New approach to calculating the News

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We consider the problem of finding the gravitational radiation output, or news, within the context of a numerical simulation of a spacetime by means of the null-cone, or characteristic, approach to numerical relativity. We develop a method for computing the news that uses an explicit coordinate transformation to a coordinate system that satisfies the Bondi conditions. The method has been implemented computationally. We present results of applying the method to certain test problems, demonstrating second order convergence of the news to the analytic value.

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I. INTRODUCTION

In many simulations in numerical relativity, an important objective is to compute the emitted gravitational radiation. Within ADM-type approaches, one uses computed data near the outer boundary of the simulation to estimate the radiation at future null infinity (I+) in the form of an expansion of spherical harmonics. The underlying theory was developed some time ago [1, 2, 3], and the computational implementation is discussed, for example, in [4, 5, 6]. However, this paper uses the characteristic, or null-cone, approach with Bondi-Sachs coordinates, in which I+ is included in the compactified grid, and therefore the gravitational radiation, or news, can in principle be found exactly [7].

Even so, the computation of the news is not a straightforward matter. The news takes a very simple form, \( N = \frac{1}{2} J_{\ell u} \), when the null-cone coordinate system satisfies the Bondi conditions (the meaning of these terms and symbols will be given later). However, the Bondi conditions are conditions at I+ (where \( \ell = 0 \)), and in any physical simulation the freedoms in the null-cone coordinates are set in the interior of the spacetime. In general, a computed metric will not satisfy the Bondi conditions. Thus, a procedure for computing the news in a general Bondi-Sachs coordinate system has been developed [8].

A motivation for this work is that some difficulties with the news computation of [8] have been reported [9], specifically significant gravitational radiation when the gravitational field is static. The cause of the difficulties is not known, and it may be that they can be resolved by technical improvements to the method and code [10], but as yet no such results have been reported. Our method has also not been tested for the problem of [9], and so it is not certain that it will be of assistance; but because of its simplicity (see below) it would be much easier to analyze, and thereby resolve, any problems.

However, our primary motivation is that the method presented here is much simpler than the previous method. Conceptually, the idea is straightforward: we construct a transformation to the “right” coordinate system, and consequently there is a means to monitor the accuracy of results obtained. The technical differences between the two methods are summarized in Sec. IV. Here we note that the present method requires two fewer evolution equations than the previous method, and also that the most complicated formula in the present method is Eq. (25), whereas the formulas for the previous method were given in an appendix taking up a whole page. In numerical work, simplicity means that there is less opportunity for the introduction of error. Of course, the previous method has been tested against available analytic solutions (as will be done for the method presented here), but these solutions are rather special. Thus, whatever the computational performance of the two methods, it would still be necessary to develop the present method and use it to validate results obtained by the previous method.

We construct an explicit coordinate transformation between the general Bondi-Sachs coordinate system and a Bondi-Sachs coordinate system that satisfies the Bondi conditions (given in Eq. (9)). The coordinate transformation on \( I^+ \) is known [8], but constructing the transformation throughout the spacetime is much more difficult. The key point of our method is that, while it is in principle possible to find the transformation throughout the spacetime, we do not need it. Rather, we need to evaluate \( \frac{1}{2} J_{\ell u} \) on I+ (where \( \ell = 0 \)), and thus we need the transformed metric off I+ only to first order in \( \ell \). We proceed by expressing the coordinate transformation as a Taylor series in \( \ell \). The algebraic calculations are quite lengthy and require the use of computer algebra.

Our news calculation has been implemented computationally, and used in problems in which an analytic result is known. Results are presented below, and it is seen that the errors are second-order convergent to zero.

The paper starts with a summary of relevant results and notation for the characteristic formulation of numerical relativity (Sec. II). The coordinate transformation and computer algebra results are given in Sec. III and then the procedure for computing the news – at both analytic and computational levels – is described in Sec. IV. The computational tests and results are described in Sec. V. The Conclusion, Sec. VI, gives a comparison between the method in this paper and that of [8], and also discusses possibilities for further work.
II. NOTATION

The formalism for the numerical evolution of Einstein’s equations, in null cone coordinates, is well known \[1, 3, 14\] (see also \[11, 12, 13, 14\]). For the sake of completeness, we give a summary of those aspects of the formalism that will be used here. We start with coordinates based upon a family of outgoing null hypersurfaces. We let \( u \) label these hypersurfaces, \( x^A (A = 2, 3) \), label the null rays and \( r \) be a surface area coordinate. In the resulting \( x^α = (u, r, x^A) \) coordinates, the metric takes the Bondi-Sachs form \[14, 15\]

\[
ds^2 = -\left(e^{2β}(1 + \frac{W}{r}) - r^2 h_{AB} U^A U^B\right) du^2 -2e^{2β} du dr - 2r^2 h_{AB} U^B du x^A + r^2 h_{AB} dx^A dx^B, \quad (1)
\]

where \( h_{AB} h_{BC} = \delta^C_A \) and \( \det(h_{AB}) = \det(q_{AB}) \), with \( q_{AB} \) a unit sphere metric. We work in stereographic coordinates \( x^A = (q, p) \) for which the unit sphere metric is

\[
q_{AB} dx^A dx^B = \frac{4}{P^2}(dq^2 + dp^2), \quad (2)
\]

where

\[
P = 1 + q^2 + p^2. \quad (3)
\]

We also introduce a complex dyad \( q_A \) defined by

\[
q^A = \frac{P}{2} (1, i), \quad q_A = \frac{2}{P} (1, i) \quad (4)
\]

with \( i = \sqrt{-1} \). For an arbitrary Bondi-Sachs metric, \( h_{AB} \) can then be represented by its dyad component

\[
J = h_{AB} q^A q^B/2, \quad (5)
\]

with the spherically symmetric case characterized by \( J = 0 \). The full nonlinear \( h_{AB} \) is uniquely determined by \( J \), since the determinant condition implies that the remaining dyad component

\[
K = h_{AB} q^A q^B/2 \quad (6)
\]

satisfies \( 1 = K^2 - J J \). We introduce the spin-weighted field

\[
U = U^A q_A, \quad (7)
\]

as well as the (complex differential) \( \bar{\alpha} \) operators \( \bar{\alpha} \) and \( \bar{\alpha} \) (see \[16\] for full details).

The news calculation is performed in conformally compactified coordinates. Specifically, \( (u, r, x^A) \rightarrow (u,\ell, x^A) \) where \( \ell = 1/r \). In \( (u,\ell, x^A) \) coordinates, the compactified metric is \( ds^2 = \ell^2 ds^2 \) where \( \ell \) is a conformal factor with future null infinity \( \mathcal{I}^+ \) given by \( \ell = 0 \). The compactified metric will be denoted by \( \hat{g}_{αβ} \), and the general compactified Bondi-Sachs metric is

\[
\hat{g}^{11} = e^{-2β} V_α, \quad \hat{g}^{1A} = e^{-2β} U^A, \quad (8)
\]

\[
\hat{g}^{10} = e^{-2β}, \quad \hat{g}^{AB} = h^{AB}, \quad \hat{g}^{00} = 0, \quad (8)
\]

where \( V_α = \ell^2 (1 + \ell W) \). In addition to the general compactified Bondi-Sachs metric Eq. (8), we will refer to the compactified Bondi-Sachs metric satisfying the Bondi conditions, and such quantities will be denoted with a suffix \([B]\), with compactified metric and coordinates \( \hat{g}_{αβ}^{[B]} \) and \( (u_{[B]}, \ell_{[B]}, x^A_{[B]}) \), respectively. On \( \mathcal{I}^+ \), i.e. \( \ell_{[B]} = 0 \), \( \hat{g}_{αβ}^{[B]} \) satisfies the Bondi conditions

\[
\hat{g}^{11}_{[B]} = 0, \quad \hat{g}^{1A}_{[B]} = 0, \quad \hat{g}^{01}_{[B]} = 1, \quad \hat{g}^{AB}_{[B]} = q^{AB}_{[B]}, \quad (9)
\]

where \( q^{AB}_{[B]} \) is a unit sphere metric with respect to the Bondi angular coordinates \( x^A_{[B]} \).

III. COORDINATE TRANSFORMATION

We define a coordinate transformation, near \( \mathcal{I}^+ \), between \( (u, \ell, x^A) \) and \( (u_{[B]}, \ell_{[B]}, x^A_{[B]}) \):

\[
u_{[B]} = u_{[B]0} + A^u \ell + C^u \ell^2, \quad \ell_{[B]} = \omega \ell + C^\ell \ell^2,
\]

\[
x^A_{[B]} = x^A_{[B]0} + A^A \ell + C^A \ell^2, \quad (10)
\]

where \( \omega, x^A_{[B]0}, A^A, u_{[B]0} \) are all functions of \( x^A \) and \( u \) only. It will turn out that the \( C^α \) are irrelevant to the news calculation, but they are included so that the Jacobian (constructed by differentiating the above) is manifestly correct to first order in \( \ell \). We also introduce complex quantities

\[
A = q_A A^A \quad \text{and} \quad X = q_A x^A_{[B]0}, \quad (11)
\]

both of which are defined to have spin weight 1. The general metric quantities are expanded as a Taylor series to first order in \( \ell \), with the \( q \) quantities being the values at \( \ell = 0 \), i.e. on \( \mathcal{I}^+ \), and the \( x \) quantities being the first \( \ell \) derivative evaluated at \( \ell = 0 \), i.e. \( (\partial_\ell x)_0 = 0 \).

\[
\beta = \beta_0 + \ell \beta_1, \quad U = U_0 + U_\ell \ell, \quad J = J_0 + \ell J_\ell, \quad K = K_0 + \ell K_\ell, \quad V_α = \ell V_{α\ell}. \quad (12)
\]

The same first-order expansion in \( \ell \) and notation will be used for the metric quantities in the coordinate system satisfying the Bondi conditions (e.g., \( J_{[B]} = J_{[B]0} + \ell J_{[B]_\ell} \)). The general compactified Bondi-Sachs metric, and the Bondi-Sachs metric satisfying the Bondi conditions, are related by

\[
\dot{g}_{αβ}^{[B]} + \ell \dot{g}_{[B]}^{αβ} = \omega^{-2} \frac{∂x^α_{[B]}}{∂x^μ} \frac{∂x^β_{[B]}}{∂x^ν} \dot{g}^{μν}, \quad (13)
\]

with the factor \( \omega^{-2} \) appearing because there is also an implicit change of compactification factor from \( \ell \) to \( \ell_{[B]} \).

We have used Maple to find \( \dot{g}_{[B]}^{αβ} \) to first order in \( \ell \), and in doing so we have imposed the conditions (c.f. Eqs. (39) and (40) in \[18\])

\[
(\partial_α + U^B_0 \partial_B) x^A_{[B]0} = 0 \quad (14)
\]
We found that the Bondi conditions
\[ g_{[B]0}^{11} = 0, \quad g_{[B]0}^{14} = 0, \quad g_{[B]0}^{01} = 1 \]  
are satisfied identically. The remaining Bondi condition is
\[ \hat{g}_{[B]}^{AB} = \hat{g}_{[B]}^{AB}. \]  
Taking the determinant of Eq. (17) leads to an explicit expression for \( \omega \)
\[ \omega = \frac{P}{P_{[B]}} \sqrt{q_{[B],q} P_{[B],p} - q_{[B],p} P_{[B],q}}, \]  
where \( P_{[B]} = 1 + q_{[B]}^2 + p_{[B]}^2 \). The remaining content of Eq. (17) can be written as
\[ J_{[B]0} = -\hat{g}_{[B]}^{AB} q_{A/B} q_{B/[B]} / 2 = 0, \]  
where \( q_{A/B} \) is the complex dyad appropriate to the metric \( g_{[B]}^{AB} \). The Maple calculation finds
\[ J_{[B]0} = 4P^2 \left[ 4J_0 X^2 \bar{c}^2 - 2K_0 \bar{d}X \bar{d}X + 4J_0 X \bar{d}X \bar{c} \right. \]
\[ \left. -4K_0 X \bar{d}X \bar{c} + \bar{J}_0 (\bar{d}X)^2 + J_0 (\bar{d}X)^2 \right] / \left(4 + P^2 X \bar{X} \right)^2, \]  
where \( \bar{c} = q - ip \). Eq. (20) is not used to determine any of the transformation parameters, but rather as a check on the accuracy of the computation, as described in Sec. IV.

In addition to satisfying the Bondi conditions, \( \hat{g}_{[B]}^{AB} \) is also a Bondi-Sachs metric and must satisfy the conditions
\[ \hat{g}_{[B]}^{00} = 0, \quad \hat{g}_{[B]}^{0A} = 0. \]  
Now, \( \hat{g}_{[B]}^{00} = 0 \) and \( \hat{g}_{[B]}^{00} = 0 \) can be expressed as linear equations in \( A^u \) and \( C^u \) respectively, but these values will not be needed. Also, \( \hat{g}_{[B]}^{0A} = 0 \) and \( \hat{g}_{[B]}^{0A} = 0 \) can be expressed as linear equations in \( A^A \) and \( C^A \) respectively: the expression for \( C^A \) will not be needed, but the expression for \( A^A \) will be used in the news calculation and, in complex form, \[ A = \left[ J_0 \bar{d}u_{[B]}(2X \bar{c} + \bar{d}X) + \bar{J}_0 \bar{d}u_{[B]} \bar{d}X \right. \]
\[ \left. -K_0 (\bar{d}u_{[B]}(2X \bar{c} + \bar{d}X) + \bar{d}u_{[B]} \bar{d}X) \right] / (2\omega). \]  
The computer algebra calculation evaluates the inverse Jacobian at \( \ell = 0 \), from which we find a result that will be used in the news calculation
\[ \frac{\partial}{\partial u_{[B]}} (u, \ell, x^A) = \frac{e^{-2\bar{d}X}}{\omega} (1, 0, U_0^A). \]  
The computer algebra yields an expression for \( J_{[B]0} \): Defining
\[ T_1 = J_0 \bar{d}X - K_0 \bar{d}A \]
\[ T_2 = J_0 (\bar{d}X)^2 + \bar{d}X (J_0 \bar{d}X - 2K_0 \bar{d}X), \]  
then
\[ J_{[B]} = P^2 (4(1 - P_{[B]})A T_2 + 2X P_{[B]}(J_0 \bar{d}X)^2 + \bar{J}_0 (\bar{d}X)^2 + 2(\bar{d}A T_1 + 2\bar{c}X J_0 \bar{d}A + \bar{d}A J_0 \bar{d}X - K_0 \bar{d}X) + 2\bar{c}X K_0 \bar{d}A \]
\[ -K_0 \bar{d}X (2\bar{c}X + \bar{d}X)) + 4\bar{c}J_0 X (\bar{c}X + \bar{d}X)) - 4AX (-4\bar{c}^2 J_0 X + P_{[B]}(e^{-2\bar{d}X}) (2\bar{U}_0 X + (U_0 \bar{d}X + \bar{U}_0 \bar{d}X) + 2A_\omega + \bar{A}V_{\bar{A}1} + (U_0 \bar{d}A + \bar{U}_0 \bar{d}A) + 2\bar{c}A V_{\bar{A}0} + 2\bar{c}T_1)) - P^2 X^2 (4\bar{c}X (\bar{c}J_0 X + T_1) + T_2) + 16\bar{c} A X T_1) / (8X P_{[B]}^3 \omega^2). \]  

\[ \text{IV. PROCEDURE FOR COMPUTING THE NEWS} \]

We can now describe the procedure used for calculating the news. First, we look at the process from an analytic viewpoint, and then we give some details of the computational implementation. As discussed in Sec. II we assume that the general Bondi-Sachs metric is known throughout the spacetime.

A. Analytic procedure

1. Solve Eq. (14) to find \( x_{[B]0}^A = x_{[B]0}^A(u, x^A) \).
2. Eq. (18) gives \( \omega = \omega(u, x^A) \).
3. Use Eq. (20) to find by how much \( J_{[B]0} \) differs from zero, and thus obtain an estimate of the accuracy of the calculation.
4. Solve Eq. (15) to find \( u_{[B]0} = u_{[B]0}(u, x^A) \).
5. Eq. (21) gives \( A = A(u, x^A) \).
6. We note that
\[
(\frac{\partial J_B}{\partial \ell})_{\ell=0} = \frac{\partial x^\mu}{\partial \ell} \frac{\partial J_B}{\partial x^\mu}
\]
\[
= \frac{\partial \ell}{\partial \ell} \frac{\partial J_B}{\partial \ell} = \frac{J_B}{\omega},
\]
(26)
because \(J_B\) is zero on \(I^+\), and so derivatives in directions other than \(\ell\) vanish. Then the news \(N = N(u, x^A)\) is
\[
N = \frac{1}{2} \frac{\partial^2 J_B}{\partial u \partial u} = \left( \frac{\partial}{\partial u} \frac{J_B}{2\omega} \right),
\]
(27)
\[
= \left( \frac{\partial u}{\partial u} \frac{\partial}{\partial u} + \frac{\partial x^A}{\partial u} \frac{\partial}{\partial x^A} \right) \frac{J_B}{2\omega}.
\]
The terms \(\partial u/\partial u\) etc. are evaluated at \(\ell = 0\) and were given in Eq. (26) above.

7. Finally, we need to express the news as a function of Bondi coordinates, i.e. to transform \(N(u, x^A)\) to \(N(u_B, x^A_B)\).

B. Computational implementation

The news module has been written to interface directly with the null gravity code in its current form. Thus we use a compactified radial coordinate \(x = r/(R + r)\), with \(R = 1\). There are \(n_r\) points in the \(x\) direction in the range \([0.5, 1]\) (corresponding to \(1 < r < \infty\)). We use stereographic angular coordinates, with the number of grid points in each angular direction \(n_\theta\).

Instead of solving equations (14) and (15), we convert them to ordinary differential equations as in reference 5, and solve them by means of the second order method
\[
(x^{A(n+1)}-x^{A(n)})i = \Delta u (U^A_{n\ell})i, (28)
\]
\[
(u^{(n+1)}_{[B]}-u^{(n)}_{[B]})i = \Delta u (\omega e^{2\phi_{n\ell}}(n+\frac{1}{2})), (29)
\]
At the initial time-slice we match the Bondi coordinates to the Bondi-Sachs coordinates: i.e. at \(u = 0, u_B = 0\) and \(x^A_B = x^A\). This also gives \(A = 0\) and \(J_B = J_\ell\) on the initial surface. The solution of Eq. (28) leads to \(x^A = x^A(u, x^A_B)\), and this needs to be inverted to give \(x^A_B = x^A_B(u, x^A)\). At this stage the inversion has not been implemented in general, because it is not needed in the test examples considered below.

Second derivatives do not appear directly in the formulas leading to the news. First angular (eth) derivatives are implemented using the central difference second order method - with second order accurate upwind/downwind at the boundaries. The first time derivative of \(A\) is needed to calculate \(J_B\), and the first time derivative of \(J_B/\omega\) appears in the Bondi news function. For both the quantities, apart from the first two time-steps, we use one sided- downwind second order accurate differentiation scheme. For the initial steps we use the standard first order scheme.

V. COMPUTATIONAL TESTS AND RESULTS

We tested our method for calculating the Bondi news function against two exact solutions: linearized Robinson-Trautman, and Schwarzschild in rotating coordinates. We discretize this analytic solutions on the required computational grid and determine the various metric components and use them as an input for the news module. The news module as such sees it as input from an equivalent numerical evolution code.

A. Linearized Robinson-Trautman solution

The Robinson-Trautman solution represents a distorted black hole emitting purely outgoing radiation. The radiation decays exponentially, and asymptotically the solution becomes Schwarzschild.

The Robinson-Trautman metric can be put in the form
\[
ds^2 = -(\kappa - \frac{2}{rW})du^2 - 2Wdudr - 2rW_{\ell\ell}dudx^A
\]
\[
+ r^2q_{\ell\ell}dx^A dx^B, (30)
\]
where \(\kappa = W^2[1 - L^2(\log W)]\), \(L^2\) is the angular momentum operator (i.e., the angular part of the Laplacian operator on the unit sphere), and \(W(u, x^A)\) has to satisfies the nonlinear differential equation
\[
12\partial_u(\log W) = W^2L^2\kappa. (31)
\]
We recover the Schwarzschild solution by putting \(W\) = constant in the above metric.

Linearized solutions to the Robinson-Trautman equation (31) are obtained by setting \(W = 1 + \phi\) and dropping nonlinear terms in \(\phi\),
\[
12\partial_u\phi = L^2(2 - L^2)\phi. (32)
\]
For a spherical harmonic perturbation \(\phi = (\lambda(u)Y_{\ell m} - \lambda(0)e^{-u(\ell+1)\ell(\ell+\beta)/12}\). The (linear) perturbation can be a linear sum of various spherical harmonics with small amplitudes.

We consider a linearized solution of the form:
\[
\phi = Re[(\lambda_{22}e^{-2u}Y_{22} + \lambda_{33}e^{-10u}Y_{33})] (33)
\]
with \(\lambda_{22} = 3 \times 10^{-7}\) and \(\lambda_{33} = 7 \times 10^{-7}\). The metric components are, \(J = 0, \beta = 0.5\phi, U = \partial\phi/r\) (so \(U = 0\) at \(r = \infty\)) and \(W = 1\). Giving these quantities as inputs at the grid points at each time level, we call the news module to calculate the Bondi news function. We calculate the \(L_2\) norm \(J_B(0)\) (which should be zero), as well as the
The computation was performed for the following grid sizes

(a) \( \Delta q = \Delta p = 0.100, \Delta x = 1/60, \Delta u = 0.04 \)
(b) \( \Delta q = \Delta p = 0.050, \Delta x = 1/120, \Delta u = 0.02 \)
(c) \( \Delta q = \Delta p = 0.025, \Delta x = 1/240, \Delta u = 0.01 \)
(d) \( \Delta q = \Delta p = 0.020, \Delta x = 1/300, \Delta u = 0.008 \)

In this case as \( \zeta_{[B]} = \zeta \) we find that \( ||J_{[B]}||^2 \) constant with \( u \) and we plot \( ||J_{[B]0}||^2 \) at \( u = 1.6 \) for the discretizations (35); the observed convergence rate is 1.99. The \( L_2 \) norm of the error in the news is plotted against \( u \) for the discretizations (35) in Fig. 2. We see convergence that, at earlier times, is approximately second-order, but at later times saturation is observed. The convergence rate between different grids as \( u \) increases is given in Table I. Further, Fig. 2 shows that the error for the finest grid (d) saturates at about \( e^{-24} \), the next finest grid (c) saturates at \( e^{-24.5} \), grid (b) saturates at \( e^{-26.5} \), while grid (a) saturates at \( e^{-28.5} \). We observe that the saturation level scales approximately with discretization length as \( \Delta^{-3} \). These matters are discussed further in Sec. VI.

The \( Y_{33} \) component initially dominates the total news function, which should therefore decay approximately as \( e^{-10u} \) i.e. with slope 10 on a log scale; while at later times the \( Y_{22} \) component dominates and the news should decay as \( e^{-2u} \). If the percentage of error remains constant in the news function, the \( L_2 \) norm of the error should also show a similar trend with increasing \( u \). We see from Fig. 1 that this is actually the case. Until saturation effects become evident, a little before \( u = 2 \), the norm of the error in the news decreases as expected at all grid resolutions.

The error in the news is significantly higher at the first two time-steps, presumably because we are using a first-order scheme since there is not enough past data to evaluate second-order accurate time derivatives (actually, at the second time-step one can implement a second-order scheme, but its error is larger).

### B. Schwarzschild in rotating coordinates

By the transformation \( \varphi \to \hat{\varphi} + \kappa u \) of the azimuthal coordinate, the Schwarzschild line element in null coor-
\[ ds^2 = -(1 - \frac{2m}{r} - \kappa^2 r^2 \sin^2 \theta)du^2 - 2dudr + 2\kappa r^2 \sin^2 \theta dud\varphi + r^2 q_{AB} dx^A dx^B. \] (36)

In this case we set \( \kappa = 4 \) This coordinate change gives a nontrivial value for \( U \) at \( I^+ \) and thus it becomes a useful test to check that the numerically calculated Bondi news function remains zero. In this case the values of the various metric components are \( \beta = 0, J = 0, U = 2I\kappa \zeta/(1 + \zeta \bar{\zeta}), W = 1 \). Giving these as inputs we calculate \( ||J_{[B]0}||_2 \) and the news function \( ||N||_2 \), both of which have zero as the analytic value. We again found that \( ||J_{[B]0}||_2 \) hardly varies with \( u \), and in Fig. 3 we plot \( ||J_{[B]0}||_2 \) at \( u = 1.6 \) for the discretizations (35), the observed convergence rate is again 1.99. The \( L^2 \) norm of the news is plotted against \( u \) for the discretizations (35) in Fig. 4. We find convergence to zero at \( u = 1.6 \) at a rate 4.97 averaged over the grids (35) with only minor variations according to the specific grid used. The higher than expected rate is presumably due to some cancellation effect in the truncation errors. It is interesting that the magnitude of the error in the news builds up quite rapidly with time. Since the error remains convergent to zero, this must mean that the truncation error constants are growing with time.

VI. CONCLUSIONS

We have presented a method for calculating the news, based upon an explicit coordinate transformation from general Bondi-Sachs coordinates to a Bondi-Sachs coordinate system that satisfies the Bondi conditions. The method has been implemented computationally and validated against known analytic solutions. There are similarities and differences between the method and that of [8]

- Both methods require \( J = 0 \) on \( I^+ \) at \( u = 0 \).
- Both methods solve evolution equations for \( x^A_{[B]0} \) (Eq. (14)) and \( u_{[B]0} \) (Eq. (15)).
- The method of [8] solves evolution equations for \( \delta \) (Eq. (8-36)) and \( \omega \) (Given after Eq. (8-31)). In our method, \( \delta \) is not required, and \( \omega \) is found explicitly (Eq. (18)).
- The formulas used here are much simpler than those of [8].
- There is a means to monitor the accuracy of results obtained: we simply find \( J_{[B]0} \) and compare the value to zero.

The saturation and reduction in convergence rate reported in Sec. VA is typically an indication that round-off error becomes significant as the magnitude of the signal decays. The saturation level scales approximately as \( \Delta^{-3} \), which corresponds to the finite difference representation of a third derivative. This is consistent with the fact that here the news calculation implicitly involves third derivatives: \( x^A_{[B]0} \) and \( u_{[B]0} \) are differentiated to obtain \( A \) (Eq. (22)); then \( A \) is differentiated to obtain \( J_{[B]0} \).
(Eq. 23); and finally $J_{[B] \ell}$ is differentiated to obtain the news $N$ (Eq. 27). The observed scaling of saturation as $\Delta^{-3}$ is for the specific case of the Robinson-Trautman solution, which has the special property $x_A^0 = x_A^0 = 0$; thus, in a general case the effect may be more severe. For future applications, it is important to be aware that round-off error may be a problem.

It will be interesting to apply our news calculation to the problem considered in [8] (for which the method of [8] was unsuccessful), but that is deferred to further work. Also deferred to further work is the possibility of extending the coordinate transformation to higher order in $\ell$, and so being able to extract the Bondi mass and angular momentum aspects – the difficulty will be that the computer algebra is likely to lead to very complicated formulas.

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