Graviton-Scalar Interaction in the PP-Wave Background

K. Bobkov

Department of Physics
University of North Carolina, Chapel Hill, NC 27599-3255

We compute the graviton two scalar off-shell interaction vertex at tree level in Type IIB superstring theory on the pp-wave background using the light-cone string field theory formalism. We then show that the tree level vertex vanishes when all particles are on-shell and conservation of $p_+$ and $p_-$ are imposed. We reinforce our claim by calculating the same vertex starting from the corresponding SUGRA action expanded around the pp-wave background in the light-cone gauge.
1. Introduction

In [1] it was conjectured that there exists a duality between the Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang Mills theory on the boundary of $AdS_5$. A concrete recipe for verifying this correspondence in the large $\lambda \equiv g_{YM}^2 N \rightarrow \infty$ limit when $\alpha' \rightarrow 0$ was given in [2-3]. There have been many checks of this duality in the supergravity ($\alpha' \rightarrow 0$) limit but the verification of the correspondence in the full string theory remains elusive. In a more recent development, the authors of [4-5] showed that in the Penrose limit the $AdS_5 \times S^5$ background turns into the pp-wave solution of Type IIB supergravity preserving all 32 supercharges

$$ds^2 = -4dx^+ dx^- - \mu^2 x_I x_I (dx^+)^2 + dx_I dx_I, \quad F_{+1234} = F_{+5678} = 2\mu,$$

where $I = 1, \ldots, 8$. Unlike the case of $AdS_5 \times S^5$, where we do not even know the free string spectrum, string theory on the pp-wave background can be solved [6-7] in the light-cone gauge, despite the presence of a non-zero Ramond-Ramond flux. Partly motivated by the conjecture in [1], Berenstein, Maldacena and Nastase [8] have argued that a particular sector of $\mathcal{N} = 4$ super Yang Mills theory containing operators with large R-charge J is dual to Type IIB string theory on the pp-wave background with Ramond-Ramond flux. The fact that string theory on the pp-wave background has been exactly solved has opened an exciting possibility to check the proposed correspondence beyond the supergravity limit. In fact, the authors of [8] succeeded in reproducing the tree level string spectrum on the pp-wave [7] from the perturbative super Yang Mills theory as the first test of a full string theory/CFT duality. The anomalous dimensions of the BMN operators arising from the Yang-Mills perturbation theory were studied in [9-12] to check the correspondence between strings on the plane wave background and the Yang-Mills theory at the level of perturbative expansions. In a separate development, Spradlin and Volovich generalized the formalism of the light-cone string field theory in Minkowski space [13-15] to the plane-wave geometry [16-17]. The subject of the pp-wave light-cone string field theory pioneered by SV in [16-17] was studied further in [18-20]. The factorization theorem for the Neumann coefficients was discussed in [19-20] and explicit formulas for the Neumann coefficients were derived in [21]. By employing the formalism of [16-17], certain three-string amplitudes were found to be in agreement with the corresponding three-point functions of the BMN operators.
A cubic light-cone interaction Hamiltonian for the chiral primary system from the Type IIB pp-wave supergravity was constructed in [24], [27]. Another interesting question is the issue of existence of the $S$-matrix interpretation for field theories on the plane wave backgrounds. The authors of [28] have demonstrated that, at least at the tree level, the field theory of scalars and scalars coupled to a gauge field do have an $S$-matrix formulation.

The motivation behind this paper is brought about by the fact that, unlike in the flat Minkowski space, the stringy modes do not decouple from the supergravity modes in the pp-wave background [22-25]. Therefore, there is a possibility that the $\alpha'$ corrections could potentially contribute to the three-point interactions at tree level. In section 2 we compute the string tree level three-point amplitude for a graviton and two scalars, i.e. we use the formalism of [16-17] to compute a matrix element of the cubic interaction Hamiltonian for the states represented by the graviton and a combination of the dilaton and axion fields of the Type IIB supergravity on the pp-wave background. We discover that the off-shell amplitude contains $\alpha'$ corrections encoded in the Neumann coefficients of the zero modes of the full string theory vertex. We show that when we impose conservation of $p_+$ and $p_-$, this amplitude vanishes on-shell. In section 3, we compute the graviton dilaton-axion cubic vertex starting from the Type IIB supergravity action expanded around the pp-wave background in the light-cone gauge. We show that it vanishes on-shell and thus verify in the $\alpha' \rightarrow 0$ limit our string theory computation.

2. Graviton axion-dilaton interaction from light cone string field theory

2.1. Some Key Results of the Light-Cone String Field Theory in PP-Wave

Following the light-cone string field theory formalism of [13-15] developed in Minkowski space, Spradlin and Volovich successfully generalized it to strings propagating in the plane wave background [16-17]. In particular, they constructed a cubic interaction Hamiltonian that can be expressed as

$$|H_3\rangle = \hat{h}_3|V\rangle,$$

(2.1)

where $|V\rangle$ is a three-string vertex satisfying the kinematic constraints given by equations (4.2)-(4.5) of [16] and the prefactor $\hat{h}_3$ must be inserted at the interaction point in order to preserve supersymmetry. More explicitly

$$|V\rangle = E_a E_b |0\rangle,$$

(2.2)
where the bosonic and fermionic zero mode parts of the vertex are
\[
E^0_a = \exp \left[ \frac{1}{2} \sum_{r,s=1}^{3} \sum_{I=1}^{8} a^+_r N^{rs} a^+_s I \right],
\]
(2.3)
\[
E^0_b = \frac{1}{8!} \epsilon_{a_1 \ldots a_8} \lambda^{a_1} \ldots \lambda^{a_8}, \quad \text{where} \quad \lambda^a = \hat{\lambda}_1^a + \hat{\lambda}_2^a + \hat{\lambda}_3^a,
\]
(2.4)
\hat{\lambda}^a's are zero modes of the fermionic conjugate momenta of the string. They are complex positive chirality \(SO(8)\) spinors with \(a = 1, \ldots, 8\). The strings are labeled by \(r, s = 1, 2, 3\).

We are only going to be concerned with the states corresponding to the Type IIB supergravity multiplet [7]. However, as was pointed out to us by M. Spradlin and was discussed in [22-25], the supergravity modes do not decouple from the string modes in the pp-wave background. Decoupling only takes place in the flat space limit when \(\mu \alpha' p^+ \to 0\). We will therefore be using the Neumann coefficients for the zero modes of the full string theory vertex derived explicitly in [21], instead of the supergravity vertex given in [16]. The zero mode Neumann coefficients are
\[
N^{rs} = (1 + \mu a k) \epsilon^r t^s \sqrt{\frac{\alpha_t \alpha_u}{\alpha_3^2}}, \quad r, s, t, u \in \{1, 2\},
\]
\[
N^{3r} = N^{r3} = -\sqrt{-\frac{\alpha_r}{\alpha_3}}, \quad r \in \{1, 2\},
\]
\[
N^{33} = 0,
\]
(2.5)
where \(\alpha = \alpha_1 \alpha_2 \alpha_3\) and \(\alpha_r \equiv 2p^+_r\) and where by \(p^+\) we really mean \(-p_-\). This point is important because in the pp-wave metric we have a non-zero \(\tilde{g}^{++}\) component and therefore strictly speaking \(p^+\) and \(-p_-\) are different. From eq.(4.18) of [16] we have the following prefactor for the zero modes
\[
\hat{h}_3 = P^I P^J v_{IJ} (\Lambda),
\]
(2.6)
where
\[
v_{IJ} (\Lambda) = \delta^{IJ} + \frac{1}{6 \alpha^2} t^{IJ}_{abcd} \Lambda^a \Lambda^b \Lambda^c \Lambda^d + \frac{16}{8! \alpha^4} \delta^{IJ} \epsilon_{abcdefgh} \Lambda^a \Lambda^b \Lambda^c \Lambda^d \Lambda^e \Lambda^f \Lambda^g \Lambda^h, \]
(2.7)
\[
\Lambda^a = \alpha_1 \hat{\lambda}_2^a - \alpha_2 \hat{\lambda}_1^a = \alpha_3 \hat{\lambda}_3^a + \alpha_1 \hat{\lambda}_3^a = \alpha_2 \hat{\lambda}_3^a - \alpha_3 \hat{\lambda}_2^a,
\]
(2.8)
\[
P^I = \alpha_1 \hat{p}_2^I - \alpha_2 \hat{p}_1^I = \alpha_3 \hat{p}_3^I + \alpha_1 \hat{p}_3^I = \alpha_2 \hat{p}_3^I - \alpha_3 \hat{p}_2^I,
\]
(2.9)
and
\[
\hat{p}^I = \sqrt{|\alpha| \mu} (a^I + \hat{a}^I). \quad (2.10)
\]
The self-dual tensor \( t_{abcd}^{IJ} \) was defined in [13] in terms of \( SO(8) \) gamma matrices as
\[
t_{abcd}^{IJ} = \gamma^{IK} \left[ \gamma_{ab} \right] [ab] \gamma_{JK} \gamma_{cd} \]
and satisfies various identities given in Appendix A of [15]. Appendix C of this paper contains some extra identities for tensor \( t_{abcd}^{IJ} \) that could be useful in future computations.

Using the constraint \( (\hat{p}_1^I + \hat{p}_2^I + \hat{p}_3^I) |V\rangle = 0 \) together with \( \alpha_1 + \alpha_2 + \alpha_3 = 0 \) it is easy to show that
\[
P^I P^J |V\rangle = -\alpha_1 \alpha_2 \alpha_3 \left[ \frac{1}{\alpha_1} \hat{p}_1^I \hat{p}_1^J + \frac{1}{\alpha_2} \hat{p}_2^I \hat{p}_2^J + \frac{1}{\alpha_3} \hat{p}_3^I \hat{p}_3^J \right] |V\rangle.
\]
(2.12)

Following [16] will assume that \( \alpha_1 \) and \( \alpha_2 \) are positive with \( \alpha_1 + \alpha_2 + \alpha_3 = 0 \). We can further substitute (2.10) for \( \hat{p}_r^I \) to obtain from (2.12) the following
\[
P^I P^J |V\rangle = -\mu \alpha_1 \alpha_2 \alpha_3 \left[ (a_1^I a_1^J) (a_1^I a_1^J) + (a_2^I a_2^J) (a_2^I a_2^J) - (a_3^I a_3^J) (a_3^I a_3^J) \right] |V\rangle.
\]
(2.13)

For \( I = J \) (2.13) can be written [25] as
\[
P^I P^I |V\rangle = -\mu \alpha_1 \alpha_2 \alpha_3 \left[ a_1^I a_1^I + a_2^I a_2^I - a_3^I a_3^I \right] |V\rangle.
\]
(2.14)

The light-cone Hamiltonian for bosonic zero modes is given by
\[
H_r = \mu \sum_{I=1}^{8} a_r^I a_r^I + \mu E_0^r,
\]
(2.15)
and the light-cone energy is
\[
p_r^+ = \mu \sum_{I=1}^{8} n_r^I + \mu E_0^r.
\]
(2.16)

### 2.2. Graviton Dilaton-Axion Vertex

Here we will calculate a three string amplitude using the formalism of [16] for a particular choice of states from the Type IIB supergravity multiplet. The superfield expansion for Type IIB supergravity in light-cone gauge originally given by equation (1) of [13] is
\[
\Phi(x, \theta) = \sum_{N=0}^{4} \frac{1}{(2N)!} \left( \hat{\theta}^+ \right)^{N-2} A_{a_1 a_2 \ldots a_{2N}} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_{2N}}
\]
\[
+ \sum_{N=0}^{3} \frac{1}{(2N+1)!} \left( \hat{\theta}^+ \right)^{N-2} \psi_{a_1 a_2 \ldots a_{2N+1}} \theta^{a_1} \theta^{a_2} \ldots \theta^{a_{2N+1}}.
\]
(2.17)
We are going to be interested in the bosonic terms corresponding to \( N = 0, 2, \) and 4 of the first sum that contain the dilaton, axion and graviton.

\[
\Phi(x, \theta) = \frac{1}{(\partial^+)^2} A^*(x) + \frac{1}{4!} A_{abcd}(x) \theta^a \theta^b \theta^c \theta^d \\
+ \frac{1}{8!} (\partial^+)^2 A(x) \epsilon_{abcdefgh} \theta^a \theta^b \theta^c \theta^d \theta^e \theta^f \theta^g \theta^h + \ldots,
\]

(2.18)

where we set \( A_{abcdefgh}(x) = A(x) \epsilon_{abcdefgh} \). The fields of interest are identified [15] as

\[
\begin{align*}
\tau(x) &= \chi(x) + ie^{-\phi(x)} = A(x), \\
\bar{\tau}(x) &= \chi(x) - ie^{-\phi(x)} = A^*(x), \\
h^{IJ}(x) &= \frac{1}{2} t_{abcd} A_{abcd}(x),
\end{align*}
\]

(2.19)

where \( \chi \) is the RR scalar (axion), \( h^{IJ} \) is symmetric and traceless (graviton), and \( \phi \) is the trace (dilaton). As prescribed by [16] it is necessary to transform the superfield (2.18) to the occupation number basis \( \{k_I\} \) for the transverse directions \( x^I, (I = 1, \ldots, 8) \) and to momentum space for \( x^- \) coordinate

\[
\Phi \left( x^+, \alpha, \theta; \{k_I\} \right) = \frac{4}{\alpha^2} \bar{\tau} \left( x^+, \alpha; \{k_I\} \right) + \frac{1}{4!} A_{abcd} \left( x^+, \alpha; \{k_I\} \right) \theta^a \theta^b \theta^c \theta^d \\
+ \frac{\alpha^2}{4} \tau \left( x^+, \alpha; \{k_I\} \right) \frac{1}{8!} \epsilon_{abcdefgh} \theta^a \theta^b \theta^c \theta^d \theta^e \theta^f \theta^g \theta^h + \ldots,
\]

(2.20)

where we used (2.19) to replace \( A \) and \( A^* \) with \( \tau \) and \( \bar{\tau} \). The expression that we are about to evaluate has the form

\[
\langle \Phi(1) | \langle \Phi(2) | \langle \Phi(3) | H_3 \rangle,
\]

(2.21)

where we are only going to be interested in the terms proportional to \( A_{abcd} \bar{\tau} \bar{\tau} \) that contain the graviton coupled to the dilaton-axion pair. We will first deal with the fermionic zero modes and use in our calculation the following conditions on \( \hat{\theta}^a \) and its conjugate momentum \( \hat{\lambda}^a \)

\[
\begin{align*}
\hat{\theta}^a |0\rangle &= 0, \\
\langle 0 | \hat{\lambda}^a &= 0, \\
\{\hat{\theta}^a, \hat{\lambda}^b\} &= \delta^{ab}.
\end{align*}
\]

(2.22)

By counting the number of \( \hat{\lambda} \)'s on the right hand side to saturate the number of \( \hat{\theta} \)'s on the left hand side, we see that only the second term in (2.7) will contribute to the \( h \bar{\tau} \bar{\tau} \)
interaction. Suppressing the \( x^+ \) dependence, we have the following expression for the graviton scalar vertex

\[
A_{h\tau\bar{\tau}}(1, 2, 3) = \langle \{k_1^I\} \rangle \hat{\theta}_1^{a_1} \hat{\theta}_2^{a_2} \hat{\theta}_3^{a_3} \hat{\theta}_4^{a_4} \frac{1}{4!} A_{a_1a_2a_3a_4} \left( \alpha_1, \{k_1^I\} \right) \langle \{k_2^I\} \rangle \frac{1}{\alpha_2^2} \bar{\tau} \left( \alpha_2, \{k_2^I\} \right)
\]

\[
\times \langle \{k_3^I\} \rangle \hat{\partial}_1^{b_1} \hat{\partial}_2^{b_2} \hat{\partial}_3^{b_3} \hat{\partial}_4^{b_4} \hat{\partial}_5^{b_5} \hat{\partial}_6^{b_6} \hat{\partial}_7^{b_7} \hat{\partial}_8^{b_8} \frac{1}{8!} \epsilon_{b_1b_2b_3b_4b_5b_6b_7b_8} \frac{\alpha_3^2}{4} \tau \left( \alpha_3, \{k_3^I\} \right)
\]

\[
\times \frac{\alpha_2^4}{6(\alpha_1\alpha_2\alpha_3)^2} \lambda_I^{c_1} \lambda_3^{c_3} \lambda_4^{c_4} \times \frac{1}{8!} \epsilon_{d_1d_2d_3d_4d_5d_6d_7d_8} \lambda_2^{d_2} \lambda_3^{d_3} \lambda_4^{d_4} \lambda_5^{d_5} \lambda_6^{d_6} \lambda_7^{d_7} \lambda_8^{d_8} P^J P^K E_a^0|0\rangle + \text{c.c.}
\] (2.23)

where the occupation number states are defined as

\[
|\{k_I^I\} \rangle = \prod_{I=1}^{8} (-i)^{k_I^I} \frac{(a_f^I)^{k_f^I}}{\sqrt{k_f^I!}} |0\rangle,
\]

\[
\langle \{k_I^I\} | = \langle 0 | \prod_{I=1}^{8} (i)^{k_I^I} \frac{(a_c^I)^{k_c^I}}{\sqrt{k_c^I!}},
\]

with \([a_f^I, a_c^J] = \delta^{IJ} \delta_{rs}\).

Definitions (2.24) are based on the definitions of \( a_f^I \) and \( a_c^I \) given in [16]. Following the standard procedure to bring all the \( \lambda^a \)'s to the left and all the \( \hat{\theta}^a \)'s to the right and using (2.22), we have from (2.23)

\[
A_{h\tau\bar{\tau}}(1, 2, 3) = \frac{1}{3\alpha_1^2} \langle \{k_1^I\} | \langle \{k_2^I\} | \langle \{k_3^I\} | \langle \{k_3^I\} | P^J P^K E_a^0 |0\rangle
\]

\[
\times \frac{1}{2} \lambda^{c_1} \lambda_3^{c_3} \lambda_4^{c_4} \lambda_2^{d_2} \lambda_3^{d_3} \lambda_4^{d_4} \lambda_5^{d_5} \lambda_6^{d_6} \lambda_7^{d_7} \lambda_8^{d_8} P^J P^K E_a^0 |0\rangle + \text{c.c.}
\] (2.25)

We can now identify the graviton in (2.25) using (2.19) and use the explicit representation for the occupation number states given by (2.24) to obtain from (2.25)

\[
A_{h\tau\bar{\tau}}(1, 2, 3) = \frac{1}{3\alpha_1^2} \sum_{I=1}^{8} \frac{i^{k_1^I+k_2^I+k_3^I}}{\sqrt{k_1^I k_2^I k_3^I!}} \langle 0 | (a_f^I)^{k_f^I} (a_c^I)^{k_c^I} P^J P^K E_a^0 |0\rangle
\]

\[
\times \lambda^{J} \lambda_2^{k_2^I} \lambda_3^{k_3^I} \lambda_4^{k_4^I} P^J P^K E_a^0 |0\rangle + \text{c.c.}
\] (2.26)

An explicit derivation of the Type IIB supergravity spectrum in the pp-wave background was found in [7]. In particular, the \( SO(8) \) light-cone gauge degrees of freedom of the graviton were classified according to their \( SO(4) \times SO'(4) \) decomposition. Based on those
results we can express various components of the graviton in terms of the mass eigenstates defined in [7] as follows

\[ h_{ij} = h_{ij}^\perp + \frac{1}{8}\delta_{ij}(h + \bar{h}) , \]
\[ h_{i'j'} = h_{i'j'}^\perp - \frac{1}{8}\delta_{i'j'}(h + \bar{h}) , \]
\[ h_{ij'} = h_{ij'}^\perp = \frac{1}{2}(h_{ij'} + \bar{h}_{ij'}) , \]  

(2.27)

where \( i, j = 1, \ldots, 4 \) and \( i', j' = 5, \ldots, 8 \). Expressed in terms of the mass eigenstates, the amplitude (2.26) becomes

\[
A_{h\tau\bar{\tau}}(1, 2, 3) = \frac{1}{3\alpha_1^2} \prod_{l=1}^{8} \frac{j_{k_1^l+k_2^l+k_3^l}}{\sqrt{k_1^l!k_2^l!k_3^l!}} \times \\
\times \left( \prod_{l=1}^{8} \langle 0 | (a_1^l)^{k_1^l} (a_2^l)^{k_2^l} (a_3^l)^{k_3^l} P^i P^j E_a^0 | 0 \rangle h_{1ij}^\perp \bar{\tau}_2 \tau_3 \right) \\
+ \prod_{l=1}^{8} \langle 0 | (a_1^l)^{k_1^l} (a_2^l)^{k_2^l} (a_3^l)^{k_3^l} P^{i'} P^{j'} E_a^0 | 0 \rangle h_{1i'j'}^\perp \bar{\tau}_2 \tau_3 \right) \\
+ \prod_{l=1}^{8} \langle 0 | (a_1^l)^{k_1^l} (a_2^l)^{k_2^l} (a_3^l)^{k_3^l} P^{i} P^{j} E_a^0 | 0 \rangle \bar{h}_{1ij} \bar{\tau}_2 \tau_3 \\
+ \prod_{l=1}^{8} \langle 0 | (a_1^l)^{k_1^l} (a_2^l)^{k_2^l} (a_3^l)^{k_3^l} P^{i'} P^{j'} E_a^0 | 0 \rangle \bar{h}_{1i'j'} \bar{\tau}_2 \tau_3 \\
+ \frac{1}{8} \prod_{l=1}^{8} \langle 0 | (a_1^l)^{k_1^l} (a_2^l)^{k_2^l} (a_3^l)^{k_3^l} \left( P^i P^j - P^{i'} P^{j'} \right) E_a^0 | 0 \rangle \bar{h}_{1i} \bar{\tau}_2 \tau_3 \\
+ \frac{1}{8} \prod_{l=1}^{8} \langle 0 | (a_1^l)^{k_1^l} (a_2^l)^{k_2^l} (a_3^l)^{k_3^l} \left( P^i P^j - P^{i'} P^{j'} \right) E_a^0 | 0 \rangle \bar{h}_{1i'} \bar{\tau}_2 \tau_3 \right) + c.c. .
\]

(2.28)

For the last two lines in (2.28) we can combine (2.14) together with (2.15) and (2.16) to obtain

\[
\left( P^i P^j - P^{i'} P^{j'} \right) = -\alpha_1 \alpha_2 \alpha_3 \left( \left( p_+^{1||} + p_+^{2||} - p_+^{3||} \right) - \left( p_+^{1\perp} + p_+^{2\perp} - p_+^{3\perp} \right) \right)
\]

(2.29)

and use the notation of [24] to define

\[
E_{123}^\parallel = p_+^{1||} + p_+^{2||} - p_+^{3||}, \quad E_{123}^\perp = p_+^{1\perp} + p_+^{2\perp} - p_+^{3\perp},
\]

(2.30)

where \( \parallel \) means \( i = 1, \ldots, 4 \) and \( \perp \) means \( i' = 5, \ldots, 8 \). Notice that the zero point energy contributions in (2.29) from \( \parallel \) and \( \perp \) cancelled each other. In order to proceed with
further computations we will need to evaluate expectation values of the type

\[
\prod_{l=1}^{8} \frac{i k^1_l + k^2_l + k^3_l}{\sqrt{K_l^1 k^2_l k^3_l}} \langle 0 | (a_1^I k^1_l) (a_2^I k^2_l) (a_3^I k^3_l) P^J P^K E^0_\alpha | 0 \rangle ,
\]

(2.31)

for both \( J = K \) and \( J \neq K \). Because of this distinction, we will split the first and second lines of (2.28) into the two cases and write the sums over \( i, i', j, j' \) explicitly. We can use (2.13) and (2.14) in combination with (2.29) and (2.30) and apply formulas (A.5)-(A.9) from Appendix A to obtain the final expression for the graviton dilaton-axion off-shell

\[
A_{\mu \nu \rho \{n^1_i, n^2_i, n^3_i\}} (\alpha_1, \alpha_2; \alpha_3) = (-\mu \alpha_1 \alpha_2 \alpha_3)
\]

\[
\times \frac{1}{3 \alpha^2} \left[ \sum_{i \neq j} [G^i_1 G^j_1 + G^i_2 G^j_2 - G^i_3 G^j_3] h^1_{ij} \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3) \right]
\]

\[
+ \sum_i [n^1_i + n^2_i - n^3_i] h^1_{ii} \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3)
\]

\[
+ \sum_{i' \neq j'} [G^i_1 G^j_1 + G^i_2 G^j_2 - G^i_3 G^j_3] h^1_{i'j'} \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3)
\]

\[
+ \sum_i [n^1_i + n^2_i - n^3_i] h^1_{i} \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3)
\]

\[
+ \sum_{i' \neq j'} [G^i_1 G^j_1 + G^i_2 G^j_2 - G^i_3 G^j_3] h^1_{i'j'} \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3)
\]

\[
+ \sum_i [n^1_i + n^2_i - n^3_i] h^1_{i} \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3)
\]

\[
+ \frac{1}{8 \mu} \left[ E_{123} \right] h_{1} \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3)
\]

\[
+ \frac{1}{8 \mu} \left[ E_{123} \right] \bar{h}_1 \bar{\tau}_2 \tau_3 \prod_{l=1}^{8} K_{\{n^1_l, n^2_l, n^3_l\}} (\alpha_1, \alpha_2; \alpha_3) + c.c.
\]

Here the following conditions on the occupation numbers must hold for the individual terms in (2.32) to be non-zero

\[
\sum_{l=1}^{8} (n^3_l - n^1_l - n^2_l) \leq 2 \quad \text{for terms in lines 1, 3, 5, and 6}
\]

(2.33)
and
\[
\sum_{I=1}^{8} (n_{I}^{3} - n_{I}^{1} - n_{I}^{2}) \leq 0 \quad \text{for terms in lines 2, 4, 7, and 8.} \quad (2.34)
\]

In our next step we will apply on-shell conditions together with conservation laws and show that in that case the amplitude (2.32) will vanish. Combining the conservation law \( p_{+}^{1} + p_{+}^{2} = p_{+}^{3} \) and (2.16) we obtain the following condition on the occupation numbers
\[
\sum_{I=1}^{8} (n_{I}^{3} - n_{I}^{1} - n_{I}^{2}) = E_{0}^{1} + E_{0}^{2} - E_{0}^{3}. \quad (2.35)
\]

In further analysis we are going to use the results listed in TABLE I of section 3.4 of [7] containing the spectrum of bosonic physical degrees of freedom of Type IIB supergravity on the plane wave background. In particular, we will be using the values of \( E_{0} \) in order to analyse condition (2.35) for various terms in (2.32)
\[
E_{0} (h) = 0, \quad E_{0} (h_{ij'}) = 2, \quad E_{0} (h_{ij}^+) = E_{0} (h_{ij'}^+) = 4,
\]
\[
E_{0} (\tau) = E_{0} (\bar{\tau}) = 4, \quad E_{0} (\tilde{h}_{ij'}) = 6, \quad E_{0} (\tilde{h}) = 8. \quad (2.36)
\]

For the terms in lines 1 through 4 of (2.32) condition (2.35) will read
\[
\sum_{I=1}^{8} (n_{I}^{3} - n_{I}^{1} - n_{I}^{2}) = 4, \quad (2.37)
\]
which clearly violates the non-zero conditions (2.33) and (2.34) implying that the terms in lines 1 through 4 vanish. After performing a similar check for the other terms and excluding all the terms that violate conditions (2.33) and (2.34) the amplitude reads
\[
A_{h\tau\bar{\tau}\{\{n_{1}\},\{n_{2}\};\{n_{3}\}\}} (\alpha_{1}, \alpha_{2};\alpha_{3}) = \left( -\frac{\mu \alpha_{1} \alpha_{2} \alpha_{3}}{3} \right) \\
\times \left( \frac{1}{\alpha_{1}^{2}} \sum_{i,j'} \left[ G_{1}^{i} G_{1}^{j')} + G_{2}^{i} G_{2}^{j')} - G_{3}^{i} G_{3}^{j')} \right] h_{1 \bar{i} j'} \bar{\tau}_{2} \tau_{3} \prod_{I=1}^{8} K_{\{n_{1}, n_{2}; n_{3}\}} (\alpha_{1}, \alpha_{2}; \alpha_{3}) \right)
\]
\[
+ \frac{1}{8 \mu \alpha_{1}^{2}} \left[ E_{123}^{||} - E_{123}^{\perp} \right] h_{1} \tau_{2} \bar{\tau}_{3} \prod_{I=1}^{8} K_{\{n_{1}, n_{2}; n_{3}\}} (\alpha_{1}, \alpha_{2}; \alpha_{3}) \right) \right). \quad (2.38)
\]

For the terms in (2.38), condition (2.35) will read
\[
\sum_{I=1}^{8} (n_{I}^{3} - n_{I}^{1} - n_{I}^{2}) = 2 \quad \text{for terms in line 1,} \quad (2.39)
\]
and
\[ \sum_{I=1}^{8} \left( n_I^3 - n_I^1 - n_I^2 \right) = 0 \quad \text{for terms in line 2.} \quad (2.40) \]

For the terms in the first line of (2.38) we can arbitrarily choose two particular directions \( i \) and \( j' \) for which \( n_i^3 - n_i^1 - n_i^2 = 1 \) and \( n_{j'}^3 - n_{j'}^1 - n_{j'}^2 = 1 \) and apply formula (A.10) while for the remaining six directions we will have \( n_I^3 - n_I^1 - n_I^2 = 0 \) where \( I \neq i, j' \) and apply formula (A.4). For the terms in the second line we have the condition \( n_I^3 - n_I^1 - n_I^2 = 0 \) in all eight directions and can therefore apply formula (A.4). We will therefore have no summation over \( i \) and \( j' \) for in the first line of (2.38). It will be proportional to

\[
\left[ \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3} - \frac{|\alpha_3|}{\alpha_3} \right] (n_i^1 + n_i^2 + 1)^{\frac{1}{2}} (n_{j'}^1 + n_{j'}^2 + 1)^{\frac{1}{2}} \\
\times \prod_{I=1}^{8} \sqrt{\frac{(n_I^1 + n_I^2)!}{n_I^1! n_I^2!}} \left( -\frac{\alpha_1}{\alpha_3} \right)^{n_I^1} \left( -\frac{\alpha_2}{\alpha_3} \right)^{n_I^2} \quad (2.41)
\]

and vanish due to the conservation law \( \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1 + \alpha_2 - |\alpha_3| = 0 \). The second line of (2.38) proportional to

\[
\left( E_{123}^\parallel - E_{123}^\perp \right) = \mu \left( \sum_{i=1}^{4} \left( n_i^3 - n_i^1 - n_i^2 \right) - \sum_{i'=5}^{8} \left( n_{i'}^3 - n_{i'}^1 - n_{i'}^2 \right) \right) \quad (2.42)
\]

will also vanish because \( n_I^3 - n_I^1 - n_I^2 = 0 \) for all \( I = 1, \ldots, 8 \). Therefore, the on-shell graviton dilaton-axion quantum mechanical amplitude in the pp-wave background vanishes at tree level. As a result of this analysis we have to modify conditions (2.33) and (2.34) by replacing the \( \leq \) with \( < \) for the off-shell amplitude (2.32) to be non-zero.
3. Graviton axion-dilaton coupling in pp-wave background from Type IIB supergavity in light-cone gauge

In this section we will calculate the graviton dilaton-axion cubic interaction vertex in the pp-wave background starting from the Type IIB supergravity action. We will see that much of the analysis of the previous section will be carried over to this section. The relevant piece of the Type IIB action is

\[ S_{IIB} = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \frac{g^{\mu\nu} \partial_\mu \tau \partial_\nu \bar{\tau}}{2(\text{Im}\tau)^2}, \]  

(3.1)

where again

\[ \tau(x) = \chi(x) + ie^{-\phi(x)} \]  

(3.2)

is a combination of the dilaton and axion. Expanding the dilaton-axion field around \( \phi = 0, \chi = 0 \) as \( \tau(x) = i + 2\kappa \tau'(x) \) and expanding the metric around the pp-wave background as \( g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) + 2\kappa h_{\mu\nu}(x) \), we obtain from (3.1) the following cubic vertex

\[ S_{h\tau\bar{\tau}} = 2\kappa \int d^{10}x \sqrt{-\tilde{g}} \tilde{g}^{\lambda\mu} \tilde{g}^{\rho\nu} h_{\lambda\rho} \partial_\mu \tau \partial_\nu \bar{\tau}, \]  

(3.3)

where we suppressed the prime. Following Metsaev and Tseytlin [7] we impose the light-cone gauge conditions

\[ h_{-} = 0, \quad h_{-\mu} = 0, \quad h_{+I} = \frac{2}{\partial_-} \partial_J h_{IJ}, \quad h_{++} = \frac{4}{\partial_-} \partial_I \partial_J h_{IJ}, \quad h_{II} = 0, \]  

(3.4)

and substitute for the background metric

\[ \tilde{g}^{++} = \frac{1}{4} \mu^2 x_I^2, \quad \tilde{g}^{+-} = \tilde{g}^{-+} = -\frac{1}{2}, \quad \tilde{g}^{IJ} = \delta^{IJ}, \]  

(3.5)

where \( I, J = 1, ..., 8 \) and the determinant of the background metric \( \tilde{g} = -4 \). We then obtain from (3.3) the following expression for the cubic graviton dilaton-axion interaction in the light-cone gauge

\[ S_{lc} = 4\kappa \int dx^+ dx^- d^8x \left[ \left( \frac{1}{(\partial_-)^2} \partial_I \partial_J h_{IJ} \right) \partial_- \bar{\tau} \partial_- \tau - \left( \frac{1}{\partial_-} \partial_J h_{JJ} \right) \partial_- \bar{\tau} \partial_I \tau \right. \]

\[ - \left. \left( \frac{1}{\partial_-} \partial_J h_{JJ} \right) \partial_I \bar{\tau} \partial_- \tau + h_{IJ} \partial_I \bar{\tau} \partial_J \tau \right]. \]  

(3.6)

\footnote{This is slightly different from [7] since we are using a different metric convention.}
Notice that the result (3.6) is $\mu$ independent and is exactly the same as in flat space ($\mu = 0$). The same feature was found in [24] where as a functional of classical fields, the three-scalar cubic interaction Hamiltonian in the light-cone gauge on the pp-wave was found to be identical to that in flat space. However, as the authors of [24] have pointed out, the quantum mechanical amplitudes on the pp-wave will have an explicit $\mu$ dependence coming from the frequencies of harmonic oscillator modes. The Fock spaces for the flat and the pp-wave backgrounds are very different. In one case we have a collection of free particles, in the other case we have bound states confined by the gravitational potential well and described by the harmonic oscillator wave functions. Since the $p_+$ and $p_-$ are conserved in the pp-wave, we will Fourier transform the fields in the light-cone directions $x^-$ and $x^+$ as follows

$$h_{IJ}(x^-, x^+; \vec{x}) = \frac{1}{2\pi} \int dp_+ \int \frac{d\alpha}{\sqrt{|\alpha|}} h_{IJ}(\alpha, p_+; \vec{x}) e^{-i(\alpha x^- + p_+ x^+)} ,$$

$$\tau(x^-, x^+; \vec{x}) = \frac{1}{2\pi} \int dp_+ \int \frac{d\alpha}{\sqrt{|\alpha|}} \tau(\alpha, p_+; \vec{x}) e^{-i(\alpha x^- + p_+ x^+)} ,$$

$$\bar{\tau}(x^-, x^+; \vec{x}) = \frac{1}{2\pi} \int dp_+ \int \frac{d\alpha}{\sqrt{|\alpha|}} \bar{\tau}(\alpha, p_+; \vec{x}) e^{-i(\alpha x^- + p_+ x^+)} ,$$

where $\alpha \equiv 2p_-$, and obtain from (3.6)

$$S_{lc} = \frac{2\kappa}{3} \frac{1}{2\pi} \int dp_+^3 \int \frac{d\alpha_1}{\sqrt{|\alpha_1|}} \frac{d\alpha_2}{\sqrt{|\alpha_2|}} \frac{d\alpha_3}{\sqrt{|\alpha_3|}} \int d^8 x \delta(p_1^2 + p_2^2 + p_3^2)$$

$$\times \left[ \frac{\alpha_2 \alpha_3}{\alpha_1} \left( \partial_I \partial_J h_{IJ}(\alpha_1, p_1^+; \vec{x}) \right) \bar{\tau}(\alpha_2, p_2^+; \vec{x}) \tau(\alpha_3, p_3^+; \vec{x}) \right.$$

$$\times \left. \frac{\alpha_2}{\alpha_1} \left( \partial_J h_{IJ}(\alpha_1, p_1^+; \vec{x}) \right) \bar{\tau}(\alpha_2, p_2^+; \vec{x}) \partial_I \tau(\alpha_3, p_3^+; \vec{x}) \right.$$ 

$$\left. - \frac{\alpha_2}{\alpha_1} \left( \partial_J h_{IJ}(\alpha_1, p_1^+; \vec{x}) \right) \bar{\tau}(\alpha_2, p_2^+; \vec{x}) \partial_I \tau(\alpha_3, p_3^+; \vec{x}) \right) \right] + c.c. .$$

After integrating by parts and using the conservation law $\alpha_1 + \alpha_2 + \alpha_3 = 0$ we finally get

$$S_{lc} = \frac{2\kappa}{3} \frac{1}{2\pi} \int dp_+^3 \int \frac{d\alpha_1}{\sqrt{|\alpha_1|}} \frac{d\alpha_2}{\sqrt{|\alpha_2|}} \frac{d\alpha_3}{\sqrt{|\alpha_3|}} \int d^8 x \delta(p_1^2 + p_2^2 + p_3^2)$$

$$\times \left[ \frac{1}{\alpha_1^2} \left( \partial_I \partial_J h_{IJ}(\alpha_1, p_1^+; \vec{x}) \right) \bar{\tau}(\alpha_2, p_2^+; \vec{x}) \tau(\alpha_3, p_3^+; \vec{x}) \right.$$

$$\times \left. \frac{1}{\alpha_1} \left( \partial_J h_{IJ}(\alpha_1, p_1^+; \vec{x}) \right) \bar{\tau}(\alpha_2, p_2^+; \vec{x}) \partial_I \tau(\alpha_3, p_3^+; \vec{x}) \right.$$ 

$$\left. + \frac{1}{\alpha_2} h_{IJ}(\alpha_1, p_1^+; \vec{x}) \left( \partial_I \partial_J \bar{\tau}(\alpha_2, p_2^+; \vec{x}) \right) \tau(\alpha_3, p_3^+; \vec{x}) \right) \right] + c.c. .$$
Just as we did in the previous section, we will use the results of [7] to express various components of the graviton in terms of the mass eigenstates (2.27). Then the action becomes

$$S_{lc} = \frac{2\kappa}{3} \frac{1}{2\pi} \int dp_+^1 dp_+^2 dp_+^3 \int \frac{d\alpha_1}{\sqrt{|\alpha_1|}} \frac{d\alpha_2}{\sqrt{|\alpha_2|}} \frac{d\alpha_3}{\sqrt{|\alpha_3|}} \int d^8 x \delta(p_+^1 + p_+^2 + p_+^3) \times \delta(\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2\alpha_3)$$

$$+ \frac{1}{\alpha_1^2} \left[ \frac{1}{\alpha_1} \left( \partial_i \partial_j h_{ij}^+ \right) \bar{\tau}_2 \tau_3 + \frac{1}{\alpha_2} h_{1ij}^+ \left( \partial_i \partial_j \bar{\tau}_2 \right) \tau_3 + \frac{1}{\alpha_3} h_{ij}^+ \bar{\tau}_2 \partial_i \partial_j \tau_3 \right. $$

$$+ \frac{1}{\alpha_1} \left( \partial_i \partial_j \bar{h}_{1ij} \right) \bar{\tau}_2 \tau_3 + \frac{1}{\alpha_2} h_{1ij} \left( \partial_i \partial_j \bar{\tau}_2 \right) \tau_3 + \frac{1}{\alpha_3} h_{ij} \bar{\tau}_2 \partial_i \partial_j \tau_3$$

$$+ \frac{1}{\alpha_1} \left( \partial_i \partial_j \bar{h}_{1ij} \right) \bar{\tau}_2 \tau_3 + \frac{1}{\alpha_2} h_{1ij} \left( \partial_i \partial_j \bar{\tau}_2 \right) \tau_3 + \frac{1}{\alpha_3} h_{ij} \bar{\tau}_2 \partial_i \partial_j \tau_3$$

$$+ \frac{1}{8\alpha_1} \left( \left( \partial_i^2 - \partial_\nu^2 \right) h_1 \right) \bar{\tau}_2 \tau_3 + \frac{1}{8\alpha_2} h_1 \left( \left( \partial_i^2 - \partial_\nu^2 \right) \bar{\tau}_2 \right) \tau_3 + \frac{1}{8\alpha_3} h_1 \bar{\tau}_2 \left( \partial_i^2 - \partial_\nu^2 \right) \tau_3$$

$$+ \frac{1}{8\alpha_1} \left( \left( \partial_i^2 - \partial_\nu^2 \right) \bar{h}_1 \right) \bar{\tau}_2 \tau_3 + \frac{1}{8\alpha_2} \bar{h}_1 \left( \left( \partial_i^2 - \partial_\nu^2 \right) \bar{\tau}_2 \right) \tau_3 + \frac{1}{8\alpha_3} \bar{h}_1 \bar{\tau}_2 \left( \partial_i^2 - \partial_\nu^2 \right) \tau_3$$

$$+ c.c.$$

In further calculations we assume that state 3 is incoming and states 1 and 2 are outgoing. This is easily achieved by relabelling $\alpha_3 \rightarrow -\alpha_3$ and $p_+^3 \rightarrow -p_+^3$. The conservation laws will then read $\alpha_1 + \alpha_2 - \alpha_3 = 0$ and $p_+^1 + p_+^2 - p_+^3 = 0$ where all $\alpha$’s and $p_+$’s are now positive definite. The dynamics of the fields in the transverse directions is governed by the light-cone Hamiltonian [16] given by

$$P_+ = -\frac{1}{\alpha} \partial_\nu^2 + \frac{\mu^2 \alpha}{4} x_\nu^2 + \left( E_0 - 4 \right) \mu .$$

(3.11)

This can be split into two contributions from the two $SO(4)$ directions as

$$P_+ = P_+^\parallel + P_+^\perp;$$

(3.12)

where we have defined

$$P_+^\parallel = -\frac{1}{\alpha} \partial_\nu^2 + \frac{\mu^2 \alpha}{4} x_\nu^2 + \left( E_0 - 2 \right) \mu ,$$

$$P_+^\perp = -\frac{1}{\alpha} \partial_\nu^2 + \frac{\mu^2 \alpha}{4} x_\nu^2 + \left( E_0 - 2 \right) \mu .$$

(3.13)
Following the line of argument presented in [24] we notice that if we use the conservation law \( \alpha_1 + \alpha_2 - \alpha_3 = 0 \), we can insert into the action terms proportional to

\[
\frac{\hbar^2}{4} (\alpha_1 + \alpha_2 - \alpha_3)(x_i^2 - x_i^2)
\]  

(3.14)

without changing it. This allows us to combine \( \hbar \bar{\tau} \tau \) terms as

\[
- \frac{1}{8\alpha_1} \left( (\partial_i^2 - \partial_i^2) h_1 \right) \bar{\tau}_2 \tau_3 - \frac{1}{8\alpha_2} h_1 (\partial_i^2 - \partial_i^2) \bar{\tau}_2 \tau_3 \\
+ \frac{1}{8\alpha_3} h_1 \bar{\tau}_2 (\partial_i^2 - \partial_i^2) \tau_3 = \frac{1}{8} \left( (P_{||} - P_{\perp}) h_1 \right) \bar{\tau}_2 \tau_3 \\
+ \frac{1}{8} h_1 \left( (P_{||} - P_{\perp}) \bar{\tau}_2 \right) \tau_3 - \frac{1}{8} h_1 \bar{\tau}_2 \left( P_{||} - P_{\perp} \right) \tau_3 \\
= \frac{1}{8} \left( (p_{||}^1 + p_{||}^2 - p_{||}^3) - (p_{\perp}^1 + p_{\perp}^2 - p_{\perp}^3) \right) h_1 \bar{\tau}_2 \tau_3,
\]

(3.15)

where we also used the fact that the zero point energy contributions from the two \( SO(4) \) directions cancel. Similarly, we can combine \( \bar{\hbar} \bar{\tau} \tau \) terms to obtain the following expression for the action

\[
S_{lc} = \frac{2\kappa}{3} \frac{1}{2\pi} \int dp_1^1 dp_1^2 dp_1^3 \int \frac{d\alpha_1 d\alpha_2 d\alpha_3}{\sqrt{\alpha_1 \alpha_2 \alpha_3}} \int d^8 x \delta (p_1^1 + p_1^2 - p_1^3) \\
\times \delta (\alpha_1 + \alpha_2 - \alpha_3) (\alpha_1 \alpha_2 \alpha_3) \\
\times \frac{1}{\alpha_1^4} \left[ - \frac{1}{\alpha_1} (\partial_i h_{1 i j}^1) \bar{\tau}_2 \tau_3 - \frac{1}{\alpha_2} h_{1 i j}^1 (\partial_i \partial_j \bar{\tau}_2) \tau_3 + \frac{1}{\alpha_3} h_{1 i j}^1 \bar{\tau}_2 \partial_i \partial_j \tau_3 \\
- \frac{1}{\alpha_1} (\partial_i h_{1 i j}^1 \bar{\tau}_2 \tau_3 - \frac{1}{\alpha_2} h_{1 i j}^1 (\partial_i \partial_j \bar{\tau}_2) \tau_3 + \frac{1}{\alpha_3} h_{1 i j}^1 \bar{\tau}_2 \partial_i \partial_j \tau_3 \\
- \frac{1}{\alpha_1} (\partial_i h_{1 i j}^1 \bar{\tau}_2 \tau_3 - \frac{1}{\alpha_2} h_{1 i j}^1 (\partial_i \partial_j \bar{\tau}_2) \tau_3 + \frac{1}{\alpha_3} h_{1 i j}^1 \bar{\tau}_2 \partial_i \partial_j \tau_3 \\
- \frac{1}{\alpha_1} (\partial_i h_{1 i j}^1 \bar{\tau}_2 \tau_3 - \frac{1}{\alpha_2} h_{1 i j}^1 (\partial_i \partial_j \bar{\tau}_2) \tau_3 + \frac{1}{\alpha_3} h_{1 i j}^1 \bar{\tau}_2 \partial_i \partial_j \tau_3 \\
+ \frac{1}{8} \left( (p_{||}^1 + p_{||}^2 - p_{||}^3) - (p_{\perp}^1 + p_{\perp}^2 - p_{\perp}^3) \right) h_1 \bar{\tau}_2 \tau_3 \\
+ \frac{1}{8} \left( (p_{||}^1 + p_{||}^2 - p_{||}^3) - (p_{\perp}^1 + p_{\perp}^2 - p_{\perp}^3) \right) \bar{\hbar} \bar{\tau}_2 \tau_3 \right] + c.c.
\]

(3.16)

Corresponding to the Hamiltonian (3.11) are eight-dimensional harmonic oscillator wave functions \( \psi_k (\sqrt{\frac{\hbar}{2}} \vec{x}) \) written explicitly in Appendix B. They form a complete basis. It is
natural to expand our interacting fields in such a basis

\[ h_{ij}^\perp (\alpha, p_+; \vec{x}) = \sum_{\vec{k}} h_{ij}^\perp (\alpha, p_+; \vec{k}) \psi_{\vec{k}} \left( \sqrt{\frac{\mu \alpha}{2}} \vec{x} \right), \]

\[ h_{i'j'}^\perp (\alpha, p_+; \vec{x}) = \sum_{\vec{k}} h_{i'j'}^\perp (\alpha, p_+; \vec{k}) \psi_{\vec{k}} \left( \sqrt{\frac{\mu \alpha}{2}} \vec{x} \right), \]

\[ h_{ij'}^\perp (\alpha, p_+; \vec{x}) = \sum_{\vec{k}} h_{ij'}^\perp (\alpha, p_+; \vec{k}) \psi_{\vec{k}} \left( \sqrt{\frac{\mu \alpha}{2}} \vec{x} \right), \]

\[ h (\alpha, p_+; \vec{x}) = \sum_{\vec{k}} h (\alpha, p_+; \vec{k}) \psi_{\vec{k}} \left( \sqrt{\frac{\mu \alpha}{2}} \vec{x} \right), \]

\[ \tau (\alpha, p_+; \vec{x}) = \sum_{\vec{k}} \tau (\alpha, p_+; \vec{k}) \psi_{\vec{k}} \left( \sqrt{\frac{\mu \alpha}{2}} \vec{x} \right), \]

(3.17)

and similarly for \( h_{i'j'}, \bar{h} \) and \( \bar{\tau} \). The following analysis will be analogous to that of the previous section but now we will use the properties of the eight-dimensional harmonic oscillator wave functions given in Appendix B in place of the Fock space amplitudes of Appendix A. If we were now to calculate an off-shell interaction vertex from (3.16) using expansions (3.17) we would have to use the most general case expression for (B.5) given in [28] and formulas (B.9)-(B.11). We state in Appendix A, that the general case formula for the string calculation (A.2) will reduce to (B.5) up to a normalization factor if we use the Neumann coefficients for the supergravity vertex given in [16]. Likewise, certain products of the string formulas (A.6)-(A.8) will reduce to the general case for (B.9)-(B.11). We can therefore conclude that up to a normalization factor, the off-shell interaction amplitude (2.32) containing the \( \alpha' \) corrections will simply reduce for \( \alpha' \to 0 \) to an expression that we could also have obtained from (3.16) which would have no string corrections. Although further analysis is almost identical to the one we carried out at the end the previous section, we will nevertheless include it for the purpose of completeness. Once we insert the expansions (3.17) into the action (3.16), we will apply conditions (B.6) and (B.12) in combination with (2.35) and (2.36) to see which terms survive. For instance, let us consider the first nine terms in (3.16) of the form \( h_{ij}^\perp \bar{\tau} \bar{\tau} \) and \( h_{i'j'}^\perp \bar{\tau} \bar{\tau} \) with various second derivatives. For all those terms (2.35) reads

\[ \sum_{i=1}^{8} (k_i^3 - k_i^1 - k_i^2) = 4. \]

(3.18)
After we substitute (3.17) into (3.16) all those terms will result in integrals of type (B.9)-(B.11) and will automatically vanish because (3.18) implies (B.12). Performing a similar analysis for the other terms and dropping all those that vanish the action now reads

$$S_{lc} = \frac{2\kappa}{3} \frac{1}{2\pi} \int dp_+^1 dp_+^2 dp_+^3 \int \frac{da_1 da_2 da_3}{\sqrt{\alpha_1 \alpha_2 \alpha_3}} \int d^8x \delta(p_+^1 + p_+^2 - p_+^3)$$

$$\times \delta(\alpha_1 + \alpha_2 - \alpha_3)(\alpha_1 \alpha_2 \alpha_3)$$

$$\times \left[ \frac{1}{\alpha_1^2} \left( - \frac{1}{\alpha_1} (\partial_i \partial_j h_{1_{ij'}}) \bar{\tau}_2 \tau_3 - \frac{1}{\alpha_2} h_{1_{ij'}} (\partial_i \partial_j \bar{\tau}_2) \tau_3 + \frac{1}{\alpha_3} h_{1_{ij'}} \bar{\tau}_2 \partial_i \partial_j \tau_3 \right) \right.$$

$$+ \frac{1}{8\alpha_1^2} \left( E_{123}^\| - E_{123}^\perp \right) h_{1_{ij}} \bar{\tau}_2 \tau_3 \left] \right. \right) \quad (3.19)$$

Here we again used the notation of [24] to define

$$E_{123}^\| = p_+^1 + p_+^2 - p_+^3, \quad E_{123}^\perp = p_+^1 - p_+^2 + p_+^3. \quad (3.20)$$

We see that all the remaining terms constitute special cases described in Appendix B. Namely, for the three terms in (3.19) containing second derivatives, condition (2.35) reads

$$\sum_{I=1}^{8} (k_3^I - k_1^I - k_2^I) = 2, \quad (3.21)$$

while for the last terms with no derivatives it reads

$$\sum_{I=1}^{8} (k_3^I - k_1^I - k_2^I) = 0. \quad (3.22)$$

We notice immediately that (3.21) implies (B.13) so we can use (B.14)-(B.16) to evaluate the integrals (B.9)-(B.11) appearing in the terms of (3.19) containing second derivatives. Since (3.22) implies (B.7) we can use (B.8) to evaluate the last terms. The sum of the second derivative terms in (3.19) will then be proportional to

$$\left( - \frac{1}{\alpha_1} I_1 - \frac{1}{\alpha_2} I_2 + \frac{1}{\alpha_3} I_3 \right)$$

$$= \frac{\mu}{4\alpha_3} (-\alpha_1 - \alpha_2 + \alpha_3) (k_1^1 + k_2^2 + 1)^{\frac{1}{2}} (k_{j'}^1 + k_{j'}^2 + 1)^{\frac{1}{2}}$$

$$\times F_{\{\bar{k}_1, \bar{k}_2, k_1 + k_2\}} (\alpha_1, \alpha_2; \alpha_3), \quad (3.23)$$

and will vanish due the conservation law $\alpha_1 + \alpha_2 - \alpha_3 = 0$. The last term in (3.19) is proportional to

$$\left( E_{123}^\| - E_{123}^\perp \right) = \mu \left( \sum_{i=1}^{4} (k_3^i - k_1^i - k_2^i) - \sum_{i'=5}^{8} (k_3^{i'} - k_1^{i'} - k_2^{i'}) \right), \quad (3.24)$$

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where the zero point energy contributions cancelled each other and since \( k_1^I + k_2^I = k_3^I \) for \( I = 1, \ldots, 8 \), the term proportional to \( (E_{123}^\parallel - E_{123}^\perp) \) vanishes because individual terms inside the sums in (3.24) are zero.

4. Conclusion

In this paper we explicitly calculated the graviton dilaton-axion three-point vertex in the light-cone gauge in the pp-wave background. In section 2 we employed the light-cone string field theory formalism to obtain the off-shell vertex containing the stringy \( \alpha' \) corrections. Through a careful analysis we showed that the vertex vanishes when all the particles are on-shell and the conservation laws are imposed. In section 3 we approached the same problem from the low energy limit and expanded a particular sector of the Type IIB supergravity action around the pp-wave background in the light-cone gauge. We then analysed the supergravity graviton dilaton-axion vertex using the properties of the eight-dimensional harmonic oscillator wave functions and showed that the interaction vertex vanishes when the conservation laws are combined with the on-shell conditions.

The authors of [28] have investigated a possibility of the S-matrix formulation for scalar field theories as well as a gauge theory coupled to scalars on the plane-wave background. In section (3.2.1) of [28] it was shown that for gauge theory coupled to scalars, the corresponding interaction vertex for the on-shell “photon” exchange vanishes after some non-trivial cancellations. This result was used to argue that the propagator for the exchanged particle in a four-point amplitude will never blow-up since the exchanged particle will always be off-shell. This, together with the condition of convergence for the four-point amplitudes allowed them to conclude that the S-matrix interpretation for field theories on the plane wave background exists, at least at the tree level.

Our result is in the spirit of [28] and we can similarly conclude that in four-point amplitudes involving scalars coupled to gravity on the pp-wave background, the propagator of the exchanged particle will never blow up and the potentially dangerous “graviton” will always be off-shell. Moreover, our result is true not only in the case of a field theory such as Type IIB supergravity (section 3), but also in the case of the full Type IIB string theory (section 2).
Appendix A. Three-Particle Bosonic Amplitudes

For our computations we need to evaluate the expectation value given by

\[ K_{\{n^1_1,n^2_1,n^3_1\}}(\alpha_1, \alpha_2; \alpha_3) = \frac{i^{n^1_1+n^2_1+n^3_1}}{\sqrt{n^1_1!n^2_1!n^3_1!}} (0 | (a^I_1)^{n^1_1} (a^I_2)^{n^2_1} (a^I_3)^{n^3_1} E^0_a | 0) , \]  

(A.1)

where \( E^0_a \) is given in (2.3). Taking into account the fact that \( \mathcal{N}_{33} = 0 \), after a careful analysis we obtain the following formula for the general case

\[ K_{\{n^1_1,n^2_1,n^3_1\}}(\alpha_1, \alpha_2; \alpha_3) = i^{n^1_1+n^2_1+n^3_1} \sqrt{\frac{n^1_1!n^2_1!n^3_1!}{2^{n^1_1+n^2_1+n^3_1}}} \sum_{l_1=0}^{n^1_1} \sum_{l_2=0}^{n^2_1} \frac{(N_{11})^{n^1_1-l_1}}{(n^2_1-l_1)!} \frac{(N_{12})^{n^1_1-l_1}}{(n^1_1-l_1)!} \frac{(N_{13})^{n^1_1-l_1}}{(n^1_1-l_1)!} \frac{(N_{22})^{n^2_1-l_2}}{(n^2_1-l_2)!} \frac{(N_{23})^{n^2_1-l_2}}{(n^2_1-l_2)!} \frac{(N_{33})^{n^2_1-l_2}}{(n^2_1-l_2)!} \times \]  

(A.2)

where \( \frac{n^3_1+l_1+l_2}{2} \) is an integer and \( \frac{n^3_1+l_1+l_2}{2} \geq \max\{n^2_1, l_1, l_2\} \) and both \( n^1_1 - l_1 \) and \( n^2_1 - l_2 \) must be even. Therefore \( K_{\{n^1_1,n^2_1,n^3_1\}}(\alpha_1, \alpha_2; \alpha_3) \) is non-zero only for \( n^2_1 - n^1_1 - n^3_1 \leq 0 \).

If we use the Neumann coefficients for the supergravity vertex given in [16], expression (A.2) will reduce to formula (A.5) given in Appendix A of [28] up to a factor of \( \sqrt{\pi} \sqrt{2^{n^1_1+n^2_1+n^3_1}} (n^1_1!n^2_1!n^3_1!) \). Formula (B.5) from the next section will represent precisely such case. For a special case when \( n^2_1 - n^1_1 - n^3_1 = 0 \) formula (A.2) will reduce to

\[ K_{\{n^1_1,n^2_1,n^3_1\}}(\alpha_1, \alpha_2; \alpha_3) = (-1)^{n^1_1+n^2_1} \sqrt{\frac{(n^1_1+n^3_1)!}{n^1_1!n^3_1!}} \mathcal{N}_{13} \mathcal{N}^{n^2_1}_{23} , \]  

(A.3)

which was derived in [25]. Notice, that the Neumann coefficients for the zero modes of the string vertex in (2.5) coincide with those of the supergravity vertex \( M_{13} \) and \( M_{23} \) given in [16]. Substituting explicitly for \( \mathcal{N}_{13} \) and \( \mathcal{N}_{23} \) we obtain from (A.3)

\[ K_{\{n^1_1,n^2_1,n^3_1\}}(\alpha_1, \alpha_2; \alpha_3) = \sqrt{\frac{(n^1_1+n^2_1)!}{n^1_1!n^2_1!}} \left( -\frac{\alpha_1}{\alpha_3} \right)^{n^1_1} \left( -\frac{\alpha_2}{\alpha_3} \right)^{n^2_1} , \]  

(A.4)

which coincides with formula (A.6) given in Appendix A of [28] up to a factor of \( \sqrt{\pi} 2^{n^1_1+n^2_1} \sqrt{\frac{(n^1_1+n^3_1)!}{n^1_1!n^3_1!}} \). Another expectation value that we would like to evaluate is

\[ G^I_{r\{n^1_1,n^2_1,n^3_1\}}(\alpha_1, \alpha_2; \alpha_3) = \frac{i^{n^1_1+n^3_1+n^3_1}}{\sqrt{n^1_1!n^2_1!n^3_1!}} (0 | (a^I_1)^{n^1_1} (a^I_2)^{n^2_1} (a^I_3)^{n^3_1} (a^I_r + a^I_r) E^0_a | 0) . \]  

(A.5)
Using the commutation relation in (2.24) in combination with the definition (A.1), we obtain for $n^I_3 - n^I_1 - n^I_2 \leq 1$

\[
G^I_{\{n^I_1,n^I_2,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3) = -i \sqrt{n^I_1 + 1} K_{\{n^I_1+1,n^I_2,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3) \\
+ i \sqrt{n^I_1} K_{\{n^I_1+1,n^I_2,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3), \tag{A.6}
\]

\[
G^I_{\{n^I_1,n^I_2,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3) = -i \sqrt{n^I_2 + 1} K_{\{n^I_1,n^I_2+1,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3) \\
+ i \sqrt{n^I_2} K_{\{n^I_1,n^I_2+1,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3), \tag{A.7}
\]

\[
G^I_{\{n^I_1,n^I_2,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3) = -i \sqrt{n^I_3 + 1} K_{\{n^I_1,n^I_2,n^I_3+1\}}(\alpha_1, \alpha_2; \alpha_3) \\
+ i \sqrt{n^I_3} K_{\{n^I_1,n^I_2,n^I_3+1\}}(\alpha_1, \alpha_2; \alpha_3), \tag{A.8}
\]

and

\[
G^I_{\{n^I_1,n^I_2,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3) = 0, \text{ if } n^I_3 - n^I_1 - n^I_2 > 1. \tag{A.9}
\]

For a special case when $n^I_3 - n^I_1 - n^I_2 = 1$, (A.5) becomes

\[
G^I_{\{n^I_1,n^I_2,n^I_3\}}(\alpha_1, \alpha_2; \alpha_3) = -i \text{ sign}(\alpha_r)(n^I_1 + n^I_2 + 1)^{\frac{1}{2}} \left( \frac{\lvert \alpha_r \rvert}{\alpha_3} \right)^{\frac{1}{2}} \\
x K_{\{n^I_1,n^I_2,n^I_3+1\}}(\alpha_1, \alpha_2; \alpha_3). \tag{A.10}
\]

**Appendix B. Integrals Involving Harmonic Oscillator Wave Functions**

Here we will list a few useful formulas and identities involving the eight-dimensional harmonic oscillator wave functions. This section will contain expressions that can be easily derived based on the formulas given in the Appendix A of [28] as well as in [29]. The lightcone Hamiltonian for a physical field is

\[
P_+ = -\frac{1}{\alpha} \partial_I^2 + \frac{\mu^2 \alpha}{4} x_I^2 + (E_0 - 4) \mu, \tag{B.1}
\]

where $(E_0 - 4) \mu$ is a contribution to the zero point energy coming from the fermionic zero modes. The corresponding wave function is

\[
\psi^+_k \left( \sqrt{\frac{\mu \alpha}{2}} \right) = \prod_{I=1}^{8} \psi_{k_I} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right), \tag{B.2}
\]

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where
\[ \psi_{k_I} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) = \left( \frac{\alpha \mu}{2 \pi} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^{k_I} k_I!}} e^{-\mu \alpha x_I^2/4} H_{k_I} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right), \]
with the energy
\[ p_+ = \mu \left( \sum_{I=1}^{8} k_I + E_0 \right). \]

Following the notation of [28] we define
\[ F\{\vec{k}_1, \vec{k}_2, \vec{k}_3\}(\alpha_1, \alpha_2; \alpha_3) = \prod_{I=1}^{8} F\{k_I^1, k_I^2, k_I^3\}(\alpha_1, \alpha_2; \alpha_3), \]
\[ F\{k_I^1, k_I^2, k_I^3\}(\alpha_1, \alpha_2; \alpha_3) = \int \psi_{k_I^1} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \psi_{k_I^2} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \psi_{k_I^3} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \, dx_I, \]
where \( \alpha_3 = \alpha_1 + \alpha_2 \). The general expression for \( F\{k_I^1, k_I^2, k_I^3\}(\alpha_1, \alpha_2; \alpha_3) \) when \( k_I^3 - k_I^1 - k_I^2 \leq 0 \) was given in [28] and it was found that it vanishes if \( k_I^3 - k_I^1 - k_I^2 > 0 \). Therefore we have the following condition
\[ F\{\vec{k}_1, \vec{k}_2, \vec{k}_3\}(\alpha_1, \alpha_2; \alpha_3) = 0, \text{ if } \sum_{I=1}^{8} (k_I^3 - k_I^1 - k_I^2) > 0. \]

Of particular interest will be the case when \( k_I^3 = k_I^1 + k_I^2 \). In this special case also given in [28],
\[ \sum_{I=1}^{8} (k_I^1 + k_I^2 - k_I^3) = 0, \]
\[ F\{\vec{k}_1, \vec{k}_2, \vec{k}_1 + \vec{k}_2\}(\alpha_1, \alpha_2; \alpha_3) = \prod_{I=1}^{8} \left( \frac{\mu \alpha_1 \alpha_2}{2 \pi \alpha_3} \right)^{\frac{1}{4}} \sqrt{\frac{(k_I^1 + k_I^2)!}{k_I^1! k_I^2!}} \left( \frac{\alpha_1}{\alpha_3} \right)^{k_I^1} \left( \frac{\alpha_2}{\alpha_3} \right)^{k_I^2}. \]

Other cases of interest will involve integrals with second derivatives of the wave functions. The cases relevant to our calculations will involve
\[ I_1 = \prod_{I=1}^{8} \int \left( \partial_J \partial_K \psi_{k_I^1} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \right) \psi_{k_I^2} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \psi_{k_I^3} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \, dx_I, \]
\[ I_2 = \prod_{I=1}^{8} \int \psi_{k_I^1} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \left( \partial_J \partial_K \psi_{k_I^2} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \right) \psi_{k_I^3} \left( \sqrt{\frac{\mu \alpha}{2}} x_I \right) \, dx_I, \]
\[
I_3 = \prod_{l=1}^{8} \int \psi_{k^3_l} \left( \frac{\sqrt{\mu \alpha_1}}{2} \cdot x_l \right) \psi_{k^2_l} \left( \frac{\sqrt{\mu \alpha_2}}{2} \cdot x_l \right) \left( \partial_J \partial_K \psi_{k^1_l} \left( \frac{\sqrt{\mu \alpha_3}}{2} \cdot x_l \right) \right) dx_l. \quad (B.11)
\]

Using formulas (A.5)-(A.7) of [28] it is straightforward to show that

\[
I_1 = I_2 = I_3 = 0, \quad \text{if} \quad \sum_{l=1}^{8} (k^3_l - k^1_l - k^2_l) > 2. \quad (B.12)
\]

For purposes of the calculation it is important to note that (B.12) is true for both \( J = K \) and \( J \neq K \). For a special case

\[
\sum_{l=1}^{8} (k^3_l - k^1_l - k^2_l) = 2, \quad (B.13)
\]

we have

\[
I_1 = \frac{\mu \alpha_1^2}{4 \alpha_3} (k^1_J + k^2_J + 1)^{\frac{3}{2}} \left( k^1_K + k^2_K + 1 \right)^{\frac{3}{2}} F_{\{\bar{k}_1, \bar{k}_2; \bar{k}_1 + \bar{k}_2\}} (\alpha_1, \alpha_2; \alpha_3), \quad (B.14)
\]

\[
I_2 = \frac{\mu \alpha_2^2}{4 \alpha_3} (k^1_J + k^2_J + 1)^{\frac{3}{2}} \left( k^1_K + k^2_K + 1 \right)^{\frac{3}{2}} F_{\{\bar{k}_1, \bar{k}_2; \bar{k}_1 + \bar{k}_2\}} (\alpha_1, \alpha_2; \alpha_3), \quad (B.15)
\]

\[
I_3 = \frac{\mu \alpha_3^2}{4} (k^1_J + k^2_J + 1)^{\frac{3}{2}} \left( k^1_K + k^2_K + 1 \right)^{\frac{3}{2}} F_{\{\bar{k}_1, \bar{k}_2; \bar{k}_1 + \bar{k}_2\}} (\alpha_1, \alpha_2; \alpha_3), \quad (B.16)
\]

where \( J \neq K \) and the occupation numbers must satisfy the condition

\[
k^3_I = \begin{cases} k^1_I + k^2_I + 1 & \text{when } I = J \text{ or } I = K \\ k^1_I + k^2_I & \text{otherwise} \end{cases} \quad (B.17)
\]

### Appendix C. Some \( \gamma \) matrix identities

This section contains some \( SO(8) \) \( \gamma \) matrix identities that we derived using MathTensor and FeynCalc packages for Mathematica. The gamma matrices satisfy

\[
\gamma^J_{a\dot{c}} \gamma^J_{\dot{c}b} + \gamma^J_{a\dot{c}} \gamma^I_{\dot{c}b} = 2 \delta^{IJ} \delta_{ab}, \quad (C.1)
\]

and

\[
\gamma^I_{ab} = \frac{1}{2} \left( \gamma^I_{a\dot{c}} \gamma^J_{\dot{c}b} - \gamma^J_{a\dot{c}} \gamma^I_{\dot{c}b} \right). \quad (C.2)
\]

The self dual tensor \( t_{abcd}^{IJ} \) is defined as follows

\[
t_{abcd}^{IJ} = \gamma_{[ab}^{JK} \gamma_{cd]}^{IJ}. \quad (C.3)
\]
Based on the definitions (C.1)-(C.3), we have derived the following two identities

\[ t_{abcd}^{IJ} t_{abcd}^{KL} = 192 \delta^{IL} \delta^{KJ} - 48 \delta^{IJ} \delta^{KL} + 192 \delta^{IK} \delta^{JL}, \]  

(C.4)

and for a more complicated case we have

\[ t_{abcd}^{MN} \gamma_a^I \gamma_b^J \gamma_c^K \gamma_d^L t_{hbcd}^{PQ} = \]

\[ -96 \delta^{ML} \delta^{PK} \delta^{NQ} \delta^{IJ} + 96 \delta^{MK} \delta^{PL} \delta^{NQ} \delta^{IJ} + 96 \delta^{MQ} \delta^{PL} \delta^{NQ} \delta^{IK} \]

\[ -96 \delta^{MK} \delta^{PL} \delta^{NQ} \delta^{JK} - 96 \delta^{ML} \delta^{PK} \delta^{NQ} \delta^{JK} - 96 \delta^{MP} \delta^{NL} \delta^{QJ} \delta^{IK} \]

\[ + 96 \delta^{MP} \delta^{NL} \delta^{QJ} \delta^{JK} + 96 \delta^{MP} \delta^{NL} \delta^{QJ} \delta^{IK} \]

\[ -96 \delta^{MQ} \delta^{PK} \delta^{NJ} \delta^{IL} - 96 \delta^{MQ} \delta^{PK} \delta^{NJ} \delta^{IL} - 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} \]

\[ + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} + 96 \delta^{MP} \delta^{NL} \delta^{QJ} \delta^{JK} - 96 \delta^{MJ} \delta^{PK} \delta^{NJ} \delta^{IL} - 96 \delta^{MJ} \delta^{PK} \delta^{NJ} \delta^{IL} \]

\[ -96 \delta^{MP} \delta^{NL} \delta^{QJ} \delta^{JK} + 96 \delta^{MJ} \delta^{PK} \delta^{NJ} \delta^{IL} + 96 \delta^{MJ} \delta^{PK} \delta^{NJ} \delta^{IL} \]

\[ + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} - 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} - 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} \]

\[ + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} \]

\[ + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} \]

\[ + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} + 96 \delta^{MN} \delta^{PL} \delta^{QJ} \delta^{JK} \]

(C.5)

Although (C.4)-(C.5) were not used in this paper, they may prove to be very useful in future calculations. For a more detailed list of properties of SO(8) $\gamma$ matrices see [15].

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