Form-field Gauge Symmetry in M–theory

J. Kalkkinen$^1$ and K.S. Stelle$^2$ §

$^1$ Institut des Hautes Études Scientifiques, Le Bois-Marie
35, Route de Chartres, Bures-sur-Yvette F-91440, France
$^2$ The Blackett Laboratory, Imperial College, London SW7 2AZ, UK

Abstract: We show how to cast an interacting system of M–branes into manifestly gauge-invariant form using an arrangement of higher-dimensional Dirac surfaces. Classical M–theory has a cohomologically nontrivial and noncommutative set of gauge symmetries when written using a “doubled” formalism containing 3-form and 6-form gauge fields. We show how the arrangement of Dirac surfaces allows an integral subgroup of these symmetries to be preserved at the quantum level. The proper context for discussing these large gauge transformations is relative cohomology, in which the 3-form transformation parameters become exact when restricted to the five-brane worldvolume. This structure yields the correct lattice of M-theory brane charges.

1 Introduction

The bosonic sector of $D = 11$ supergravity is derived from the action

The form-field $C_3$ has the field equation

Rewriting the $C_3$ field equation as $d(*G_4 + \frac{1}{2} C_3 \wedge G_4) = 0$, notice that one can introduce a dual field strength

This doubled system has a noncommutative ring of large gauge transformations:

$$\delta C_3 = \Lambda_3 , \quad \delta \tilde{C}_6 = \Lambda_6 - \frac{1}{2} \Lambda_3 \wedge C_3$$

$$[\delta_{\Lambda_3}, \delta_{\Lambda'_3}] = \delta_{\Lambda_6} \quad \text{with} \quad \Lambda_6 = \Lambda_3 \wedge \Lambda'_3$$

$$[\delta_{\Lambda_3}, \delta_{\Lambda_6}] = [\delta_{\Lambda_6}, \delta_{\Lambda'_3}] = 0 .$$

This is the cohomology ring for 3-forms and 6-forms on the underlying spacetime. These cohomologies are taken for the time being to be defined over the real numbers, but they will soon be restricted to integral cohomologies when we consider the corresponding Dirac quantization conditions.

§corresponding author : k.stelle@imperial.ac.uk
In the rest of this article, which is based on [?], we will investigate the way in which the algebra (1) is preserved a) in the presence of 2–branes and 5–branes, and b) at the quantum level.

2 Current couplings to 2–branes and 5–branes

For an $M_2$–brane worldvolume $W_3$ ending on an $M_5$–brane worldvolume $W_6$, one has $\partial W_3 \neq 0$, so the basic $M_2$ coupling $\int_{W_3} C_3$ fails to be gauge invariant even for small gauge transformations $\Lambda_3 = d\lambda_2$. The cure for this problem is provided by a form-field that exists on the $W_6$ worldvolume: the self-dual 3–form $h_3$, which has a potential $b_2$. Using the latter, one can take the combination $\int_{W_3} C_3 - \int_{\partial W_3} b_2$, which is invariant under small gauge transformations when taken together with a compensating Green-Schwarz mechanism, $\delta C_3 = d\lambda_2$, $\delta B_2 = \lambda_2$.

Note that a self-dual 3–form field strength is precisely what is needed in order to complete the bosonic part of the $(2,0)$ worldvolume fluctuation-field supermultiplet. The transverse oscillations of the 5–brane provide 5 worldvolume scalar bosonic degrees of freedom, while the 16 broken supersymmetries contribute 8 worldvolume fermionic degrees of freedom (taking into account that the fermionic equations of motion are of first-order). Thus, in order to have a bose–fermi balance on the $W_6$ worldvolume, one needs to have an additional 3 bosonic degrees of freedom. This is what is contributed by the self-dual 3-form, which contributes precisely $\frac{1}{2}(4 \cdot 3/2) = 3$ degrees of freedom.

The gauge-invariant field strength for the $C_3$ gauge field is accordingly the bulk $\oplus$ worldvolume combination

The action for the $M_2$, $M_5$ system [?] can be written

In order to determine the values of the coefficients in (??–??), one has to impose the requirements of gauge invariance but also take into account the fact that the “magnetically” charged 5–brane and the string boundary of the 2–brane give rise to violations of the normal Bianchi identities for the corresponding bulk and worldvolume form-fields.

3 Relative homology and cohomology, Bianchi identities and gauge invariance

In order to express the violated Bianchi identities compactly, it is convenient to use the language of relative cohomology. Consider a pair of form-fields of adjacent rank, $(C_k, C_{k-1})$. The first element in a pair is taken to be valued in the bulk spacetime, but the second element is taken to be valued in the subspace with respect to which the relative cohomology is being defined, in this case the 5–brane worldvolume $W_6$. Using the pull-back $i^*$ from the bulk spacetime to the worldvolume $W_6$ as above, one can define the relative exterior derivative (or coboundary operator) as

The pair of field strengths $(G_4, h_3)$, valued respectively in the bulk and in the worldvolume $W_6$, is thus given locally in terms of the exterior derivative (??): $(G_4, h_3) = d(C_3, b_2)$.

\footnote{For some original applications of relative cohomology to branes, see [?, ?].}
so the naïve Bianchi identity is \( d(G_4, h_3) = 0 \). In the presence of magnetically charged sources, however, this must be violated on the corresponding source loci:

After some analysis [?], the various conditions for small gauge invariance, taken together with the form (??) of the violated Bianchi identities plus use of the field equations yield the following relationships between the coefficients in (??–??):

\[
T_3 = -a = 2k \\
T_6 = -2e = 6f \\
T_{2\rightarrow 6} = \frac{k}{e} = \frac{T_3}{T_6} \\
b = 0 .
\]

These relations are fully consistent with the “brane surgery” relations [?, ?] expected on the basis of charge conservation for a \((q-1)\) brane intersecting a \((p-1)\) brane over a \((k-1)\) brane,

In addition to the coefficient relations (2–5), one obtains also the following homology relations from the gauge invariance requirements:

Homology relations are of course dual to cohomology relations, and so one has an appropriate boundary relation that is dual to the relative exterior derivative/coboundary operator defined in (??). For cycles \((W_k, W_{k-1})\) in \((X, W_6)\) one has the relative boundary relation

Pairs of cycles and pairs of forms can be integrated as follows:

Relative cohomology language can now be used to describe the sense in which the gauge transformations (1) remain “large” in the presence of 2–branes and 5–branes. The transformation parameters \((\Lambda_3, \lambda_2)\) for the form-fields \((C_3, b_2)\) are taken to be elements of \(H^3(X, W_6; \mathbb{R})\). Thus, \(\Lambda_3\) may remain cohomologically nontrivial on \(X - W_6\), but it must reduce to an exact form \(d\lambda_2\) when restricted to the 5–brane worldvolume \(W_6\).

4 Dirac-Schwinger-Zwanziger quantization relations

The above real relative homology and cohomology groups become restricted to integral subgroups when quantum effects are taken into account. The basic requirement is that adiabatic deformations of a dual pair of electric and magnetic solitons through a closed deformation path should not produce any change in the quantum generating functional path-integral. Thus, variations of the action \(I\) by amounts \(2\pi k, k \in \mathbb{Z}\) are allowed since the integrand \(\exp(iI)\) becomes multiplied in that case just by \(\exp(2\pi ik) = 1\). A Wu-Yang style argument [?] then shows that if an \(M_2\) brane worldvolume \(W_3\) is deformed through a
closed path $\Sigma_4$ around an $M_5$ brane worldvolume $W_6$, one must for quantum consistency have at most a phase change

Taking then $\Sigma_4 = \partial D_5$ and using the violated Bianchi identity $dG = \kappa T_6 \delta(W_6)$ yields the $M_2$-$M_5$ brane Dirac quantization rule for the cell units of the brane charge/tension lattice:

In addition, one has a quantization relation for the self-dual (i.e. dyonic) string on the 5–brane worldvolume $W_6$. Firstly, note that the dimension here is in the sequence $d = 4n + 2$, $n \in \mathbb{Z}$, for which the Dirac-Schwinger-Zwanziger quantization condition for dyons with (electric,magnetic) charges $(e_i, g_i)$ is symmetric [?]:

### 5 $D = 12$ Formulation and Dirac Surfaces

The bulk plus brane-source action discussed so far for M–theory needs to be completed by gravitational counterterms in order to cancel diffeomorphism anomalies on the 5–brane worldvolume $W_6$ that arise from loops of worldvolume chiral fermion modes [?]. In order to write these, it is convenient to introduce a $D = 12$ spacetime $Y$ such that

The main advantage of the $D = 12$ formulation, however, is in the way it gives to express the Dirac surfaces needed to maintain manifest gauge invariance. For this purpose, we need to introduce two surfaces bounded by $W_6$: $V_7$, which extends into $Y$ in such a way that

A similar construction can be made on the 5–brane worldvolume for surfaces bounded by $W_2$. Letting $i : W_6 \hookrightarrow W_7$, introduce $U_3 \subset W_7$ with $\partial U_3 = W_2$ such that

Taken all together, one has the following relative homology relations for the Dirac surfaces:

$$
\begin{align*}
\partial (V_8, W_7) &= (V_7, W_6) \\
\partial (U_4, U_3) &= (W_3, W_2).
\end{align*}
$$

At this point, we can also state the gauge invariance requirement for the integration domain in the remaining term $\frac{1}{2} \int_{W_8} G_4 \wedge G_4$ in the action (??), with a coefficient to be understood in the sense of Eq. (??):

Taken all together, the gauge-field part of the action is then

$$
I_{\text{gauge}} = \int_X \frac{1}{2\kappa} G_4 \wedge \ast G_4 + T_3 G_4 \wedge \Omega_7(R) - \int_Y \frac{1}{6\kappa} G_4 \wedge G_4 \wedge G_4
$$

$$
- \frac{1}{2} T_6 \int_{(V_8, W_7)} \left( G_4 \wedge G_4 - 2\kappa T_3 \Omega_8, h_3 \wedge \ast G_4 \right)
$$

$$
+ \frac{T_6}{4} \int_{W_6} h_3 \wedge \ast h_3 + T_3 \int_{U_4} G_4 - \frac{T_3}{2} \int_{U_3} h_3 ,
$$