Conformal Einstein equations and Cartan conformal connection

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Abstract

Necessary and sufficient conditions for a space-time to be conformal to an Einstein space-time are interpreted in terms of curvature restrictions for the corresponding Cartan conformal connection.
In Ref. [1] we gave necessary and sufficient conditions for a 4-dimensional metric to be conformal to an Einstein metric. One of these conditions, the vanishing of the Bach tensor of the metric, has been discussed by many authors [1, 2, 3, 4]. In particular, it was interpreted as being equivalent to the vanishing of the Yang-Mills current of the corresponding Cartan conformal connection. The other condition, which is given in terms of rather complicated equation on the Weyl tensor of the metric has not been analyzed from the point of view of the corresponding Cartan conformal connection. The purpose of this letter is to fill this gap.

Let $M$ be a 4-dimensional manifold equipped with the conformal class of metrics $[g]$. Here we will be assuming that $g$ has Lorentzian signature, but our results are also valid in the other two signatures.

Given a conformal class $[g]$ on $M$ we choose a representative $g$ for the metric. Let $\theta^\mu$, $\mu = 1, 2, 3, 4$, be a null (or orthonormal) coframe for $g$ on $M$. This, in particular, means that $g = \eta_{\mu\nu}\theta^\mu\theta^\nu$, with all the coefficients $\eta_{\mu\nu}$ being constants. We define $\eta^{\mu\nu}$ by $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$ and we will use $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to rise and lower the Greek indices, respectively. The metric

In the following we will also need 1-forms

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Remarks

- Conditions (i) and (ii) are independent. In particular, metrics with vanishing Bach tensor and not conformal to Einstein metrics are known [5].
- If $C^2 = 0$ the condition (ii) must be replaced by another condition for the above theorem to be true. This another condition depends on the algebraic type of the Weyl tensor and is given in [1].
- Baston and Mason [3] gave another version of the above theorem in which condition (ii) was replaced by the vanishing of a different tensor than $N_{\nu\rho\sigma}$. Unlike $N_{\nu\rho\sigma}$, which is cubic in the Weyl tensor, the Baston-Mason tensor $E_{\nu\rho\sigma}$, is only quadratic in $C^\mu_{\nu\rho\sigma}$.
- Merkulov [4] interpreted condition (ii) as the vanishing of the Yang-Mills current of the Cartan normal conformal connection $\omega$ associated with the metric $g$. Following him, Baston and Mason [3] interpreted the condition $E_{\nu\rho\sigma} = 0$ in terms of curvature condition for $\omega$.

Although in the context of Theorem 1 conditions (ii) and $E_{\nu\rho\sigma} = 0$ are equivalent, the tensors $N_{\nu\rho\sigma}$ and $E_{\mu\nu\rho}$ are quite different. In addition to cubic versus quadratic dependence on the Weyl tensor, one can mention the fact that it is quite easy to express tensor $E_{\nu\rho\sigma}$ in the spinorial language and quite complicated in the tensorial language. Totally opposite situation occurs for the tensor $N_{\nu\rho\sigma}$. One of the motivation for the present letter is the existence of the normal conformal connection interpretation for the condition $E_{\nu\rho\sigma} = 0$. As far as we know such interpretation of $N_{\nu\rho\sigma} = 0$ has not been discussed. To fill this gap we first give the formal definition of the Cartan normal conformal connection. In order to do this we first, introduce the $6 \times 6$ matrix
The curvature of $\omega$ is

$$R = d\omega + \omega \wedge \omega$$

and has the rather simple form

$$R = \begin{pmatrix}
0 & (D\tau_\mu)' \\
0 & C^{\nu\mu} \\
0 & 0
\end{pmatrix}$$

with the 2-forms $C^{\nu\mu}$ and $(D\tau_\mu)'$ defined by

$$C^{\nu\mu} = \Lambda^{-1\rho} \Lambda^\rho C^{\nu\sigma} \Lambda^\sigma_{\mu}$$

$$(D\tau_\mu)' = e^{\phi} D\tau_\nu \Lambda^\nu_{\mu} - \xi\Lambda^{-1\rho} C^{\rho\nu} \Lambda^\nu_{\mu}.$$  \hspace{1cm} (2)

Similarly to the properties of $\omega$, the curvature $R$ can be used to extract the transformations of $D\tau_\mu$ and $C^{\nu\mu}$ under the conformal rescaling of the metrics. If $g \to g' = e^{-2\phi} g$ these transformations are given by (2) with $\xi = -\nabla'_\mu \phi$. In particular, if we freeze the Lorentz transformations of the tetrad, $\Lambda^\mu_{\nu} = \delta^\mu_{\nu}$, then we see that the Weyl 2-forms $C^{\nu\mu}$ constitute the conformal invariant.

The curvature $R$ of the Cartan normal connection $\omega$ is horizontal which, in other words, means that it has only $\theta^\mu \wedge \theta^\nu$ terms in the decomposition onto the basis of forms $(\theta^\mu, d\phi, \Lambda^{-1\rho}_\mu d\Lambda^\rho_{\nu}, d\xi_{\mu})$. Thus, the Hodge $\ast$ operator associated with $g$ on $M$ is well defined acting on $R$ and in consequence the Yang-Mills equations for $R$ can be written.

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**References**


