Lagrangian Aspects of Quantum Dynamics on a Noncommutative Space*

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Abstract

In order to evaluate the Feynman path integral in noncommutative quantum mechanics, we consider properties of a Lagrangian related to a quadratic Hamiltonian with noncommutative spatial coordinates. A quantum-mechanical system with noncommutative spatial coordinates is equivalent to another one with commutative coordinates. We found connection between quadratic classical Lagrangians of these two systems. We also shown that there is a subclass of quadratic Lagrangians, which includes harmonic oscillator and particle in a constant field, whose connection between ordinary and noncommutative regimes can be expressed as a linear change of position in terms of a new position and velocity.

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1 Introduction

Quantum theories with noncommuting spatial coordinates have been investigated intensively during the recent years. M(atrix) theory compactification on noncommutative tori, strings in some constant backgrounds, quantum Hall effect and IR/UV mixing are some of the most popular themes (for a review of noncommutative quantum field theory and some related topics, see e.g. [1]). Most of the research has been done in noncommutative field theory, including noncommutative extension of the Standard Model [2]. Since quantum mechanics can be regarded as the one-particle nonrelativistic sector of quantum field theory, it is also important to study its noncommutative aspects including connection between ordinary and noncommutative regimes. Because of possible phenomenological realization, noncommutative quantum mechanics (NCQM) of a charged particle in the presence of a constant magnetic field has been mainly considered on two- and three-dimensional spaces (see, e.g. [3] and references therein).

Recall that to describe quantum-mechanical system theoretically one uses a Hilbert space \( L^2(\mathbb{R}^n) \) in which observables are linear self-adjoint operators. In ordinary quantum mechanics (OQM), by quantization, classical canonical variables \( x_k, p_j \) become Hermitean operators \( \hat{x}_k, \hat{p}_j \) satisfying the Heisenberg commutation relations
\[
[\hat{x}_k, \hat{p}_j] = i \hbar \delta_{kj}, \quad [\hat{x}_k, \hat{x}_j] = 0, \quad [\hat{p}_k, \hat{p}_j] = 0, \quad k, j = 1, 2 \ldots n. \tag{1}
\]

In a very general NCQM one has that \( [\hat{x}_k, \hat{p}_j] = i \hbar \delta_{kj} \), but \( [\hat{x}_k, \hat{x}_j] \neq 0 \) and \( [\hat{p}_k, \hat{p}_j] \neq 0 \). However, we consider here the most simple and usual NCQM which is based on the following algebra:
\[
[\hat{x}_k, \hat{p}_j] = i \hbar \delta_{kj}, \quad [\hat{x}_k, \hat{x}_j] = i \hbar \theta_{kj}, \quad [\hat{p}_k, \hat{p}_j] = 0, \tag{2}
\]
where \( \Theta = (\theta_{kj}) \) is the antisymmetric matrix with constant elements.

To find \( \Psi(x, t) \) as elements of the Hilbert space in OQM, and their time evolution, it is usually used the Schrödinger equation
\[
i \hbar \frac{\partial}{\partial t} \Psi(x, t) = \hat{H} \Psi(x, t),
\]
which realizes the eigenvalue problem for the corresponding Hamiltonian operator \( \hat{H} = H(\hat{p}, x, t) \), where \( \hat{p}_k = -i \hbar \frac{\partial}{\partial x_k} \) and \( \hbar = \frac{\hbar}{2\pi} \). However, there is...
another approach based on the Feynman path integral method \[4\]

\[K(x'',t'';x',t') = \int_{(x',t')}^{(x'',t'')} \exp \left( \frac{i}{\hbar} S[q] \right) \mathcal{D}q, \tag{3}\]

where \(K(x'',t'';x',t')\) is the kernel of the unitary evolution operator \(U(t)\), functional \(S[q] = \int_{t'}^{t''} L(\dot{q}, q, t) dt\) is the action for a path \(q(t)\) in the classical Lagrangian \(L(\dot{q}, q, t)\), and \(x'' = q(t'')\), \(x' = q(t')\) with the following notation \(x = (x_1, x_2, \ldots, x_n)\) and \(q = (q_1, q_2, \ldots, q_n)\). The kernel \(K(x'',t'';x',t')\) is also known as quantum-mechanical propagator, Green’s function, and the probability amplitude for a quantum particle to come from position \(x'\) at time \(t'\) to another point \(x''\) at \(t''\). The integral in (3) has a symbolic meaning of an intuitive idea that a quantum-mechanical particle may propagate from \(x'\) to \(x''\) using infinitely many paths which connect these two points and that one has to take all of them into account. Thus the Feynman path integral means a continual (functional) summation of single transition amplitudes \(\exp \left( \frac{i}{\hbar} S[q] \right)\) over all possible paths \(q(t)\) connecting \(x' = q(t')\) and \(x'' = q(t'')\).

In direct calculations, it is the limit of an ordinary multiple integral over \(N - 1\) variables \(q_i = q(t_i)\) when \(N \to \infty\). Namely, the time interval \(t'' - t'\) is divided into \(N\) equal subintervals and integration is performed for every \(q_i \in (-\infty, +\infty)\) and fixed time \(t_i\). In fact, \(K(x'',t'';x',t')\), as the kernel of the unitary evolution operator, can be defined by equation

\[\Psi(x'',t'') = \int \mathcal{K}(x'', t''; x', t') \Psi(x', t') dx' \tag{4}\]

and then Feynman’s path integral is a method to calculate this propagator.

Eigenfunctions of the integral equation (4) and of the above Schrödinger equation are the same. Note that the Feynman path integral approach, not only to quantum mechanics but also to whole quantum theory, is intuitively more attractive and more transparent in its connection with classical theory than the usual canonical operator formalism. In gauge theories and some other cases, it is also the most suitable method of quantization.

The Feynman path integral for quadratic Lagrangians can be evaluated analytically and the exact result for the propagator is

\[K(x'', t''; x', t') = \frac{1}{(i\hbar)^{\frac{3}{2}}} \left| \det \left( -\frac{\partial^2 \bar{S}}{\partial x_k' \partial x_j'} \right) \right| \exp \left( \frac{2\pi i}{\hbar} \bar{S}(x'', t''; x', t') \right), \tag{5}\]

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where \( \bar{S}(x'', t''; x', t') \) is the action for the classical trajectory which is solution of the Euler-Lagrange equation of motion.

In this article we search the form of a Lagrangian which corresponds to a system with noncommutative spatial coordinates. This is necessary to know before to employ Feynman’s path integral method in NCQM. To this end, let us note that algebra (2) of operators \( x_k, p_j \) can be replaced by the equivalent one

\[
[q_k, p_j] = i \hbar \delta_{kj}, \quad [q_k, q_j] = 0, \quad [p_k, p_j] = 0, \quad (6)
\]

where linear transformation

\[
\hat{x}_k = q_k - \theta_{kj} \hat{p}_j \quad (7)
\]

is used, while \( \hat{p}_k \) are remained unchanged, and summation over repeated indices is assumed. According to (7), NCQM related to the classical phase space \((x, p)\) can be regarded as an OQM on the other phase space \((q, p)\). Thus, in \(q\)-representation, \( \hat{p}_k = -i \hbar \frac{\partial}{\partial q_k} \) in the equations (6) and (7). It is worth noting that Hamiltonians \( H(\hat{p}, \hat{x}, t) = H(-i \hbar \frac{\partial}{\partial q_k}, q_k + \frac{i \hbar \theta_{kj}}{2} \frac{\partial}{\partial x_j}, t) \), which are more than quadratic in \( x \), will induce Schrödinger equations which contain derivatives higher than second order and even of the infinite order. This leads to a new part of a modern mathematical physics of partial differential equations with arbitrary higher-order derivatives (for the case of an infinite order, see, e.g. \([5]\) and \([6]\)). In this paper we restrict our consideration to the case of quadratic Langrangians (Hamiltonians).

2 Quadratic Lagrangians

2.1 Classical case

Let us start with a classical system described by a quadratic Lagrangian which the most general form in three dimensions is:

\[
L(\dot{x}, x, t) = \alpha_{11} \dot{x}_1^2 + \alpha_{12} \dot{x}_1 \dot{x}_2 + \alpha_{13} \dot{x}_1 \dot{x}_3 + \alpha_{22} \dot{x}_2^2 + \alpha_{23} \dot{x}_2 \dot{x}_3 + \alpha_{33} \dot{x}_3^2 + \beta_{11} \dot{x}_1 x_1 + \beta_{12} \dot{x}_1 x_2 + \beta_{13} \dot{x}_1 x_3 + \beta_{21} \dot{x}_2 x_1 \\
+ \beta_{22} \dot{x}_2 x_2 + \beta_{23} \dot{x}_2 x_3 + \beta_{31} \dot{x}_3 x_1 + \beta_{32} \dot{x}_3 x_2 + \beta_{33} \dot{x}_3 x_3 + \gamma_{11} x_1^2 + \gamma_{12} x_1 x_2 + \gamma_{13} x_1 x_3 + \gamma_{22} x_2^2 + \gamma_{23} x_2 x_3 + \gamma_{33} x_3^2 \\
+ \delta_1 \dot{x}_1 + \delta_2 \dot{x}_2 + \delta_3 \dot{x}_3 + \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \phi, \quad (8)
\]
where the coefficients $\alpha_{ij} = \alpha_{ij}(t)$, $\beta_{ij} = \beta_{ij}(t)$, $\gamma_{ij} = \gamma_{ij}(t)$, $\delta_i = \delta_i(t)$, $\xi_i = \xi_i(t)$ and $\phi = \phi(t)$ are some analytic functions of the time $t$.

If we introduce the following matrices,

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix},$$

assuming that the matrix $\alpha$ is nonsingular (regular) and if we introduce vectors

$$\delta = (\delta_1, \delta_2, \delta_3), \quad \xi = (\xi_1, \xi_2, \xi_3), \quad \dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3), \quad x = (x_1, x_2, x_3),$$

one can express the Lagrangian (8) in the following, more compact, form:

$$L(\dot{x}, x, t) = \langle \alpha \dot{x}, \dot{x} \rangle + \langle \beta x, \dot{x} \rangle + \langle \gamma x, x \rangle + \langle \delta, \dot{x} \rangle + \langle \xi, x \rangle + \phi, \quad (9)$$

where $\langle \cdot, \cdot \rangle$ denotes standard scalar product. Solving the equations

$$p_j = \frac{\partial L}{\partial \dot{x}_j}, \quad j = 1, 2, 3, \quad (10)$$

one can express $\dot{x}$ as

$$\dot{x} = \frac{1}{2} \alpha^{-1} (p - \beta x - \delta). \quad (11)$$

Then, the corresponding classical Hamiltonian

$$H(p, x, t) = \langle p, \dot{x} \rangle - L(\dot{x}, x, t) \quad (12)$$

becomes also quadratic, i.e.,

$$H(p, x, t) = \langle Ap, p \rangle + \langle B x, p \rangle + \langle C x, x \rangle + \langle D, p \rangle + \langle E, x \rangle + F, \quad (13)$$

where:

$$A = \frac{1}{4} \alpha^{-1}, \quad B = -\frac{1}{2} \alpha^{-1} \beta,$$

$$C = \frac{1}{4} \beta^T \alpha^{-1} \beta - \gamma, \quad D = -\frac{1}{2} \alpha^{-1} \delta,$$

$$E = \frac{1}{2} \beta^T \alpha^{-1} \delta - \xi, \quad F = \frac{1}{4} \langle \delta, \alpha^{-1} \delta \rangle - \phi. \quad (14)$$
Here, $\beta^\tau$ denotes transpose map of $\beta$.

Let us mention now that matrices $A$ and $C$ are symmetric ($A^\tau = A$ and $C^\tau = C$), since the matrices $\alpha$ and $\gamma$ are symmetric. Also, if the Lagrangian $L(\dot{x}, x, t)$ is nonsingular ($\det \alpha \neq 0$), then the Hamiltonian $H(p, x, t)$ is also nonsingular ($\det A \neq 0$).

The above calculations can be considered as a map from the space of quadratic nonsingular Lagrangians $\mathcal{L}$ to the corresponding space of quadratic nonsingular Hamiltonians $\mathcal{H}$. More precisely, we have $\varphi : \mathcal{L} \rightarrow \mathcal{H}$, given by

$$\varphi(L(\alpha, \beta, \gamma, \delta, \xi, \phi, \dot{x}, x)) = H(\varphi_1(L), \varphi_2(L), \varphi_3(L), \varphi_4(L), \varphi_5(L), \varphi_6(L), \varphi_7(L), \varphi_8(L))$$

$$= H(A, B, C, D, E, F, p, x).$$

(15)

From relation (12) it is clear that inverse of $\varphi$ is given by the same relations (14). This fact implies that $\varphi$ is essentially involution, i.e. $\varphi \circ \varphi = id$.

2.2 Noncommutative case

In the case of noncommutative coordinates $[\hat{x}_k, \hat{x}_j] = i \hbar \theta_{kj}$, one can replace these coordinates using the following ansatz (7),

$$\hat{x} = \hat{q} - \frac{1}{2} \Theta \hat{p},$$

(16)

where $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$, $\hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)$, $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ and

$$\Theta = \begin{pmatrix}
0 & \theta_{12} & \theta_{13} \\
-\theta_{12} & 0 & \theta_{23} \\
-\theta_{13} & -\theta_{23} & 0
\end{pmatrix}.$$

Now, one can easily check that $\hat{q}_i$ for $i = 1, 2, 3$ are mutually commutative operators (but do not commute with operators of momenta, $[\hat{q}_k, \hat{p}_j] = i \hbar \delta_{kj}$).

If we start with quantization of the nonsingular quadratic Hamiltonian given by (13), i.e., $\hat{H} = H(A, B, C, D, E, F, \hat{p}, \hat{x})$ and then apply the change of coordinates (16), we will again obtain quadratic quantum Hamiltonian, $\hat{H}_\Theta$ :

$$\hat{H}_\Theta = \langle A_\Theta \hat{p}, \hat{p} \rangle + \langle B_\Theta \hat{q}, \hat{p} \rangle + \langle C_\Theta \hat{q}, \hat{q} \rangle + \langle D_\Theta \hat{p}, \hat{p} \rangle + \langle E_\Theta \hat{q}, \hat{q} \rangle + F_\Theta,$$

(17)
where
\[ A_\Theta = \left( A - \frac{1}{2} \Theta B - \frac{1}{4} \Theta C \Theta \right)_{sym}, \quad B_\Theta = B - \Theta C, \]
\[ C_\Theta = C, \quad D_\Theta = D + \frac{1}{2} \Theta E, \quad E_\Theta = E \quad F_\Theta = F, \quad (18) \]
and \( sym \) denotes symmetrization of the corresponding operator. Let us note that for the nonsingular Hamiltonian \( \hat{H} \) and for sufficiently small \( \theta_{kj} \) the Hamiltonian \( \hat{H}_\Theta \) is also nonsingular.

In the process of calculating path integrals, we need classical Lagrangians. It is clear that to an arbitrary quadratic quantum Hamiltonian we can associate classical one replacing operators by the corresponding classical variables. Then, by using equations
\[ \dot{q}_k = \frac{\partial H_\Theta}{\partial p_k}, \quad k = 1, 2, 3, \]
from such Hamiltonian we can come back to the corresponding Lagrangian
\[ L_\Theta(\dot{q}, q, t) = \langle p, \dot{q} \rangle - H_\Theta(p, q, t), \]
where
\[ p = \frac{1}{2} A^{-1}_\Theta (\dot{q} - B_\Theta q - D_\Theta) \]
is replaced in \( H_\Theta(p, q, t) \). In fact, our idea is to find connection between Lagrangians of noncommutative and the corresponding commutative quantum mechanical systems (with \( \theta = 0 \)). This implies to find the composition of the following three maps:
\[ L_\Theta = (\varphi \circ \psi \circ \varphi)(L), \quad (19) \]
where \( L_\Theta = \varphi(H_\Theta) \), \( H_\Theta = \psi(H) \) and \( H = \varphi(L) \) (here we use facts that \( \varphi \) is an involution given by formulas (14), and \( \psi \) is given by (18)). More precisely, if
\[ L(\dot{x}, x, t) = \langle \alpha \dot{x}, \dot{x} \rangle + \langle \beta x, \dot{x} \rangle + \langle \gamma x, x \rangle + \langle \delta, \dot{x} \rangle + \langle \xi, x \rangle + \phi, \]
and
\[ L_\Theta(\dot{q}, q, t) = \langle \alpha_\Theta \dot{q}, \dot{q} \rangle + \langle \beta_\Theta q, \dot{q} \rangle + \langle \gamma_\Theta q, q \rangle + \langle \delta_\Theta, \dot{q} \rangle + \langle \xi_\Theta, q \rangle + \phi_\Theta, \]
then the connections are given by

\[
\alpha_\Theta = (\alpha^{-1} - \frac{1}{2} (\beta^\tau \alpha^{-1} \Theta - \Theta \alpha^{-1} \beta) + \Theta \gamma \Theta - \frac{1}{4} \Theta \beta^\tau \alpha^{-1} \beta \Theta)^{-1},
\]
\[
\beta_\Theta = \alpha_\Theta (\alpha^{-1} \beta + \frac{1}{2} \Theta \beta^\tau \alpha^{-1} \beta - 2 \Theta \gamma),
\]
\[
\gamma_\Theta = \frac{1}{4} (\beta^\tau \alpha^{-1} - \frac{1}{2} \beta^\tau \alpha^{-1} \beta \Theta + 2 \gamma \Theta) \alpha_\Theta (\alpha^{-1} \beta + \frac{1}{2} \Theta \beta^\tau \alpha^{-1} \beta
\]
\[
- 2 \Theta \gamma) - \frac{1}{4} \beta^\tau \alpha^{-1} \beta + \gamma,
\]
\[
\delta_\Theta = \alpha_\Theta (\alpha^{-1} \delta - \frac{1}{2} \Theta \beta^\tau \alpha^{-1} \delta + \Theta \xi),
\]
\[
\xi_\Theta = \frac{1}{2} (\beta^\tau \alpha^{-1} - \frac{1}{2} \beta^\tau \alpha^{-1} \beta \Theta + 2 \gamma \Theta) \alpha_\Theta (\alpha^{-1} \delta
\]
\[
- \frac{1}{2} \Theta \beta^\tau \alpha^{-1} \delta + \Theta \xi) - \frac{1}{2} \beta^\tau \alpha^{-1} \delta + \xi,
\]
\[
\phi_\Theta = \frac{1}{4} \langle \delta_\Theta, \alpha^{-1} \delta - \frac{1}{2} \Theta \beta^\tau \alpha^{-1} \delta + \Theta \xi \rangle - \frac{1}{4} \langle \alpha^{-1} \delta, \delta \rangle + \phi.
\]

It is clear that formulas (20) are very complicated and that to find explicit exact relations between elements of matrices in general case is a very hard task. However, the relations (20) are quite useful in all particular cases.

### 2.3 Linearization and some examples

In this section we try to introduce the linear change of coordinates to gain $L_\Theta(\dot{q}, q, t)$ directly from $L(\dot{x}, x, t)$ for some simple examples. Let $L(\dot{x}, x, t)$ be a nonsingular quadratic Lagrangian given by (9). If we make the following change of variables:

\[
\dot{x} = \dot{q}, \quad x = U(\Theta) \dot{q} + V(\Theta) q + W(\Theta) = U \dot{q} + V q + W,
\]

we obtain again quadratic Lagrangian $L_\Theta(\dot{q}, q, t)$, where

\[
\alpha_\Theta = (\alpha + U^\tau \gamma U + \beta U)_{\text{sym}}, \quad \beta_\Theta = 2 U^\tau \gamma V + \beta V,
\]
\[
\delta_\Theta = \beta W + 2 U^\tau \gamma W + \delta + U^\tau \xi, \quad \gamma_\Theta = V^\tau \gamma V,
\]
\[
\xi_\Theta = 2 V^\tau \gamma W + V^\tau \xi, \quad \phi_\Theta = \langle \gamma W + \xi, W \rangle + \phi.
\]
and $U$, $V$, $W$ are matrices.

It is clear that formulas (22) are simpler than (20). We will see that for some systems, by this linear change of the coordinates, it is possible to obtain connection between Lagrangians $L(\dot{x}, x, t)$ and $L_\Theta(\dot{q}, q, t)$

**Example 1.** Harmonic oscillator, $n = 2$. In this case we have

$$
\alpha = \frac{m}{2} \text{Id}, \quad \beta = 0, \quad \gamma = k \text{Id}, \quad \Theta = \theta \mathbf{J},
$$

(23)

where $k < 0$ is related to an ordinary harmonic oscillator,

$$
\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{J}^2 = -\text{Id}, \quad \delta = 0, \quad \xi = (\xi_1, \xi_2), \quad \phi = 0,
$$

and $\text{Id}$ is $2 \times 2$ unit matrix.

Using formulas (20), one can easily find

$$
\alpha_\Theta = \frac{m}{2 - k m \theta^2} \text{Id}, \quad \beta_\Theta = \frac{-2 k m \theta}{2 - k m \theta^2} \mathbf{J}, \quad \gamma_\Theta = \frac{2 k}{2 - k m \theta^2} \text{Id}.
$$

(24)

From (22) it follows

$$
\alpha_\Theta = \alpha + U^\top \gamma U = \alpha + k U^\top U, \quad \text{and consequently}
$$

$$
\alpha_\Theta - \alpha = \frac{k m^2 \theta^2}{2 (2 - k m \theta^2)} \text{Id}.
$$

(25)

So, it is clear that matrix $U$ is proportional to an orthogonal operator, i.e, $U = \lambda \tilde{U}$. Now, from (25) we obtain

$$
\lambda = \varepsilon_\lambda \frac{m \theta}{\sqrt{2}} \frac{1}{\sqrt{2 - k m \theta^2}}, \quad \text{where } \varepsilon_\lambda = \pm 1 \text{ and } 2 - k m \theta^2 > 0.
$$

(26)

Similarly, from (24) and $\gamma_\Theta = V^\top \gamma V$, we find that matrix $V$ is also proportional to an orthogonal operator $V = \mu \tilde{V}$, where

$$
\mu = \varepsilon_\mu \frac{\sqrt{2}}{\sqrt{2 - k m \theta^2}}, \quad \varepsilon_\mu = \pm 1 \text{ and } 2 - k m \theta^2 > 0.
$$

(27)

It is known that

$$
\tilde{U} = R(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \quad \text{or} \quad \tilde{U} = R_1(\psi) = \begin{pmatrix} -\cos \psi & \sin \psi \\ \sin \psi & \cos \psi \end{pmatrix},
$$

$$
\tilde{V} = R(\omega) = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \quad \text{or} \quad \tilde{V} = R_1(\omega) = \begin{pmatrix} -\cos \omega & \sin \omega \\ \sin \omega & \cos \omega \end{pmatrix},
$$

(28)

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where $0 \leq \psi, \omega < 2\pi$. From (22) and (24), we find
\[
\beta_\Theta = 2 U^\tau \gamma V = \frac{-2 km\theta}{2 - km\theta^2} J,
\]
and using (26)-(28), we have
\[
\tilde{U}^\tau \tilde{V} = -\varepsilon_\lambda \varepsilon_\mu J. \tag{29}
\]
From (29), depending on $\varepsilon_\lambda \varepsilon_\mu$ (-1 or 1), we have
\[
\omega = \psi - \frac{\pi}{2} \quad \text{or} \quad \omega = \psi + \frac{\pi}{2}. \tag{30}
\]
If $\xi = 0$, from the relation $\phi_\Theta = 0$, we obtain that $W = 0$. It implies that in this case the transition from Lagrangian $L(\dot{x}, x, t)$ to the Lagrangian $L_\Theta(\dot{q}, q, t)$ is given by a linear change of the coordinates.

Finally, let us show that in the case $\xi \neq 0$, it is not possible to obtain the transition from Lagrangian $L(\dot{x}, x, t)$ to the Lagrangian $L_\Theta(\dot{q}, q, t)$ by a linear change of the coordinates (21). From (20) and (22), we have
\[
\delta_\Theta = 2 U^\tau \gamma W + U^\tau \xi
\]
and consequently
\[
W = \frac{1}{2 \lambda k} \tilde{U} \delta_\Theta - \frac{1}{2 k} \xi. \tag{31}
\]
Similarly, from the relation $\xi_\Theta = 2 V^\tau \gamma W + V^\tau \xi$, one can find
\[
W = \frac{1}{2 \mu k} \tilde{V} \xi_\Theta - \frac{1}{2 k} \xi. \tag{32}
\]
The relations (31) and (32) imply
\[
\delta_\Theta = \frac{\lambda}{\mu} \tilde{U}^\tau \tilde{V} \xi_\Theta = -\frac{|\lambda|}{|\mu|} J \xi_\Theta, \tag{33}
\]
From the other side, the relations (20) and (24) give
\[
\delta_\Theta = \frac{m\theta}{2 - km\theta^2} J \xi, \quad \xi_\Theta = 2 \frac{1 - km\theta^2}{2 - km\theta^2} \xi. \tag{34}
\]
Combining the relations (33) and (34), we have

\[ \frac{1}{2 - k m \theta^2} = - \frac{1 - k m \theta^2}{2 - k m \theta^2}, \]

and consequently \( 2 - k m \theta^2 = 0 \), but it is impossible, since \( 2 - k m \theta^2 > 0 \), according to (26) and (27).

In the first case \( (\xi = 0) \), the corresponding Lagrangian \( L_\Theta(\dot{q}, q, t) \) is

\[
L_\Theta = \langle \alpha_\Theta \dot{q}, \dot{q} \rangle + \langle \beta_\Theta q, \dot{q} \rangle + \langle \gamma_\Theta q, q \rangle + \langle \delta_\Theta, \dot{q} \rangle + \langle \xi_\Theta, q \rangle + \phi_\Theta
= \frac{1}{2 - k m \theta^2} [m (\dot{q}_1^2 + \dot{q}_2^2) + 2k (q_1^2 + q_2^2) - 2k m \theta (\dot{q}_1 q_2 - q_1 \dot{q}_2)].
\]

From (35), we obtain the Euler-Lagrange equations,

\[ m \ddot{q}_1 - 2m k \theta \dot{q}_2 - 2k q_1 = 0 \quad \text{and} \quad m \ddot{q}_2 + 2m k \theta \dot{q}_1 - 2k q_2 = 0. \]  

Let us remark that the Euler-Lagrange equations (36) form a coupled system of second order differential equations, which is more complicated than in commutative case \( (\theta = 0) \).

**Example 2.** A particle in a constant field, \( n = 2 \).

This example is defined by the following data:

\[
\alpha = \frac{m}{2} Id, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0, \quad \xi = (\xi_1, \xi_2), \quad \phi = 0.
\]

Using the general composition formula (20), one can easily find

\[
\alpha_\Theta = \frac{m}{2} Id, \quad \beta_\Theta = 0, \quad \gamma_\Theta = 0, \quad \xi_\Theta = \xi, \quad \delta_\Theta = \frac{m \theta}{2} J \xi, \quad \text{and} \quad F_\theta = \frac{m \theta^2}{8} \langle \xi, \xi \rangle,
\]

where \( J \) is the same as in the first example (see (23)).

Now, using the method of the linear change of coordinates, from the relation (22), we have

\[
\alpha_\Theta = \frac{m}{2} Id, \quad \beta_\Theta = 0, \quad \gamma_\Theta = 0, \quad \xi_\Theta = V^\tau \xi, \quad \delta_\Theta = U^\tau \xi, \quad \text{and} \quad \phi_\Theta = \langle \xi, W \rangle.
\]
From (38) and (39), it is easy to see that for
\[ U = -\frac{m \theta}{2} J, \quad V = Id \quad \text{and} \quad W = \frac{m \theta^2}{8} \xi, \quad (40) \]
the linearization gives the same as the general formula.

In this case, it is easy to find the classical action. The Lagrangian \( L_\Theta(\dot{q}, q, t) \) is
\[ L_\Theta = \frac{m}{2} (\ddot{q}_1^2 + \ddot{q}_2^2 - \theta (\xi_1 \dot{q}_2 - \xi_2 \dot{q}_1)) + \xi_1 q_1 \]
\[ + \xi_2 q_2 + \frac{m \theta^2}{8} (\xi_1^2 + \xi_2^2). \quad (41) \]
The Lagrangian given by (41) implies the Euler-Lagrange equations,
\[ m \dddot{q}_1 = \xi_1 \quad \text{and} \quad m \dddot{q}_2 = \xi_2. \quad (42) \]
Their solutions are:
\[ q_1(t) = \frac{\xi_1 t^2}{2 m} + t C_2 + C_1 \quad \text{and} \quad q_2(t) = \frac{\xi_2 t^2}{2 m} + t D_2 + D_1, \quad (43) \]
where \( C_1, C_2, D_1 \) and \( D_2 \) are constants which have to be determined from conditions:
\[ q_1(0) = x_1', \quad q_1(T) = x_1'' \quad q_2(0) = x_2', \quad q_2(T) = x_2''. \quad (44) \]
After finding the corresponding constants, we have
\[ q_j(t) = x_j' + \frac{\xi_j t^2}{2 m} + t \left( \frac{1}{T} (x_j'' - x_j') - \frac{\xi_j T}{2 m} \right), \quad j = 1, 2, \quad (45) \]
\[ \dot{q}_j(t) = \frac{\xi_j t}{m} + \frac{1}{T} (x_j'' - x_j') - \frac{\xi_j T}{2 m}, \quad j = 1, 2. \quad (46) \]
Using (45) and (46), we finally calculate the corresponding action
\[ S_\Theta(x'', T; x', 0) = \int_0^T L_\Theta(\dot{q}, q, t) \, dt = \frac{m}{2 T} \left[ (x_1'' - x_1')^2 + (x_2'' - x_2')^2 \right] \]
\[ + \frac{T}{2} \left[ \xi_1 (x_1'' + x_1') + \xi_2 (x_2'' + x_2') \right] - \frac{m \theta}{2} \left[ \xi_1 (x_2'' - x_2') \right. \]
\[ - \xi_2 (x_1'' - x_1') \left] - \frac{T^3}{24 m} \left( \xi_1^2 + \xi_2^2 \right) - \frac{m T \theta}{8} \left( \xi_1^2 + \xi_2^2 \right). \quad (47) \]
3 Concluding Remarks

Note that almost all results obtained in Section 2 for three-dimensional case allow straightforward generalization to any \( n \geq 2 \) spatial dimension.

According to the formula (5), the propagator in NCQM with quadratic Lagrangians can be easily written down when we have the corresponding classical action \( \bar{S}_\Theta(x'', t''; x', t') = \int_{t'}^{t''} L(\dot{q}, q, t) dt \), where \( q = q(t) \) is solution of the Euler-Lagrange equations of motion. As a simple example we calculated \( \bar{S}_\Theta(x'', T; x', 0) \) for a noncommutative regime of a particle on plane in a constant field.

Note that the path integral approach to NCQM has been considered in the context of the Aharonov-Bohm effect [7], Aharonov-Bohm and Casimir effects [8], and a quantum system in a rotating frame [9]. Our approach includes all systems with quadratic Lagrangians (Hamiltonians) and some new results on path integrals on noncommutative spaces will be presented elsewhere.

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