}. \quad (39)
\]

7 Field Strength Components

In section 5 we saw that the Nahm transform singled out two points in the \( x_\perp = 0 \) slice. This suggests that they may be the locations of some kind of constituent. As they are isolated points in \( \mathbb{T}^2 \times \mathbb{R}^2 \) rather than lines it appears that we are not dealing with monopole constituents. Indeed if we take (15) at face value the constituents appear to be BPST instantons, one with size \( \lambda = \sqrt{\kappa L_1 L_2/\pi} \) at \( x_\parallel = x_\perp = 0 \), the other with size \( \lambda = \sqrt{(1 - \kappa) L_1 L_2/\pi} \) at the second soft point. This description shares some features with early attempts to describe \( SU(2) \) instantons in terms of \( 2k \) ‘instanton quarks’ [13].

To see if this is realised we must compute the field strengths. This can be done explicitly in the \( x_\perp = 0 \) slice. Inserting the power series (39) into the ansatz (37) it is straightforward to compute \( F_{x_\parallel \bar{x}_\parallel} \) and \( F_{x_\parallel \bar{x}_\perp} \)
\[
F_{x_\parallel \bar{x}_\parallel} \bigg|_{x_\perp = 0} = \tau_3 \partial_{x_\parallel} \partial_{\bar{x}_\parallel} \log \rho_0, \quad F_{x_\parallel \bar{x}_\perp} \bigg|_{x_\perp = 0} = 2\pi i (\tau_1 + i \tau_2) \kappa \rho_0 \partial_{x_\parallel}^2 \frac{\nu_0}{\rho_0}. \quad (40)
\]
The other components are fixed by self-duality, i.e. \( F_{x_\parallel \bar{x}_\parallel} + F_{x_\parallel \bar{x}_\perp} = 0 \) and \( F_{x_\parallel \bar{x}_\perp} = 0 \). Both \( \rho_0 \) and \( \nu_0 \) diverge at \( x_\parallel = 0 \), but the field strengths should not. A careful analysis of \( \rho_0 \) and \( \nu_0 \) in the neighbourhood of \( x_\parallel = 0 \) shows that the field strengths are well defined at this point. Alternatively, one can just note that \( \rho \) and \( \nu \) are well behaved at the second soft point and invoke (21). Using these results the action density can be computed
\[
-\frac{1}{2} \text{Tr} F_{\mu \nu} F^{\mu \nu} \bigg|_{x_\perp = 0} = 16 \left( \partial_{x_\parallel} \partial_{\bar{x}_\parallel} \log \rho_0 \right)^2 + 128 \pi^2 \kappa^2 \rho_0^2 \left| \partial_{x_\parallel}^2 \frac{\nu_0}{\rho_0} \right|^2. \quad (41)
\]
From (34) one can derive integral representations of $\rho_0$ and $\nu_0$ where the integrands are expressible in terms of standard functions. These representations allow us to plot the action density within the $x_\perp = 0$ slice for different values of the parameters $\kappa$, $\omega_1$ and $\omega_2$. The two periods $L_1$ and $L_2$ can also be varied. However, the physically important quantity is the ratio of the lengths

$$a = \frac{L_1}{L_2},$$

(42)

since scaling identically both lengths equates to a trivial scaling of the action density:

$$\beta^4 \text{Tr} F^2(\beta x; \beta L_1, \beta L_2; \beta^{-1} \omega_1, \beta^{-1} \omega_2; \kappa) = \text{Tr} F^2(x; L_1, L_2; \omega_1, \omega_2; \kappa).$$

(43)

In [9] plots for different $\kappa$ in the equal length case, $a = 1$, were presented. There we found that the action density has either one or two peaks. A single peak is observed when the two soft points are quite close or when one size is somewhat larger than the other. In the latter case the smaller sized instanton dominates. Even when two peaks are resolved there is quite a strong overlap. In fact such an overlap is ‘necessary’ since otherwise the instanton would have charge two rather than one. The constituent locations then correspond to the ‘cores’ of the overlapping instantons.

Here we also consider rectangular tori with $a \neq 1$. Without loss of generality we set $L_2 = 1$ and $L_1 = a$. To begin with consider $\kappa = \frac{1}{2}$ where the two constituents are identical. Now choose $(\omega_1, \omega_2) = (\frac{1}{2} \pi a^{-1}, \frac{1}{2} \pi)$. This means that there is a core at the centre of the torus and another of equal size at the corners. Starting with $a = 1$ increase the value of $a$ thereby stretching the torus in the $x_1$ direction.

In fig. [1] the action density is plotted for some values of $a$ ranging from 1 to 3. The first plot ($a = 1$) is just the equal length case clearly showing two instanton-like peaks exactly at the expected locations. The second plot ($a = \frac{3}{2}$) shows that as the torus is stretched the two cores become stretched as well; they no longer have the approximate spherical symmetry of the $a = 1$ case. On stretching the torus further the now deformed cores seek to overlap with the periodic copies of themselves. This has already started to happen in the third plot where $a = 2$; the soft point locations are no longer maxima of the action density. Finally, at $a = 3$ the self-overlap has generated two monopole like objects; the solution has a very weak dependence on $x_2$ and there are peaks at two values of $x_1$. Note that the two monopole ‘worldlines’ do not cross the soft points. Increasing $a$ more weakens further the dependence on $x_2$. In the limit $a \to \infty$ a periodic monopole (i.e. a self dual solution on $S^1 \times \mathbb{R}^2$) will form. However, this is a singular limit.

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What is happening is that the doubly periodic instantons are becoming more caloron-like (caloron-like objects have been observed recently in lattice-based approaches \[14, 15\]). It is helpful to recall some of the basic features of the simplest caloron solutions. An \(\text{SU}(2)\) caloron with charge one comprises two monopole constituents separated by a distance \(\frac{\pi \lambda^2}{L_1}\) (taking \(x_2\) to be the compact coordinate). Here \(\omega_2\) (there is no \(\omega_1\)) determines the mass ratio of the two monopoles. For the \(\mathbb{T}^2 \times \mathbb{R}^2\) problem \(\pi \lambda^2 = \kappa L_1 L_2\) suggesting that the separation of the monopoles should be simply \(\kappa L_1\) for \(\kappa \leq \frac{1}{2}\) and \((1 - \kappa) L_1\) for \(\kappa \geq \frac{1}{2}\). In fig. 1 we took \(\kappa = \frac{1}{2}\) and the worldlines are indeed separated by a half-period. In fig. 2 we have taken \(\omega\) as above but increased \(\kappa\) to \(\frac{5}{8}\). The first plot is the equal length case showing
a peak at the centre of the torus. At the origin of the torus, the location of the larger constituent, there is a broader but much shallower peak. The second plot is for $a = 3$ which clearly shows two monopoles. Moreover, the distance between them is consistent with $\frac{3}{8}L_1$. Though the dependence on $x_2$ is not quite as weak as for the corresponding plot of fig. 1 away from $\kappa = \frac{1}{2}$ more stretching is required to access the monopole regime.

Figure 2: action density $-\frac{1}{2}\text{Tr} F^2$ for $x_\perp = 0$, $\kappa = \frac{5}{8}$ and $\omega_1 = \frac{1}{2}\pi, \omega_2 = \frac{1}{2}\pi a$ with $a = 1, 3$.

We have argued that for $a$ at or near to 1 the constituent core picture best describes the action density while for larger (or smaller) $a$ almost static monopole-like objects form. In that respect, the instanton core and monopole constituent pictures are complementary. However, we have an example displaying features of both descriptions:

Figure 3: Plot of action density for $x_\perp = 0$, $\kappa = \frac{1}{2}$, $a = 1$ and $\omega_1 = \frac{1}{2}\pi, \omega_2 = 0$. 
Above we plot the action density for $\kappa = \frac{1}{2}$, $\omega_1 = \frac{1}{2}\pi$ and $\omega_2 = 0$. This configuration shows essentially no dependence on $x_2$ even though the lengths are equal. What seems to have occurred is that the two cores at $x_{\parallel} = 0$ and $x_{\parallel} = \frac{1}{2}i$ have merged into a single monopole. Unlike in the previous examples the monopole worldline passes through the soft points. That there is only one worldline is not inconsistent with the constituent monopole picture since by virtue of $\omega_2 = 0$ the ‘second’ monopole should be massless. The monopoles observed in figs. 1 and 2 have equal masses in accord with the symmetric value of $\omega_2$.

8 Charge One Half

When $\kappa = \frac{1}{2}$ the two instanton cores are identical. This has an interesting consequence. If we choose the constituent locations so that they are separated by half periods the charge one instanton can be ‘cut’ to yield two copies of a charge $\frac{1}{2}$ instanton (see also [16]). This happens when $(\omega_1, \omega_2)$ is $(\frac{1}{2}\pi/L_1, 0)$, $(0, \frac{1}{2}\pi/L_2)$ or $(\frac{1}{2}\pi/L_1, \frac{1}{2}\pi/L_2)$, see fig. 4. After cutting we have a twist $Z_{12} = -\mathbb{1}$ in the half torus. But to produce a half integer topological charge another (non-orthogonal) twist is required; this is most simply achieved with $Z_{03} = -\mathbb{1}$. Such a twist would have the novel feature of being associated with the non-compact $x_0$ and $x_3$ directions. Far away from $x_{\perp} = 0$ the potential must be pure gauge

$$A_\mu(x) \sim V^{-1}(x_1, x_2, \theta) \partial_\mu V(x_1, x_2, \theta), \quad r \to \infty,$$

where $x_{\perp} = re^{i\theta}$. The non-compact twist translates into a double valued gauge function,

$$V(x_1, x_2, \theta + 2\pi) = -V(x_1, x_2, \theta).$$

The examples considered in fig. 4 are all ‘doubled’ half instantons. Here the cut tori are diamond shaped (see fig. 4) apart from the first ($a = 1$) plot, where it is square. This case has another interesting feature. Compare the constituent locations with the half instanton obtained via $L_1 = \sqrt{2}$, $L_2 = 1/\sqrt{2}$, $\omega_1 = 0$, $\omega_2 = \pi/\sqrt{2}$. On cutting both yield charge $\frac{1}{2}$ instantons on square tori with length $1/\sqrt{2}$. By analogy with the four torus case, for a given set of twists and periods, charge $\frac{1}{2}$ instantons are expected to be unique up to translations. Yet these two cases have different values of $a$, one has $a = 1$ where the core picture should work while the other ($a = 2$) is expected to have some monopole characteristics. As can clearly be seen from the first plot in fig. 4 the action density has maxima at the soft points. We have computed the $a = 2$ case and found maxima not coincident with the soft points. The soft points correspond to saddle points of the action density. But allowing for translations the two half instantons are identical. What seems to be happening is that when $a = 2$ the
monopoles are still not developed, and in this intermediate state there are lumps of action density which while not at the soft points are identical to those at the soft points of the $a=1$ case. More precisely the two action densities satisfy

$$
\text{Tr} \, F^2_{a=2}(x_1, x_2) = \text{Tr} \, F^2_{a=1}\left(\frac{1}{\sqrt{2}}(x_1 + x_2) - \frac{1}{4}, \frac{1}{\sqrt{2}}(x_2 - x_1) - \frac{1}{4}\right).
$$

The $a=1$ plot is obtained by a 45 degree rotation of the $a=2$ plot followed by a translation. This reflects the fact that the two Nahm potentials are also related by a 45 degree rotation.

### 9 Asymptotics

In this section we consider the large $|x_\perp|$ limit of the zero mode equations, and deduce the large $|x_\perp|$ behaviour of the gauge potential and field strength. This allows us to check that the charge $\frac{1}{2}$ instantons have the expected properties. In particular, we see that the gauge potential has asymptotic behaviour commensurate with a twist, $Z_{03} = -I$. Furthermore, the exponential decay of the action density is derived. In fact, all the $\kappa = \frac{1}{2}$ solutions have both the twist and exponential decay even if they are not ‘doubled’ $\frac{1}{2}$ instantons.
When $|x_\perp|$ is much larger than the two periods the zero modes of $D_x^\dagger (\hat{A})$ become strongly localised about the two flux singularities. In this regime we can approximate the zero modes with $\mathbb{R}^2$ solutions. A single flux in $\mathbb{R}^2$ corresponds to $\phi = -\frac{1}{2} \kappa \log y\bar{y}$. The Weyl operator is then
\[ -\frac{i}{2} D_x^\dagger = \begin{pmatrix} \frac{1}{2} \bar{x}_\perp & \partial_y - \frac{\kappa}{2y} - \frac{i}{2} \bar{x}_\parallel \\ \partial_y + \frac{\kappa}{2y} - \frac{i}{2} \bar{x}_\parallel & \frac{1}{2} \bar{x}_\perp \end{pmatrix}. \] (46)

When $x_\perp \neq 0$, $D_x^\dagger$ has a normalisable zero mode. In the special cases $\kappa = \pm \frac{1}{2}$ the normalised modes take rather simple forms:
\[ \psi_\frac{1}{2} (z) = \frac{e^{ix \cdot z - |x_\perp| \sqrt{y\bar{y}}}}{\sqrt{2\pi |x_\perp|}} \left( \begin{pmatrix} (y\bar{y})^{-\frac{1}{4}} x_\perp \\ (y\bar{y})^{\frac{1}{4}} |x_\perp| \end{pmatrix} \right), \quad \psi_{\frac{-1}{2}} (z) = \frac{e^{ix \cdot z - |x_\perp| \sqrt{y\bar{y}}}}{\sqrt{2\pi |x_\perp|}} \left( \begin{pmatrix} (y\bar{y})^{\frac{1}{4}} x_\perp \\ (y\bar{y})^{-\frac{1}{4}} |x_\perp| \end{pmatrix} \right), \] (47)
where $x \cdot z = x_1 z_1 + x_2 z_2$. For sufficiently large $|x_\perp|$ we can take
\[ \psi^1(z) = \psi_\frac{1}{2} (z - \omega), \quad \psi^2(z) = \psi_{\frac{-1}{2}} (z + \omega), \] (48)
and extend the integration region from $\mathbb{R}^2$ to $\mathbb{T}_x^2$. For this approximation to work it is important that the flux separation $2|\omega|$ is less than both dual periods, $2\pi/L_1$ and $2\pi/L_2$. This is because we assume that the ‘interference’ between the fluxes dominates the effect of the periodicity for large $|x_\perp|$. Note that
\[ \int_{\mathbb{R}^2} d^2 z \psi^\dagger_\frac{1}{2} (z - \omega) \psi_{-\frac{1}{2}} (z + \omega) = 0. \] (49)

Inserting the zero modes into the inverse Nahm transform produces a $U(2)$ potential; we discard the pure gauge $U(1)$ part. The remaining $SU(2)$ piece turns out to be in a different gauge to that implicit in section 6. An explicit computation (see appendix C) shows that the $SU(2)$ potential has asymptotics corresponding to a twist and the action density has the exponential decay
\[ -\frac{1}{2} \text{Tr} F_\mu F_\mu \sim \frac{32|\omega|^3}{\pi |x_\perp|} e^{-4|\omega||x_\perp|}. \] (50)

We can apply this to the charge $\frac{1}{2}$ case. Consider such an instanton in a square box of length $L$. This is a one instanton in the doubled box with $L_1 = L$, $L_2 = 2L$. Here $\omega_1 = \pi/(2L)$, $\omega_2 = 0$, and so the $\frac{1}{2}$ instanton has the fall-off
\[ -\frac{1}{2} \text{Tr} F_\mu F_\mu \sim \frac{4\pi^2}{L^3 |x_\perp| L} e^{-2\pi |x_\perp|/L}. \] (51)
For a rectangular box simply replace $L$ with the longest length. In [3] a decay exponent of 6.5 was reported for a unit box. This is within 5 percent of the analytic value $2\pi$. The decay formula is reminiscent of similar results for Bogomolnyi vortices in the abelian Higgs model. There the decay exponent is straightforward to obtain while an analytic computation of the prefactor is still an open problem (see however [17, 18]).

One can repeat the analysis for $\kappa \neq \frac{1}{2}$. The corresponding $\mathbb{R}^2$ zero modes are more complicated; they can be written in terms of modified Bessel functions. Here the asymptotics of the gauge potential does not correspond to a four torus twist. More precisely, $\rho$ has the power decay

$$\rho = \frac{C}{|x_\perp|^{2\kappa}} + \text{exponentially decaying remainder,}$$

where $C$ is a (dimensionful) constant. This translates into a twist only if $\kappa = \frac{1}{2}$.

The question of the form of the decay was an important consideration in [7]. There it was assumed that doubly periodic instantons have quadratically decaying field strength components. This translates into a quartic decay of the action density. Yet the solutions we have discussed show a much more rapid exponential decay. This is not necessarily a contradiction since our discussion of the asymptotics was confined to the radial solutions whereas [7] was concerned with the whole charge one sector. The more generic non-radial solutions may decay quartically. Indeed the two monopole constituents are expected to be separated in the non-compact directions. This can give rise to an algebraic tail in the action density.

### 10 Outlook

We have developed an instanton core picture of the simplest doubly periodic instantons. Numerical calculations of the action density correlate well with this picture for square tori. However, in rectangular tori the solutions become caloron-like when the ratio of the two periods is increased; the cores merge with periodic copies of themselves in the short direction to form monopole-like tubes of action density. The basic properties of these monopoles, such as their mass ratios and spatial separation, follow the pattern of the charge one $SU(2)$ calorons. It would be interesting to develop this monopole constituent description in a more direct manner. One approach would be to exploit the fact that the radially symmetric solutions considered here fall into a class of axially-symmetric multi-calorons discussed recently [19] if one allows for infinite topological charge.
There are a number of obvious ways to extend the results in this paper. An explicit treatment of the $x_\perp \neq 0$ zero mode equation is still lacking. In the non-radially symmetric case ($\alpha \neq 0$) the zero mode equations have not yielded solutions for any points in $\mathbb{T}^2 \times \mathbb{R}^2$. Even in the absence of explicit zero modes it might be possible to extract some information about possible instanton core or monopole constituents. A more straightforward extension would be to generalise the results regarding radially symmetric one instantons to $SU(N)$. That these solutions decay exponentially indicates they can be compactified further to $\mathbb{C}^3 \times \mathbb{R}$ or even $\mathbb{C}^4$. The latter is only possible for the $\kappa = \frac{1}{2}$ case where the asymptotics of the gauge field correspond to a twist.

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A Properties of $f$

Inserting the zero modes (35) into the definition of $f$ (31) yields

$$f(z, z'; x) = \frac{1}{2}(\sigma_0 + i\sigma_3)e^{-\phi(z)}g_+(z, z'; x)e^{-\phi(z')} + \frac{1}{2}(\sigma_0 - i\sigma_3)e^{\phi(z)}g_-(z, z'; x)e^{\phi(z')} ,$$

(53)

where

$$g_\pm(z, z'; x) = K_\pm(z, z'; x) - \frac{1}{\rho(x)}K_\pm(z, \mp \omega; x)K_\pm(\mp \omega, z'; x).$$

(54)

It is not obvious that $f$ commutes with $\sigma_1$ and $\sigma_2$. This only holds if the coefficients of the projectors $\sigma_0 \pm i\sigma_3$ match, that is if

$$e^{-\phi(z)}g_+(z, z'; x)e^{-\phi(z')} = e^{\phi(z)}g_-(z, z'; x)e^{\phi(z')} .$$

(55)

A proof of (55) for $x_\perp = 0$ was given in [8]. It is sufficient to show that the left and right hand sides of (55) have the same asymptotics at the fluxes since when $z, z' \neq \pm \omega$ they satisfy the same differential equation. Consider $g_\pm(z, z'; x)$ in the neighbourhood of $z = \omega$. It follows immediately from (54) that $g_-(\omega, z'; x) = 0$ whereas $g_+(\omega, z'; x)$ is non-zero. This does not contradict (55) since $e^{\phi(z)}$ diverges ($\propto |y - \omega_1 - i\omega_2|^\kappa$) at $z = \omega$. A short calculation gives

$$g_-(z, z'; x) \sim \frac{|y - \omega_1 - i\omega_2|^\kappa}{4\pi \kappa \rho(x)}K_-(\omega, z'; x) .$$

(56)
This is compatible with (55) if

\[ g_+(\omega, z'; x) = \frac{e^{2\phi(z')}}{4\pi \kappa \rho(x)} K_-(\omega, z'; x). \]  

(57)

To check this, note that for \( z' \neq \pm \omega \) both sides satisfy the same differential equation and then examine them in the neighbourhoods of \( z' = \pm \omega \). From this discussion of the asymptotics of \( g_\pm \) it should be clear that \( f(z, z'; x) \), unlike \( (D^\dagger_x D_x)^{-1}(z, z') \) is always finite. Indeed \( f(z, z'; x) \) tends to zero as \( z \) or \( z' \) approaches a flux point.

**B Computation of (37)**

Here we outline the computation of \( A_{x||} \); the other components can be dealt with in much the same way. Inserting the zero modes into the inverse Nahm transform yields

\[ A_{x||}^{11} = \frac{1}{\sqrt{\rho}} \int_{\mathcal{T}^2} d^2 z K_-(\omega, z; x) e^{\phi(z)} \left(D^\dagger_x (\hat{A}) \partial_{x||} D_x (\hat{A})\right)^{11} \frac{e^{\phi(z)}}{\sqrt{\rho}} K_-(\omega, z; x). \]

(58)

Using \( (D^\dagger_x \partial_{x||} D)^{11} = \partial_{x||} (D^\dagger_x D)^{11} \) and the representation (33) of \( (D^\dagger_x D)^{-1} \)

\[ A_{x||}^{11} = \frac{1}{\sqrt{\rho}} \int_{\mathcal{T}^2} d^2 z K_-(\omega, z; x) \partial_{x||} \delta^2(z - \omega) \frac{1}{\sqrt{\rho}} = -\frac{1}{2} \partial_{x||} \log \rho, \]

(59)

as required. For \( A_{x||}^{21} \) we proceed similarly

\[ A_{x||}^{21} = \frac{1}{\sqrt{\rho}} \int_{\mathcal{T}^2} d^2 z K_+(\omega, z; x) e^{-\phi(z)} \left(D^\dagger_x (\hat{A}) \partial_{x||} D_x (\hat{A})\right)^{21} \frac{e^{\phi(z)}}{\sqrt{\rho}} K_-(\omega, z; x) \]

(60)

\[ = \frac{i x_\perp}{\rho} \int_{\mathcal{T}^2} d^2 z K_+(\omega, z; x) K_-(z, \omega; x). \]

The key step is to use (57)

\[ A_{x||}^{21} = 4\pi i \kappa x_\perp \int_{\mathcal{T}^2} d^2 z K_+(\omega, z; x) e^{-2\phi(z)} g_+(z, \omega; x). \]

(61)

Now use (54) to write \( g_+ \) in terms of \( K_+ \) and the formula

\[ \int_{\mathcal{T}^2} d^2 s K_+(z, s; x) e^{-2\phi(s)} K_+(s, z'; x) = -\frac{\partial}{\partial |x_\perp|^2} K_+(z, z'; x), \]

(62)

to get the result \( A_{x||}^{21} = -4\pi i \kappa \partial_{x_\perp} (\nu^*/\rho) \).

21
C Exponential Decay of the Action Density

We consider the \( A_{x\perp} \) derived from the asymptotic zero modes given in section 9. A straightforward calculation gives

\[
A_{x\perp}^{11} - A_{x\perp}^{22} = 0.
\]  

(63)

The off-diagonal components are more involved:

\[
A_{x\perp}^{21} = \int_{\mathbb{R}^2} d^2z \, \psi_+^\dagger (z + \omega) \frac{\partial}{\partial x_{\perp}} \psi_- (z - \omega) = \frac{\hat{x}_{\perp}}{2\pi|x_{\perp}|} e^{2ix\cdot\omega}(I + J).
\]  

(64)

Here

\[
I = \int_{\mathbb{R}^2} d^2z \, \frac{(y_2\bar{y}_2)^{\frac{1}{4}}}{\bar{y}_1^2} (y_1\bar{y}_1)^{-\frac{1}{4}} e^{-|x_{\perp}|(\sqrt{\rho y_1} + \sqrt{\rho y_2})}
\]  

and

\[
J = \frac{1}{2} \int_{\mathbb{R}^2} d^2z e^{-|x_{\perp}|(\sqrt{\rho y_1} + \sqrt{\rho y_2})}
\]

\[
\times \left[ (y_2\bar{y}_2)^{-\frac{1}{4}} \frac{(y_1\bar{y}_1)^{\frac{1}{2}}}{\bar{y}_1} - |x_{\perp}|(y_2\bar{y}_2)^{\frac{1}{2}} (y_1\bar{y}_1)^{\frac{1}{2}} - |x_{\perp}|(y_2\bar{y}_2)^{-\frac{1}{4}} (y_1\bar{y}_1)^{\frac{3}{4}} \right],
\]

where \( y_1 = y - \omega_1 - i\omega_2 \) and \( y_2 = y + \omega_1 + i\omega_2 \). As \( |x_{\perp}| \) is increased the integral \( I \) is dominated by a small neighbourhood about the line-segment joining the two fluxes. For example, if \( \omega_2 = 0 \) we have

\[
e^{-|x_{\perp}|(\sqrt{\rho y_1} + \sqrt{\rho y_2})} \sim \pi^{\frac{1}{2}} |x_{\perp}|^{-\frac{1}{2}} |\omega_1|^{-\frac{1}{2}} (\omega_1^2 - z_1^2)^{\frac{1}{2}} e^{-2|x_{\perp}||\omega_1|} \delta(z_1)
\]

for \(-\omega_1 < z_1 < \omega_1\). Integrating along the line-segment gives

\[
I \sim \frac{2\pi^{\frac{1}{2}} |\omega|^{-\frac{1}{2}} |x_{\perp}|^{-\frac{1}{2}}}{\omega_1 - i\omega_2} e^{-2|\omega||x_{\perp}|}.
\]  

(67)

\( J \) exhibits a similar exponential decay. \( A_{x\perp}^{21} \) is simply \( x_{\perp} e^{2ix\cdot\omega}/(2\pi|x_{\perp}|) \), and so

\[
F_{x_{\perp}1}^{21} = -\frac{1}{2\pi|x_{\perp}|} e^{2ix\cdot\omega} \partial_{\perp} x (\vec{x}_1 I) \sim e^{2ix\cdot\omega} e^{-2|\omega||x_{\perp}|} \frac{(\omega_1 + i\omega_2)|\omega|^{\frac{1}{2}}}{\pi^{\frac{1}{2}} |x_{\perp}|^{\frac{1}{2}}}
\]

(68)

We also have

\[
F_{x_{\perp}1}^{11} = O(e^{-4|\omega||x_{\perp}|}).
\]  

(69)

Now compare with the corresponding field strengths generated by the ansatz (37) in section 6

\[
F_{x_{\perp}1}^{21} = 2\pi i \rho \partial_{\parallel} \nu^* \rho,
\]

\[
F_{x_{\perp}1}^{11} = \partial_{x_{\perp}} \log \rho + (2\pi \rho)^2 \partial_{x_{\parallel}} \nu^* \rho.
\]  

(70)

Consider

\[
\rho = \frac{C}{|x_{\perp}|^t}, \quad \nu^* = \frac{1}{2\pi} e^{2i\omega \cdot x} e^{-2|\omega||x_{\perp}|^t} |\omega|^{-\frac{1}{2}} |x_{\perp}|^{-\frac{1}{2}},
\]

(71)
where \( t \) is some real number and \( C \) is a constant. Neglecting sub-leading terms the expressions for \( F_{x_{\perp}\bar{x}_{\perp}} \) agree up to a phase, \( \sqrt{x_{\perp}/\bar{x}_{\perp}} = e^{i\theta} \), which can be compensated for by the double-valued gauge transformation \( V = e^{i\tau_3} \). Under this gauge transformation the diagonal part of \( A_{x_{\perp}} \) becomes non-trivial
\[
A_{x_{\perp}} = \frac{\tau_3}{4x_{\perp}} + O(e^{-4|x_{\perp}|}),(72)
\]
which agrees with (67) if \( t = 1 \) in (71). This is exactly what is required to generate the twist \( Z_{03} = -\Pi \). Using the large \( |x_{\perp}| \) form of \( \rho \) and \( \nu \) it is straightforward to obtain the decay formula (50).

References


