Hybrid Superstrings in NS-NS Plane Waves

Hiroshi KUNITOMO

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-8502, Japan

Abstract

Using the hybrid formalism, superstrings in four-dimensional NS-NS plane waves are studied in a manifest supersymmetric manner. This description of the superstring is obtained through a field redefinition of the RNS worldsheet fields and defined as a topological $\mathcal{N} = 4$ string theory. The Hilbert space consists of two types of representations describing short and long strings. We study the physical spectrum to find boson-fermion asymmetry in the massless spectrum of the short string. Some massive spectra of the short string and the massless spectrum of the long string are also studied.
§1. Introduction

Plane-wave backgrounds are exact string vacua and provide examples of string theories in non-compact curved space-times. These backgrounds are obtained by taking the Penrose limit of AdS spaces and have attracted much attention in the study of the AdS/CFT duality beyond the supergravity approximation.\(^1\) Despite much progress on this subject, the analyses performed so far do not respect whole supersymmetry of these backgrounds. The superstring in R-R plane waves has been quantized in the light-cone gauge\(^2\) or using a non-covariant formalism.\(^3\) Only a part of the supersymmetry is linearly realized in these formulations. For superstrings in NS-NS plane waves, on the other hand, we can use the Ramond-Neveu-Schwarz (RNS) formalism for covariant quantization in which, however, the space-time supersymmetry is not manifest. The manifest supersymmetry is not always necessary, but it is desirable to make transparent any cancellations coming from the supersymmetry.

In this paper, we study four-dimensional superstrings in NS-NS plane waves in terms of the hybrid formalism that was developed in Refs. 6) and 7) and has been applied to several compactified string theories.\(^8\) This description of the superstring is obtained through a field redefinition of the RNS worldsheet fields and manifestly preserves all isometries of the background, including supersymmetry.

Strings in NS-NS plane waves are described by the Nappi-Witten (NW) model,\(^9\) which is the WZW model on the group manifold \(H_4\) generated by the four-dimensional Heisenberg algebra,

\[
\left[ J, P \right] = P, \quad \left[ J, P^* \right] = -P^*, \quad \left[ P, P^* \right] = F. \tag{1.1}
\]

The Hilbert space of the NW model consists of two distinct representations, discrete (types II and III) and continuous (type I). The model has a spectral flow symmetry and all flowed representations must also be included.\(^10\),\(^11\) The spectrally flowed discrete representations are regarded as describing short strings localized in the space transverse to the plane wave. The flowed continuous representations define long string states propagating in the whole four-dimensional space. This structure is similar to the spectrum of the string in AdS\(_3\),\(^12\) which in fact is obtained by taking its Penrose limit.\(^5\)

The RNS superstring in this background is described by superconformal field theories of the type \(H_4 \times \mathcal{M}\), where \(H_4\) denotes the super NW model and \(\mathcal{M}\) is represented by an arbitrary \(N = 2\) unitary superconformal field theory with \(c = 9\).\(^5\) This \(\mathcal{M}\) sector represents a Calabi-Yau compactification if we project into the integral sector of the \(U(1)_R\) charge \(I_{\mathcal{M},0}\).
However, it was found in Ref. 5) that the string vacua have enhanced supersymmetry if we apply a generalized GSO projection that restricts the total $U(1)_R$ charge to integer values, while fractional $I_{M,0}$ is allowed. We adopt this weak GSO projection throughout.

The hybrid superstrings in NS-NS plane waves are related to the RNS superstrings by a field redefinition of worldsheet fields. We show how to carry out a redefinition from RNS fields to hybrid fields making all the space-time supersymmetry manifest. The model can be formulated as an $N = 4$ topological string theory.7)

Next, we examine the physical spectrum at several lower mass levels. We find that the massless spectrum of the short string has boson-fermion asymmetry, which is allowed without breaking supersymmetry. There are two massless bosons without fermionic partners. This fact is made evident by the manifest supersymmetry. Some massive spectra of the short string and the massless spectrum of the long string are also studied in a manifestly supersymmetric manner.

This paper is organized as follows. In §2, we begin with a brief review of the super-NW model in the RNS formalism. The space-time supercharges are given in a form satisfying the conventional supersymmetry algebra. Hybrid worldsheet fields are introduced in §3 through a redefinition of the RNS worldsheet fields. The model is then reformulated as a topological $N = 4$ string theory. In §4, we construct the Hilbert space of the hybrid superstring using hybrid fields. This space consists of two sectors representing short and long strings, respectively. The physical spectrum is studied in §5. It is found that the massless spectrum of the short string has boson-fermion asymmetry. The results are summarized with some discussion in §6.

§2. RNS superstrings in NS-NS plane waves

A description of RNS superstrings propagating in four-dimensional NS-NS plane waves is provided by superconformal field theories of the type $H_4 \times \mathcal{M}$.4,5) Here, $H_4$ denotes the super-NW model described by the super WZW model on the four-dimensional Heisenberg group. The Hilbert space of this model is constructed from representations of the $H_4$ super current algebra* generated by bosonic $H_4$ currents, $(J, F, P, P^*)$, and their worldsheet superpartners, $(\psi_J, \psi_F, \psi_P, \psi_{P^*})$:

\[
J(z) P(w) \sim \frac{P(w)}{z - w}, \quad J(z) P^*(w) \sim \frac{-P^*(w)}{z - w}, \\
P(z) P^*(w) \sim \frac{1}{(z - w)^2} + \frac{F(w)}{z - w}, \quad J(z) F(w) \sim \frac{1}{(z - w)^2};
\]

* We consider only the holomorphic sector in this paper. It can be easily combined with the anti-holomorphic sector if necessary.5,10–12)
\[ \psi_p(z)\psi_{p^*}(w) \sim \frac{1}{z-w}, \quad \psi_f(z)\psi_{f^*}(w) \sim \frac{1}{z-w}, \]
\[ J(z)\psi_p(w) \sim \psi_f(z)P(w) \sim \frac{\psi_p(w)}{z-w}, \quad J(z)\psi_{p^*}(w) \sim \psi_f(z)P^*(w) \sim -\frac{\psi_{p^*}(w)}{z-w}, \]
\[ P(z)\psi_{p^*}(w) \sim \psi_p(z)P^*(w) \sim \frac{\psi_{f^*}(w)}{z-w}. \]
\[ (2.1) \]

Representations of this algebra are easily obtained by using a free-field realization,\textsuperscript{10}

\[ J = i\partial X^-, \quad F = i\partial X^+, \]
\[ P = e^{iX^+}(i\partial Z + \psi^+\psi), \quad P^* = e^{-iX^+}(i\partial Z^* - \psi^+\psi^*), \]
\[ \psi_F = \psi^+, \quad \psi_J = \psi^- , \quad \psi_P = e^{iX^+}\psi, \quad \psi_{P^*} = e^{-iX^+}\psi^*. \]
\[ (2.2) \]

where operator products of free fields are defined by

\[ X^+(z)X^-(w) \sim Z(z)Z^*(w) \sim -\log(z-w), \]
\[ \psi^+(z)\psi^-(w) \sim \psi(z)\psi^*(w) \sim \frac{1}{z-w}. \]
\[ (2.3) \]

The zero modes of bosonic currents provide generators of the space-time symmetry (1.1):

\[ J = \oint \frac{dz}{2\pi i} i\partial X^-, \quad F = \oint \frac{dz}{2\pi i} i\partial X^+, \]
\[ P = \oint \frac{dz}{2\pi i} e^{iX^+}(i\partial Z + \psi^+\psi), \quad P^* = \oint \frac{dz}{2\pi i} e^{-iX^+}(i\partial Z^* - \psi^+\psi^*). \]
\[ (2.4) \]

The model has central charge \( c = 6 \), which is the same that for a superstring in flat four-dimensional space-time.

The \( M \) sector is represented by an arbitrary unitary representation of \( N = 2 \) rational superconformal field theory with \( c = 9 \). We denote the generators of this \( N = 2 \) superconformal symmetry by \( (T_M, G_M^\pm, I_M) \).

In order to covariantly quantize the RNS superstring, fermionic ghosts \( (b, c) \) and bosonic ghosts \( (\beta, \gamma) \) must be introduced. These superconformal ghosts satisfy

\[ c(z)b(z) \sim \gamma(z)\beta(w) \sim \frac{1}{z-w}. \]
\[ (2.6) \]
and have $N = 1$ superconformal invariance generated by

$$T_{gh} = -2b\partial c - b\partial c - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\partial\beta\gamma,$$

$$G_{gh} = \frac{3}{2}\beta\partial c + \partial\beta c - 2b\gamma.$$ (2.7)

The physical Hilbert space is defined by the cohomology $\mathcal{H}_{\text{phys}} = \text{Ker}Q_{\text{BRST}}/\text{Im}Q_{\text{BRST}}$ of the BRST charge

$$Q_{\text{BRST}} = \oint \frac{dz}{2\pi i} \left( c\left(T_m + \frac{1}{2}T_{gh}\right) + \gamma\left(G_m + \frac{1}{2}G_{gh}\right) \right),$$ (2.8)

where

$$T_m = T_{H4} + T_{\mathcal{M}}, \quad G_m = G_{H4}^+ + G_{H4}^- + G_{\mathcal{M}}^+ + G_{\mathcal{M}}^-.$$ (2.9)

Then, we bosonize the worldsheet fermions and the $U(1)$ current in the $\mathcal{M}$ sector as

$$\psi^+\psi^- = i\partial H_0, \quad \psi\psi^* = i\partial H_1,$$ (2.10a)

$$I_M = -\sqrt{3}i\partial H_2,$$ (2.10b)

where the bosons $H_I(z)$ ($I = 0, 1, 2$) satisfy the standard OPEs:

$$H_I(z)H_J(w) \sim -\delta_{IJ}\log(z - w).$$ (2.11)

The superconformal ghosts are also bosonized by

$$c = e^\sigma, \quad b = e^{-\sigma},$$

$$\gamma = \eta e^\phi = e^{\phi - \chi},$$

$$\beta = e^{-\phi}\partial\xi = \partial\chi e^{-\phi + \chi},$$ (2.12)

with

$$\phi(z)\phi(w) \sim -\log(z - w),$$

$$\sigma(z)\sigma(w) \sim \chi(z)\chi(w) \sim +\log(z - w).$$ (2.13)

Here, it is important that because the zero-mode $\xi_0$ is not included in these formulas, the Hilbert space of the original bosonic ghosts $(\beta, \gamma)$ is different from that of the bosonized fields $(\phi, \xi, \eta)$ [or $(\phi, \chi)$]. The former (latter) is called the small (large) Hilbert space $\mathcal{H}_{\text{small}}$ ($\mathcal{H}_{\text{large}}$). This extension of the Hilbert space is essential to realize the supersymmetry.

5
In order to obtain a supersymmetric spectrum in the RNS formalism, we must impose a GSO projection, which guarantees the mutual locality of space-time supercharges. If we use the weak GSO condition

\[ I_0 = I_{H,0} + I_{\mathcal{M},0} \in \mathbb{Z}, \quad (2.14) \]

the model has the enhanced supersymmetry generated by the following four supercharges:\(^5\)

\[ Q_{(-\frac{1}{2})}^{\pm \pm} = \oint \frac{dz}{2\pi i} e^{-\frac{\phi}{2} e^{\pm iX^+} e^{\mp i(H_0 \pm (H_1 + \sqrt{3} H_2))}}, \]
\[ Q_{(-\frac{1}{2})}^{\pm \mp} = \oint \frac{dz}{2\pi i} e^{-\frac{\phi}{2} e^{\pm iX^+} e^{\mp i(-H_0 \pm (H_1 - \sqrt{3} H_2))}}. \quad (2.15) \]

The subscript \("(-\frac{1}{2})\)\) indicates that these operators are given in the \(-\frac{1}{2}\) picture, which is generally read from the eigenvalue of the operator

\[ \mathcal{R} = \oint \frac{dz}{2\pi i} (\xi \eta - \partial \phi). \quad (2.16) \]

We note that these supercharges form a peculiar algebra, due to the infinite degeneracy of pictures. This algebra is equivalent to supersymmetry only in the on-shell physical amplitudes. We change the picture for two of the supercharges, \((Q_{(-\frac{1}{2})}^{-\pm}), Q_{(-\frac{1}{2})}^{\mp \mp}\), to \(+\frac{1}{2}\) so that the conventional supersymmetry algebra may hold without any condition:\(^*\)

\[ Q_{(\frac{1}{2})}^{-\pm} = \oint \frac{dz}{2\pi i} \left\{ Q_{\text{BRST}}^+, \xi e^{-\frac{\phi}{2} e^{-i\chi^+} e^{\frac{\chi}{2}(H_0 - (H_1 + \sqrt{3} H_2))}} \right\}, \]

\[ = \oint \frac{dz}{2\pi i} e^{-i\chi^+} \left( \eta be^{\frac{\phi}{2} + \frac{\chi}{2}(H_0 - (H_1 + \sqrt{3} H_2))} + i\partial X^+ e^{\frac{\phi}{2} + \frac{\chi}{2}(H_0 + (H_1 + \sqrt{3} H_2))} + i\partial Z^+ e^{\frac{\phi}{2} + \frac{\chi}{2}(-H_0 - (H_1 - \sqrt{3} H_2))} - \psi^+ e^{\frac{\phi}{2} + \frac{\chi}{2}(-H_0 - (H_1 - \sqrt{3} H_2))} - G_{\mathcal{M}} e^{\frac{\phi}{2} + \frac{\chi}{2}(H_0 - (H_1 - \sqrt{3} H_2))} - G_{\mathcal{M}} e^{\frac{\phi}{2} + \frac{\chi}{2}(H_0 + (H_1 - \sqrt{3} H_2))} \right), \]

\[ Q_{(\frac{1}{2})}^{\mp \mp} = \oint \frac{dz}{2\pi i} \left\{ Q_{\text{BRST}}^+, \xi e^{-\frac{\phi}{2} e^{\frac{\chi}{2}(-H_0 + (H_1 - \sqrt{3} H_2))}} \right\}, \]

\[ = \oint \frac{dz}{2\pi i} \left( \eta be^{\frac{\phi}{2} + \frac{\chi}{2}(-H_0 + (H_1 - \sqrt{3} H_2))} - i\partial X^- e^{\frac{\phi}{2} - \frac{\chi}{2}(-H_0 - (H_1 + \sqrt{3} H_2))} + i\partial Z^+ e^{\frac{\phi}{2} - \frac{\chi}{2}(H_0 + (H_1 + \sqrt{3} H_2))} - G_{\mathcal{M}} e^{\frac{\phi}{2} - \frac{\chi}{2}(-H_0 + (H_1 + \sqrt{3} H_2))} - G_{\mathcal{M}} e^{\frac{\phi}{2} - \frac{\chi}{2}(-H_0 - (H_1 - \sqrt{3} H_2))} \right). \quad (2.17) \]

\(^*\) Rigorously speaking, the relative signs between terms in the explicit forms (2.17) are not fixed without specifying the cocycle factors, which are usually omitted. This fact makes practical calculations difficult, although it is often fixed by the Lorentz covariance of the result. This complexity disappears in the hybrid formalism, and this is actually one of the important advantages of the hybrid formalism.
Since we do not consider supercharges in other pictures, we simply write \((Q^{\pm+}_{-\frac{1}{2}}), Q^{\pm-}_{\frac{1}{2}}) = (Q^{\pm+}, Q^{\pm-})\) in the rest of this paper. These picture-changed supercharges together with (2.4) generate the supersymmetry algebra for the NS-NS plane waves,\(^5\)

\[ [\mathcal{J}, \mathcal{P}] = \mathcal{P}, \quad [\mathcal{J}, \mathcal{P}^*] = -\mathcal{P}^*, \]
\[ [\mathcal{P}, \mathcal{P}^*] = \mathcal{F}, \]
\[ [\mathcal{J}, Q^{\pm\pm}] = \pm Q^{\pm\pm}, \quad [\mathcal{J}, Q^{\pm\mp}] = 0, \]
\[ [Q^{\mp}, \mathcal{P}] = -Q^{++}, \quad [Q^{+-}, \mathcal{P}^*] = Q^{--}, \]
\[ \{Q^{++}, Q^{--}\} = \mathcal{F}, \quad \{Q^{-+}, Q^{+-}\} = -\mathcal{J}, \]
\[ \{Q^{++}, Q^{+-}\} = \mathcal{P}, \quad \{Q^{-+}, Q^{--}\} = \mathcal{P}^*. \quad (2.18) \]

Before closing this section, it is useful to reconsider physical state conditions in the RNS formalism. Although the physical states are defined by the BRST cohomology in \(\mathcal{H}_{\text{small}}\), we must extend it to \(\mathcal{H}_{\text{large}}\) to carry out a field redefinition to hybrid fields because, as mentioned above, \(\mathcal{H}_{\text{small}}\) is not sufficient to realize the space-time supersymmetry. Therefore, we must generalize the physical state conditions to

\[ Q_{\text{BRST}}|\psi\rangle = 0, \]
\[ |\psi\rangle \sim |\psi\rangle + \delta|\psi\rangle, \quad \delta|\psi\rangle = Q_{\text{BRST}}|\Lambda\rangle, \]
\[ \eta_0|\psi\rangle = \eta_0|\Lambda\rangle = 0, \quad (2.19a) \]

where \( |\psi\rangle, |\Lambda\rangle \in \mathcal{H}_{\text{large}}\). In addition to these cohomology conditions, we require that the physical states have ghost number one, i.e.,

\[ Q_{\text{gh}}|\psi\rangle = |\psi\rangle, \quad (2.19b) \]

counted by the charge\(^\ast\)

\[ Q_{\text{gh}} = \oint \frac{dz}{2\pi i}(cb - \xi\eta). \quad (2.20) \]

The conditions given in (2.19) have a natural interpretation as a topological \(N = 4\) string theory. We describe this interpretation below.

Let us note that there is the hidden twisted \(N = 4\) superconformal symmetry generated by

\[ T = T_m + T_{\text{gh}}, \]

\(^\ast\) This definition of the ghost number is related to the familiar one, \(N_c = \oint \frac{dz}{2\pi i}(cb - \gamma\beta)\), through the relation \(Q_{\text{gh}} = N_c - R\). The difference between the two definitions is a constant in a given picture.
\[ G^+ = J_{\text{BRST}}, \]
\[ = c(T_m + T_{\beta \gamma}) + \gamma G_m - \gamma^2 b + c \partial c b + \partial (c \xi \eta) + \partial^2 c, \]
\[ G^- = b, \quad \tilde{G}^+ = \eta, \quad \tilde{G}^- = \xi T - b \{ Q_{\text{BRST}}, \xi \} + \partial^2 \xi, \]
\[ I^{++} = \eta c, \quad I^{--} = b \xi, \quad I = c b - \xi \eta, \quad (2.21) \]

in \( H_{\text{large}} \). The physical state conditions (2.19a) and the ghost number condition (2.19b) can be written in terms of these \( N = 4 \) generators as

\[ G^+_0 |\psi\rangle = 0, \quad \delta |\psi\rangle = G^+_0 |\Lambda\rangle, \quad (2.22a) \]
\[ I_0 |\psi\rangle = |\psi\rangle, \quad (2.22b) \]
\[ \tilde{G}^+_0 |\psi\rangle = \tilde{G}^+_0 |\Lambda\rangle = 0, \quad (2.22c) \]

which can be naturally interpreted as physical state conditions in the topological \( N = 4 \) string theory.\(^7\) Because \( \eta_0 \)-cohomology is trivial, we can always solve Eq. (2.22) by

\[ |\psi\rangle = \tilde{G}^+_0 |V\rangle \]
\[ \text{and} \quad |\Lambda\rangle = \tilde{G}^+_0 |\Lambda^-\rangle \]

to rewrite (2.22) in the more symmetric forms

\[ G^+_0 \tilde{G}^+_0 |V\rangle = 0, \quad (2.23a) \]
\[ \delta |V\rangle = G^+_0 |\Lambda^-\rangle + \tilde{G}^+_0 |\tilde{\Lambda}^-\rangle, \quad (2.23b) \]
\[ I_0 |V\rangle = 0. \quad (2.23c) \]

In this paper, we call the condition (2.23a) the equation of motion and the condition (2.23b) the gauge transformation, using the standard terminology in string field theory. These conditions will be used to study the physical spectrum in \( \S 5 \).

\section*{§3. Hybrid superstrings in NS-NS plane waves}

In this section, we develop the hybrid formalism for superstrings in NS-NS plane waves. We first introduce hybrid fields by finding a field redefinition from RNS fields, which allows the whole space-time supersymmetry to be manifest. We rewrite worldsheet superconformal generators using these new fields to formulate the model as a topological \( N = 4 \) string theory.

As explained in the previous section, the basic fields of the super NW model are free fields, \((X^\pm, Z, Z^*, \psi^\pm, \psi, \psi^*)\), and superconformal ghosts, \((b, c, \beta, \gamma)\). We add to them the boson \( H_2 \) coming from the \( U(1) \) current (2.10b) in the \( M \) sector, which we need to define the supercharges (2.15). Although the bosonic fields \((X^\pm, Z, Z^*)\) are common to the hybrid formalism,\(^*\) the remaining fields must be rearranged to obtain the basic fields in the hybrid

\(^*\) These bosons are not exactly the same in the two formalisms, but they are related by the similarity transformation (3.6).
formalism. Describing these fields in terms of the six free bosons \((H_0, H_1, H_2, \phi, \chi, \sigma)\) with the help of the bosonization formulas (2.10) and (2.12), we carry out the linear transformation defined by
\[
\phi_- = \frac{1}{i} H_0 - \frac{1}{2} H_1 - \frac{1}{2} \sqrt{3} H_2 + \frac{1}{2} \phi,
\]
\[
\phi_+ = \frac{1}{i} H_0 + \frac{1}{2} H_1 - \frac{1}{2} \sqrt{3} H_2 + \frac{1}{2} \phi,
\]
\[
\phi_{++} = \frac{1}{i} H_0 + \frac{1}{2} H_1 + \frac{1}{2} \sqrt{3} H_2 - \frac{3}{2} \phi + \chi + \sigma,
\]
\[
\phi_{--} = \frac{1}{i} H_0 - \frac{1}{2} H_1 + \frac{1}{2} \sqrt{3} H_2 - \frac{3}{2} \phi + \chi + \sigma,
\]
\[
\rho = \sqrt{3} H_2 + 3i\phi - 2i\chi - i\sigma,
\]
\[
\hat{H}_2 = H_2 + \sqrt{3} i\phi - \sqrt{3} i\chi, \tag{3.1}
\]
and then define fermionic fields as
\[
\theta^{\alpha\alpha'} = e^{\phi^{\alpha\alpha'}}, \quad p_{\alpha\alpha'} = e^{-\phi^{\alpha\alpha'}}, \quad (\alpha, \alpha' = \pm) \tag{3.2}
\]
which satisfy
\[
\theta^{\alpha\alpha'}(z)p_{\beta\beta'}(w) \sim \delta_{\beta}^{\alpha} \delta_{\beta'}^{\alpha'}. \tag{3.3}
\]
The basic fields of the hybrid superstrings are finally defined by Green-Schwarz-like fields with an additional boson: \((X^\pm, Z, Z^*, \theta^{\alpha\alpha'}, p_{\alpha\alpha'}, \rho)\). The \(U(1)\) boson in the \(M\) sector is also modified to \(\hat{H}_2\), which requires modifications of the superconformal generators to \((\hat{T}_M, \hat{G}_{\pm M}, \hat{I}_M)\), uniquely determined by the change of the \(U(1)\) current
\[
\hat{I}_M = -\sqrt{3} i \partial \hat{H}_2. \tag{3.4}
\]
We note that these new generators completely (anti-)commute with the hybrid fields.

The space-time supercharges (2.15) are written in terms of these hybrid fields:
\[
Q^{++} = \oint \frac{dz}{2\pi i} e^{i X^+} p_{--},
\]
\[
Q^{--} = \oint \frac{dz}{2\pi i} e^{-i X^+} \left( p_{++} + i \partial X^- \theta^{--} + (i \partial Z^* + \theta^{-+} p_{++}) \theta^{+-} + e^{-i\rho^+} \hat{G}_M \right),
\]
\[
Q^{+-} = \oint \frac{dz}{2\pi i} p_{+-},
\]
\[
Q^{+\pm} = \oint \frac{dz}{2\pi i} \left( p_{-+} - i \partial X^- \theta^{+\pm} + i \partial Z \theta^{-\pm} - e^{-i\rho^{++}} \hat{G}_M \right). \tag{3.5}
\]
However, these supercharges are not symmetric. This leads to a complicated hermiticity property of hybrid fields. These fields are chiral coordinates in the sense that two of the supercharges, $Q^{++}$ and $Q^{-+}$, are simple superderivatives, $p_{--}$ and $p_{+-}$ (except for factors of $e^{\pm iX^+}$). In order to obtain symmetric supercharges and hybrid fields with proper hermiticity, we must further carry out a similarity transformation generated by

$$U = \oint \frac{dz}{2\pi i} \left( -e^{-i\theta^{++}\theta^{+-}}\hat{G}_M + \frac{1}{2}i\partial X^+\theta^{++}\theta^{--} + \frac{1}{2}i\partial Z^+\theta^{++}\theta^{--} + \frac{1}{2}i\partial X^-\theta^{++}\theta^{--} + \frac{1}{2}i\partial Z^-\theta^{++}\theta^{--} + \frac{1}{4}\theta^{--}\theta^{++}\partial(\theta^{--}\theta^{++}) \right). \tag{3.6}$$

In fact, the space-time supercharges (3.5) have the symmetric forms

$$Q^{++} = \oint \frac{dz}{2\pi i} e^{iX^+} \left( p_{--} + \frac{1}{2}i\partial X^+\theta^{++} + \frac{1}{2}(i\partial Z - \theta^{+-}p_{--})\theta^{--} + \frac{1}{8}\partial(\theta^{--}\theta^{++})\theta^{--} \right),$$

$$Q^{--} = \oint \frac{dz}{2\pi i} e^{-iX^+} \left( p_{++} + \frac{1}{2}i\partial X^-\theta^{--} + \frac{1}{2}(i\partial Z^+ + \theta^{+-}p_{++})\theta^{--} - \frac{1}{8}\partial(\theta^{--}\theta^{++})\theta^{--} \right),$$

$$Q^{-+} = \oint \frac{dz}{2\pi i} \left( p_{+-} - \frac{1}{2}i\partial X^+\theta^{++} + \frac{1}{2}i\partial Z^+\theta^{++} - \frac{1}{8}\partial(\theta^{--}\theta^{++})\theta^{--} \right),$$

$$Q^{+-} = \oint \frac{dz}{2\pi i} \left( p_{-+} - \frac{1}{2}i\partial X^-\theta^{--} + \frac{1}{2}i\partial Z^-\theta^{--} + \frac{1}{8}\partial(\theta^{--}\theta^{++})\theta^{--} \right) \tag{3.7}$$

after the similarity transformation.

We can also rewrite the topological $N = 4$ superconformal generators (2.21) using the hybrid fields. The $N = 2$ subalgebra is first given by

$$T = -\partial X^+\partial X^- - \partial Z\partial Z^+ - p_{\alpha\alpha'}\partial^{\alpha\alpha'} + \frac{1}{2}\partial\rho\partial\rho + \frac{1}{2}i\partial^2\rho + \hat{T}_M + \frac{1}{2}\partial\hat{I}_M,$$

$$G^+ = e^{-i\rho} \left( d_{--} + \frac{1}{8}\partial^2\theta^{++} + \frac{1}{8}\theta^{+-}\partial^2\theta^{++} - \frac{1}{4}\partial^2(\theta^{--}\theta^{++}) + \hat{G}^+_M \right),$$

$$G^- = e^{i\rho} \left( d_{++} + \frac{1}{8}\partial^2\theta^{--} + \frac{1}{8}\theta^{++}\partial^2\theta^{--} - \frac{1}{4}\partial^2(\theta^{--}\theta^{++}) + \hat{G}^-_M \right),$$

$$I = i\partial\rho - \sqrt{3}i\partial\hat{H}_2, \tag{3.8}$$

where

$$d_{--} = p_{--} - \frac{1}{2}i\partial X^+\theta^{++} + \frac{1}{2}i\partial Z^+\theta^{--} + \frac{1}{4}\theta^{++}\theta^{--} - \frac{1}{8}\partial(\theta^{++}\theta^{--})\theta^{++},$$

$$d_{++} = p_{++} - \frac{1}{2}i\partial X^-\theta^{--} + \frac{1}{2}i\partial Z^-\theta^{--} + \frac{1}{4}\theta^{++}\theta^{--} - \frac{1}{8}\partial(\theta^{++}\theta^{--})\theta^{--}.$$
\[ d_{+-} = p_{+-} + \frac{1}{2} i \partial X^{-} \theta^{+} - \frac{1}{2} i \partial Z^{*} \theta^{++} - \frac{1}{4} \theta^{-} \theta^{++} \partial \theta^{--} + \frac{1}{8} \partial (\theta^{-} \theta^{++}) \theta^{-} , \]
\[ d_{++} = p_{++} - \frac{1}{2} i \partial X^{+} \theta^{-} - \frac{1}{2} i \partial Z^{*} \theta^{-} - \frac{1}{4} \theta^{-} \theta^{++} \partial \theta^{--} + \frac{1}{8} \partial (\theta^{-} \theta^{++}) \theta^{-} , \]
\[ d_{-+} = p_{-+} + \frac{1}{2} i \partial X^{-} \theta^{+} - \frac{1}{2} i \partial Z^{*} \theta^{--} + \frac{1}{4} \theta^{-} \theta^{++} \partial \theta^{--} - \frac{1}{8} \partial (\theta^{-} \theta^{++}) \theta^{+} \] (3.9)

are local currents of supercovariant derivatives. It is also useful to introduce the bosonic supercovariant derivatives as

\[ \Pi^{+} = i \partial X^{+} + \frac{1}{2} \theta^{+} \partial \theta^{--} - \frac{1}{2} \partial \theta^{+} \theta^{--} , \]
\[ \Pi^{-} = i \partial X^{-} - \frac{1}{2} \theta^{++} \partial \theta^{-} + \frac{1}{2} \partial \theta^{++} \theta^{-} , \]
\[ \Pi_{Z} = i \partial Z - \frac{1}{2} \theta^{++} \partial \theta^{--} + \frac{1}{2} \partial \theta^{++} \theta^{-} , \]
\[ \Pi^{*}_{Z} = i \partial Z^{*} - \frac{1}{2} \theta^{-} \partial \theta^{--} + \frac{1}{2} \partial \theta^{-} \theta^{--} . \] (3.10)

These supercovariant derivatives and \( \partial \theta^{\alpha \alpha'} \) form a closed superalgebra:

\[ d_{-+}(z) d_{++}(w) \sim - \frac{\Pi^{+}(w)}{z - w} , \quad d_{-+}(z) d_{-+}(w) \sim - \frac{\Pi_{Z}(w)}{z - w} , \]
\[ d_{++}(z) d_{-+}(w) \sim - \frac{\Pi^{*}_{Z}(w)}{z - w} , \quad d_{++}(z) d_{++}(w) \sim \frac{\Pi^{-}(w)}{z - w} , \]
\[ \Pi^{+}(z) \Pi^{-}(w) \sim \frac{1}{(z - w)^2} , \quad \Pi^{*}_{Z}(z) \Pi^{*}_{Z}(w) \sim \frac{1}{(z - w)^2} , \]
\[ d_{-+}(z) \Pi^{-}(w) \sim - \frac{\partial \theta^{++}(w)}{z - w} , \quad d_{-+}(z) \Pi^{*}_{Z}(w) \sim - \frac{\partial \theta^{-+}(w)}{z - w} , \]
\[ d_{++}(z) \Pi^{-}(w) \sim - \frac{\partial \theta^{-+}(w)}{z - w} , \quad d_{++}(z) \Pi^{*}_{Z}(w) \sim - \frac{\partial \theta^{++}(w)}{z - w} , \]
\[ d_{-+}(z) \Pi^{+}(w) \sim \frac{\partial \theta^{--}(w)}{z - w} , \quad d_{-+}(z) \Pi^{*}_{Z}(w) \sim - \frac{\partial \theta^{--}(w)}{z - w} , \]
\[ d_{++}(z) \Pi^{+}(w) \sim \frac{\partial \theta^{++}(w)}{z - w} , \quad d_{++}(z) \Pi^{*}_{Z}(w) \sim - \frac{\partial \theta^{++}(w)}{z - w} . \] (3.11)

We note here that these supercovariant derivatives are supercovariant only in the sense that they (anti-)commute with two of the supercharges, \( Q^{\pm \pm} \).

It can be shown that the complicated forms of \( G^{\pm} \) in (3.8) can be rewritten as

\[ G^{+} = e^{-ip_{x}} d_{-+} d_{++} + \hat{G}^{+} , \]
\[ G^{-} = e^{ip_{x}} d_{++} d_{-+} + \hat{G}^{-} \] (3.12)

by introducing the new normal ordering \( \hat{\cdot} \) with respect to the currents \( d_{\alpha \alpha'} \). The generators (2.21) of the topological \( N = 4 \) superconformal symmetry are then given by

\[ T = -\partial X^{+} \partial X^{-} - \partial Z \partial Z^{*} - p_{\alpha \alpha'} \partial \theta^{\alpha \alpha'} + \frac{1}{2} \partial \rho \partial \rho + \frac{1}{2} i \partial^{2} \rho + \hat{T}_{M} + \frac{1}{2} \partial \hat{T}_{M} . \]
\begin{align*}
G^+ &= e^{-ip\times d_+d_-}\hat{G}_M^+,
G^- &= e^{ip\times d_+d_-}\hat{G}_M^-,
\tilde{G}^+ &= e^{2i\rho-\sqrt{3}i\hat{H}_2\times d_+d_-} + e^{i\rho-\sqrt{3}i\hat{H}_2\hat{G}_M^-},
\tilde{G}^- &= e^{-2i\rho+\sqrt{3}i\hat{H}_2\times d_+d_-} + e^{-i\rho+\sqrt{3}i\hat{H}_2\hat{G}_M^+},
I^{++} &= e^{i\rho-\sqrt{3}i\hat{H}_2},
I^{--} &= e^{-i\rho+\sqrt{3}i\hat{H}_2},
I^0 &= i\partial\rho - \sqrt{3}i\partial\hat{H}_2.
\end{align*}

Physical states are defined by the conditions (2.23) using the zero modes $G_0^+, \tilde{G}_0^+$ and $I_0$ of these generators. The supercharges (3.7) (anti-)commute with them. This guarantees the physical spectrum to be supersymmetric.

Finally, we rewrite the picture counting operator (2.16) interpreted in the hybrid formalism as the R-charge operator:

\begin{align*}
R &= \oint \frac{dz}{2\pi i} \left( i\partial\rho - \frac{1}{2}(\theta^{++}p_{++} + \theta^{--}p_{--} - \theta^{+-}p_{+-} - \theta^{-+}p_{-+}) \right).
\end{align*}

This is useful to determine whether each component of the superfields is a space-time boson or fermion. The field with (half-)integral R-charge is a space-time boson (fermion), because it comes from the NS-(R-)sector in the RNS formalism.

§ 4. Spectral flow and the Hilbert space of the hybrid superstring

Now we study the Hilbert space of the hybrid superstring. Using the hybrid fields, the $H_4$ currents are realized as

\begin{align*}
J &= i\partial\chi^-,
F &= i\partial\chi^+,
P &= e^{i\chi^+}(i\partial\chi - \theta^{++}p_{--}),
\quad P^* = e^{-i\chi^+}(i\partial\chi^* + \theta^{-+}p_{++}).
\end{align*}

We can extend this $H_4$ current algebra to a superalgebra, which is an analog of the super current algebra (2.1), by introducing the space-time supercoordinates (and their conjugates)

\begin{align*}
\Theta^{\pm\mp} &= \theta^{\pm\mp},
\quad \mathcal{P}^{\pm\mp} = p^{\pm\mp},
\Theta^{\pm\pm} &= e^{\pm i\chi^+}\theta^{\pm\pm},
\quad \mathcal{P}^{\pm\pm} = e^{\mp i\chi^+}p^{\pm\pm},
\end{align*}

together with an extra $U(1)$ current, $i\partial\rho$. The Hilbert space of the hybrid superstring is constructed from representations of this current superalgebra. We can expand these currents as

\begin{align*}
J(z) &= \sum_n J_n z^{-n-1},
F(z) &= \sum_n F_n z^{-n-1},
\end{align*}

12
\[ P(z) = \sum_n P_n z^{-n-1}, \quad \Theta^\pm(z) = \sum_n \Theta_n^\pm z^{-n}, \quad \Theta^{\pm\mp}(z) = \sum_n \Theta_n^{\pm\mp} z^{-n}, \]
\[ P^*(z) = \sum_n P_n^* z^{-n-1}, \quad \Theta^\pm(z) = \sum_n (P_n^\pm) z^{-n-1}, \quad \Theta^{\pm\mp}(z) = \sum_n (P_n^{\pm\mp}) z^{-n-1}, \]
\[ \Theta^{\pm\pm}(z) = \sum_n \Theta_n^{\pm\pm} z^{-n}, \quad \Theta^{\pm\mp}(z) = \sum_n (P_n^{\pm\mp}) z^{-n-1}, \]
\[ i\partial\rho(z) = \sum_n \rho_n z^{-n-1}, \quad \Theta^{\pm\mp}(z) = \sum_n (P_n^{\pm\mp}) z^{-n-1}, \]

where the mode operators satisfy the superalgebra

\[ [J_n, P_m] = P_{n+m}, \quad [J_n, P_m^*] = -P_{n+m}, \]
\[ [J_n, F_m] = n\delta_{n+m,0}, \quad [P_n, P_m^*] = F_{n+m} + n\delta_{n+m,0}, \]
\[ [J_n, \Theta_m^\pm] = \pm\Theta_{n+m}^{\pm\mp}, \quad [J_n, (P_{\pm\mp})_m] = \mp(P_{\pm\mp})_{n+m}, \]
\[ [P_n, \Theta_m^\pm] = -\Theta_{n+m}^{\mp\pm}, \quad [P_n, (P_{\pm\mp})_m] = -(P_{\pm\mp})_{n+m}, \]
\[ [P_n^*, \Theta_m^{\mp\pm}] = \Theta_{n+m}^{\mp\pm}, \quad [P_n^*, (P_{\pm\mp})_m] = -(P_{\pm\mp})_{n+m}, \]
\[ \{\Theta_n^{\pm\mp}, (P_{\pm\mp})_m\} = \delta_{n+m,0}, \quad \{\Theta_n^{\pm\mp}, (P_{\pm\mp})_m\} = \delta_{n+m,0}, \]
\[ [\rho_n, \rho_m] = -n\delta_{n+m,0}. \]

(4.3)

Because the hybrid fields provide the free field realization (4.1), we can easily obtain the representations of this superalgebra (4.3). As in the \( H_4 \) (super) current algebra,\(^{10,11,5}\) the only non-trivial point is the existence of the spectral flow symmetry; i.e., the superalgebra (4.3) is preserved by the replacement

\[ J_n \longrightarrow J_n, \quad F_n \longrightarrow F_n + p\delta_{n,0}, \]
\[ P_n \longrightarrow P_{n+p}, \quad P_n^* \longrightarrow P_{n-p}^*, \]
\[ \Theta_n^{\pm\mp} \longrightarrow \Theta_n^{\pm\mp}, \quad (P_{\pm\mp})_n \longrightarrow (P_{\pm\mp})_n, \]
\[ \Theta_n^{\pm\pm} \longrightarrow \Theta_n^{\pm\pm}, \quad (P_{\pm\mp})_n \longrightarrow (P_{\pm\mp})_{n+p}, \]
\[ \rho_n \longrightarrow \rho_n, \]

(4.4)

for any integer \( p \in \mathbb{Z} \). The Hilbert space contains all spectrally flowed representations classified into two types, describing short and long strings.\(^{11}\)

### 4.1. The Hilbert space of short strings

The Hilbert space of short strings in the hybrid formalism include all spectrally flowed type II representations,\(^{10}\) \((0 < \eta < 1)\) defined by

\[ J_0|j, \eta, p, l\rangle = j|j, \eta, p, l\rangle, \quad F_0|j, \eta, p, l\rangle = (\eta + p)|j, \eta, p, l\rangle, \]
where \( l = 0, \pm 1 \). The \( \rho_0 \) eigenvalue is fixed so that the supercurrents \( G^\pm \) in (3.13) are periodic and a unique representative is selected from infinitely degenerate states whose existence is due to the pictures.

The explicit representations are easily constructed in terms of the hybrid fields by noting that the transverse fields \((Z, Z^*, \theta^{\pm \pm}, p^{\pm \pm})\) satisfy the twisted boundary conditions:

\[
\begin{align*}
&i\partial Z(e^{2\pi i z}) = e^{-2\pi i n}i\partial Z(z), \quad i\partial Z^*(e^{2\pi i z}) = e^{2\pi i n}i\partial Z^*(z), \\
&\theta^{\pm \mp}(e^{2\pi i z}) = e^{2\pi i n}\theta^{\pm \mp}(z), \quad p^{\pm \pm}(e^{2\pi i z}) = e^{2\pi i n}p^{\pm \pm}(z).
\end{align*}
\] (4.6)

Then, the hybrid fields can be expanded as

\[
\begin{align*}
&i\partial X^\pm(z) = \sum_n \alpha_n^\pm z^{-n-1}, \\
&i\partial Z(z) = \sum_n Z_{n+\eta} z^{-n-\eta-1}, \quad i\partial Z^*(z) = \sum_n Z_{n-\eta} z^{-n+\eta-1}, \\
&\theta^{\pm \mp}(z) = \sum_n \theta_n^{\pm \mp} z^{-n}, \quad p^{\pm \mp}(z) = \sum_n (p^{\pm \mp})_n z^{-n-1}, \\
&\theta^{\pm \pm}(z) = \sum_n \theta_n^{\pm \pm} z^{-n+\eta}, \quad p^{\pm \pm}(z) = \sum_n p^{\pm \pm}_n z^{-n+\eta} z^{-n\eta-1}.
\end{align*}
\] (4.7)

where the oscillators satisfy the canonical (anti-)commutation relations

\[
\begin{align*}
\{\alpha_n^+, \alpha_m^-\} &= n\delta_{n+m,0}, & [Z_{n+\eta}, Z_{m-\eta}^*] &= (n + \eta)\delta_{n+m,0}, \\
\{\theta_n^{\pm \mp}, (p^{\pm \mp})_m\} &= \delta_{n+m,0}, & \{\theta_n^{\pm \pm}, (p^{\pm \pm})^m_{m\pm}\} &= \delta_{n+m,0}.
\end{align*}
\] (4.8)

The flowed type II representations are simply realized as Fock states of these oscillators (and \( \rho_n \)) on the ground state

\[
\begin{align*}
\alpha_0^-|\eta, p, \theta, l\rangle &= j|\eta, p, \theta, l\rangle, \quad &\alpha_0^+|\eta, p, \theta, l\rangle &= (\eta + p)|\eta, p, \theta, l\rangle, \\
\alpha_n^+|\eta, p, \theta, l\rangle &= 0, \quad (n > 0) \\
Z_{n+\eta}|\eta, p, \theta, l\rangle &= 0, \quad (n \geq 0) \\
Z_{n-\eta}^*|\eta, p, \theta, l\rangle &= 0, \quad (n > 0)
\end{align*}
\]
The short string is represented by Fock states on the ground state where we have diagonalized zero modes $\alpha_0 = (p_\pm, l) \frac{\partial}{\partial \theta}$, and denoted their eigenvalues by $\rho_0 = (p_\pm, l) (n > 0), \quad \rho_n = (p_\pm, l) (n > 0)$ by using the bosonization. The charge $Q$ states by a superfield $\Psi$ can formally define a unitary representation by $\mathcal{N}$ representations describing the $\mathcal{M}$ sector. Because an arbitrary unitary representation of the $N = 2$ superconformal field theory is characterized by dimension $\Delta$ and $U(1)$ charge $Q$, we can formally define a unitary representation by

$$\{ L_{0,\mathcal{M}} | \Delta, Q \} = \Delta | \Delta, Q \},$$

$$\{ I_{0,\mathcal{M}} | \Delta, Q \} = Q | \Delta, Q \}. \quad (4.10)$$

The short string is represented by Fock states on the ground state

$$| \eta, p, \theta, l; \Delta, Q \rangle = | \eta, p, \theta, l \rangle \otimes | \Delta, Q \rangle. \quad (4.11)$$

For later use, we note that this ground state has the following eigenvalues:

$$L_0 | \eta, p, \theta, l; \Delta, Q \rangle = (\eta + p) \left( \frac{1}{2} (l - \eta)(l - \eta + 1) + \Delta - \frac{1}{2} \bar{Q} - \frac{1}{2} \bar{\eta}(1 - \eta) \right) | \eta, p, \theta, l; \Delta, Q \rangle, \quad (4.12)$$

$$I_0 | \eta, p, \theta, l; \Delta, Q \rangle = (l - \eta + Q) | \eta, p, \theta, l; \Delta, Q \rangle, \quad (4.13)$$

$$\mathcal{R} | \eta, p, \theta, l; \Delta, Q \rangle = | \eta, p, \theta, l; \Delta, Q \rangle \left( \left( l - \frac{1}{2} \right) + \frac{1}{2} \left( \theta \frac{\partial}{\partial \theta} - \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \right), \quad (4.14)$$

where the constant terms in (4.12) and (4.14) are easily derived by using the bosonization (3.2). The $U(1)$ charge condition (2.23c) together with the $I_0$ eigenvalue (4.13) requires that the charge $Q$ of the short string be fractional.
4.2. The Hilbert space of long strings

The long-string Hilbert space is given by spectrally flowed type I representations \( \eta = 0 \).\(^{10}\) Mode expansions are easily obtained by setting \( \eta = 0 \) in the previous expressions, except for the transverse coordinates \( (Z, Z^*, \theta^{\pm \pm}, p_{\pm \pm}) \) having additional zero-modes. The Fock vacuum is defined by

\[
\alpha_0 |p, q, \theta, \bar{\theta}, l\rangle = |p, q, \theta, \bar{\theta}, l\rangle, \quad \alpha_0^+ |p, q, \theta, \bar{\theta}, l\rangle = p|p, q, \theta, \bar{\theta}, l\rangle, \quad \alpha_0^- |p, q, \theta, \bar{\theta}, l\rangle = q|p, q, \theta, \bar{\theta}, l\rangle, \quad \alpha_0^{\pm} |p, q, \theta, \bar{\theta}, l\rangle = 0, \quad (n > 0)
\]

\[
Z_0 |p, q, \theta, \bar{\theta}, l\rangle = q|p, q, \theta, \bar{\theta}, l\rangle, \quad Z_0^* |p, q, \theta, \bar{\theta}, l\rangle = q^*|p, q, \theta, \bar{\theta}, l\rangle, \quad Z_n |p, q, \theta, \bar{\theta}, l\rangle = 0, \quad (n > 0)
\]

\[
(p_{\pm \pm})_0 |p, q, \theta, \bar{\theta}, l\rangle = |p, q, \theta, \bar{\theta}, l\rangle \quad \frac{\partial}{\partial \theta}, \quad \theta_0^- |p, q, \theta, \bar{\theta}, l\rangle = \frac{\partial}{\partial \bar{\theta}}, \quad \theta_0^+ |p, q, \theta, \bar{\theta}, l\rangle = 0, \quad (n > 0)
\]

\[
(p_{\pm \pm})_n |p, q, \theta, \bar{\theta}, l\rangle = 0, \quad (n > 0)
\]

\[
(p_{\pm \pm})_{n \pm 1} |p, q, \theta, \bar{\theta}, l\rangle = 0, \quad (n > 0)
\]

\[
(\rho_0 |p, q, \theta, \bar{\theta}, l\rangle = l|p, q, \theta, \bar{\theta}, l\rangle, \quad \rho_n |p, q, \theta, \bar{\theta}, l\rangle = 0, \quad (n > 0) \quad (4.15)
\]

where \( q = (q, q^*) \) and \( \bar{\theta} = (\bar{\theta}, \bar{\theta}) \) are the additional zero modes. The coefficient superfield in this sector is a function of the zero modes \( (p, q, \theta, \bar{\theta}) \). The long strings can freely propagate in the four-dimensional space \( (X^{\pm}, Z, Z^*) \).

The long string is represented by Fock states on the ground state

\[
|p, q, \theta, \bar{\theta}, l; \Delta, Q\rangle = |p, q, \theta, \bar{\theta}, l\rangle \otimes |\Delta, Q\rangle, \quad (4.16)
\]

having the following eigenvalues:

\[
L_0 |p, q, \theta, \bar{\theta}, l; \Delta, Q\rangle = (pj + qq^* - \frac{1}{2}l(l + 1) + \Delta - \frac{1}{2}Q)|p, q, \theta, \bar{\theta}, l; \Delta, Q\rangle, \quad (4.17)
\]

\[
I_0 |p, q, \theta, \bar{\theta}, l; \Delta, Q\rangle = (l + Q)|p, q, \theta, \bar{\theta}, l; \Delta, Q\rangle, \quad (4.18)
\]

\[
\mathcal{R} |p, q, \theta, \bar{\theta}, l; \Delta, Q\rangle = |p, q, \theta, \bar{\theta}, l; \Delta, Q\rangle \left( l + \frac{1}{2} \left( \theta \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial \bar{\theta}} \right) \right), \quad (4.19)
\]

The \( U(1) \) charge condition \((2.23c)\) together with the \( I_0 \) eigenvalue \((4.18)\) leads to the result that the long string must have integral \( Q \).
§5. Physical spectrum

In this section we study the physical spectrum at lower mass levels explicitly. We concentrate on the states whose $\mathcal{M}$ sector is composed of (anti-)chiral primary states characterized by $\Delta = \frac{|Q|}{2}$. Then we solve the physical state conditions (2.23).

5.1. Physical states in the short string sector

We first examine physical states at mass levels $N = 0, \eta, 1 - \eta$ in the short string sector. It can easily be shown that it is sufficient to study the $l = 1, 0$ cases, since there is no physical state with $\rho_0$-momentum $l = -1$ at these levels. The $U(1)$ charge condition (2.23c) and the chirality condition $\Delta = \frac{|Q|}{2}$ yield $\Delta = -\frac{Q}{2} = \frac{1}{2}(1 - \eta)$ for the $l = 1$ case and $\Delta = \frac{Q}{2} = \frac{\eta}{2}$ for the $l = 0$ case.

Let us start by considering the oscillator ground state $N = 0$ with $l = 1$, given by

$$|V\rangle = |1\rangle\Psi^{(\frac{1}{2})}(p, \theta).$$

We denote here the state (4.11) with $l = 1$ by $|1\rangle$ and use this abbreviation in this subsection for simplicity. A half of the supersymmetry is realized on the coefficient superfield $\Psi^{(\frac{1}{2})}$ by\(^*\)

$$Q_{-}\ = \ \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2}j\bar{\theta}, \quad Q_{+}\ = \ \frac{\partial}{\partial \theta} - \frac{1}{2}j\theta.$$  \hspace{1cm} (5.2)

The superscript “$^{(\frac{1}{2})}$” of the coefficient superfield indicates that its first component has $R$-charge $\frac{1}{2}$, which can be read from Eq. (4.14). The physical state conditions (2.23) lead to manifestly supersymmetric conditions on the superfield:

$$\tilde{D}\tilde{D}\Psi^{(\frac{1}{2})} = 0,$$

$$\delta\Psi^{(\frac{1}{2})} = \tilde{D}A^{(1)},$$  \hspace{1cm} (5.3a, 5.3b)

where $A^{(1)}$ is an arbitrary gauge parameter superfield and supercovariant derivatives are defined by

$$D = \frac{\partial}{\partial \theta} + \frac{1}{2}j\bar{\theta}, \quad \tilde{D} = \frac{\partial}{\partial \bar{\theta}} + \frac{1}{2}j\theta.$$  \hspace{1cm} (5.4)

The conditions (5.3) can be easily solved by choosing an appropriate gauge as

$$\Psi^{(\frac{1}{2})} = \bar{\theta}\bar{\phi}^{(0)}(p, j = 0).$$  \hspace{1cm} (5.5)

The physical component $\bar{\phi}^{(0)}$ is a space-time boson and is identified with the *tachyon-like* state obtained in Ref. 5). The solution (5.5) also shows that there is no fermionic massless

\(^*\) The other half is a part of the DDF operators discussed in §6. These operators relate the states with different masses.
the physical state; i.e., the physical spectrum has boson-fermion asymmetry. This is only possible for the massless \((j = 0)\) state, on which the supercharges \((5.2)\) anti-commute.

For the oscillator ground state with \(l = 0\),

\[
|V\rangle = |0\rangle \psi^{(-\frac{1}{2})}(p, \theta),
\]

physical state conditions are provided by

\[
\bar{D}D\psi^{(-\frac{1}{2})} = 0, \quad \delta\psi^{(-\frac{1}{2})} = DA^{(-1)},
\]

and the solution has a form similar to \((5.5)\):

\[
\psi^{(-\frac{1}{2})} = \theta \phi^{(0)}(p, j = 0).
\]

The massless boson \(\phi^{(0)}\) has no fermionic partner and is identified with the graviton-like state obtained in Ref. 5).

Next, we consider two massive cases, \(N = \eta\) and \(1 - \eta\). General states at the level \(N = \eta\) are expanded in three Fock states as

\[
|V\rangle = (\Pi^*_Z)_-\eta|l\rangle \psi^{(\frac{l}{2})}(p, \theta) + (d_{++}^-\eta)l\rangle \Phi^{(l)}(p, \theta) + \theta^-\eta|l\rangle \Xi^{(l)}(p, \theta).
\]

Because we use a supercovariant basis created by \(((\Pi^*_Z)_-, d_{++}^-\eta, \theta^-\eta)\), the coefficient fields are superfields; i.e. their supersymmetry transformations are generated by the supercharges \((5.2)\). The equations of motion for the \(l = 1\) case can be written

\[
D \left( \bar{D}\Xi^{(1)} + \eta\psi^{(\frac{1}{2})} \right) = 0, \\
\bar{D} \left( \Xi^{(1)} - (p + \eta)D\psi^{(\frac{1}{2})} \right) + ((p + \eta)j + \eta) \psi^{(\frac{1}{2})} = 0,
\]

with the gauge transformations

\[
\delta\psi^{(\frac{1}{2})} = \bar{D}A^{(1)}, \\
\delta\phi^{(1)} = \Sigma^{(1)}, \\
\delta\Xi^{(1)} = -\eta A^{(1)}.
\]

Choosing the \(\phi^{(1)} = \Xi^{(1)} = 0\) gauge, the physical state is described by an anti-chiral superfield satisfying \(D\psi^{(\frac{1}{2})} = 0\) and the on-shell condition \(((p + \eta)j + \eta)\psi^{(\frac{1}{2})} = 0\). The anti-chiral superfield has the explicit form

\[
\psi^{(\frac{1}{2})} = \psi^{(\frac{1}{2})} + \bar{\theta} \phi^{(0)} - \frac{1}{2} \theta \bar{\theta} j \psi^{(\frac{1}{2})}.
\]
containing one boson, \(\bar{\phi}(0)\), and one fermion, \(\psi(\frac{1}{2})\).

For the case \(l = 0\), the equations of motion

\[
\bar{D} \left( \Xi^{(0)} - (p + \eta) D\psi(\frac{-1}{2}) \right) = 0,
\]
\[
D \left( \Xi^{(0)} - (p + \eta) \Phi^{(0)} \right) = 0,
\]
\[
(p + \eta) \bar{D} D\Phi^{(0)} + \eta D\psi(\frac{-1}{2}) + [D, \bar{D}] \Xi^{(0)} = ((p + \eta) j + \eta) \Phi^{(0)},
\]
\[
(p + \eta) D \left( \bar{D} \Xi^{(0)} + \eta \psi(\frac{-1}{2}) \right) = ((p + \eta) j + \eta) \Xi^{(0)}
\]

(5.13)

and the gauge transformations

\[
\delta \psi(\frac{-1}{2}) = \Sigma^{(-\frac{1}{2})} + D\lambda^{(-1)},
\]
\[
\delta \Phi^{(0)} = D \Sigma^{(-\frac{1}{2})},
\]
\[
\delta \Xi^{(0)} = (p + \eta) D \Sigma^{(-\frac{1}{2})}
\]

(5.14)

can be solved by choosing \(\psi(\frac{-1}{2}) = 0\) gauge as

\[
\Xi^{(0)} = (p + \eta) \frac{1}{j} \bar{D} D\Phi^{(0)}.
\]

(5.15)

The physical state is identified with an unconstrained superfield \(\Phi^{(0)}\) satisfying the on-shell condition \(((p + \eta) j + \eta) \Phi^{(0)} = 0\).

We can easily repeat the above analysis to study the physical spectrum at the level \(N = 1 - \eta\). The states at this level are generally

\[
|V\rangle = (\Pi Z)_{-1 + \eta |l\rangle} \psi(l\frac{-1}{2})(p, \theta) + (d_{-})_{-1 + \eta |l\rangle} \Phi^{(l-1)}(p, \theta) + \theta^{++}_{-1 + \eta |l\rangle} \Xi^{(l-1)}(p, \theta).
\]

(5.16)

For \(l = 1\), the equations of motion and the gauge transformations are given by

\[
D \left( \Xi^{(0)} - (p + \eta) \bar{D} \psi(\frac{1}{2}) \right) = 0,
\]
\[
(p + \eta) \bar{D} D\Phi^{(0)} + (1 - \eta) \bar{D} \psi(\frac{1}{2}) - [D, \bar{D}] \Xi^{(0)} = ((p + \eta) j + 1 - \eta) \Phi^{(0)},
\]
\[
(p + \eta) D \left( D \Xi^{(0)} + (1 - \eta) \psi(\frac{1}{2}) \right) = ((p + \eta) j + 1 - \eta) \Xi^{(0)},
\]
\[
D \left( \Xi^{(0)} - (p + \eta) \Phi^{(0)} \right) = 0,
\]

(5.17)

and

\[
\delta \psi(\frac{1}{2}) = \Sigma(\frac{1}{2}) + \bar{D}\lambda^{(1)},
\]
\[
\delta \Phi^{(0)} = \bar{D} \Sigma(\frac{1}{2}),
\]
\[
\delta \Xi^{(0)} = (p + \eta) D \Sigma(\frac{1}{2}),
\]

(5.18)
respectively. The physical state is given by an unconstrained superfield $\Phi^{(0)}$ satisfying

$$((p + \eta)j + 1 - \eta) \Phi^{(0)} = 0.$$  

The superfield $\Psi^{(\frac{1}{2})}$ can be gauged away, and $\Xi^{(0)}$ is expressed in terms of $\Phi^{(0)}$.

We can also solve the physical state conditions for $l = 0,$

$$D \left( \Xi^{(-1)} - (p + \eta) \bar{D} \Psi^{(-\frac{1}{2})} \right) + ((p + \eta)j + 1 - \eta) \Psi^{(-\frac{1}{2})} = 0,$$

$$D \left( D \Xi^{(-1)} + (1 - \eta) \bar{D} \Psi^{(-\frac{1}{2})} \right) = 0,$$

and

$$\delta \Psi^{(-\frac{1}{2})} = DA^{(-1)},$$

$$\delta \phi^{(-1)} = \Sigma^{(-1)},$$

$$\delta \Xi^{(-1)} = -(1 - \eta) A^{(-1)},$$

in terms of a chiral superfield $\Psi^{(-\frac{1}{2})}$ satisfying

$$\bar{D} \Psi^{(-\frac{1}{2})} = 0,$$

$$((p + \eta)j + 1 - \eta) \Psi^{(-\frac{1}{2})} = 0.$$  

The explicit form of this chiral superfield is

$$\Psi^{(-\frac{1}{2})} = \psi^{(-\frac{1}{2})} + \theta \phi^{(0)} + \frac{1}{2} \theta \bar{\theta} j \psi^{(-\frac{1}{2})}.$$  

In short, the physical spectrum at these massive levels contains two types of multiplets, (anti-)chiral and unconstrained. The latter is reducible and decomposes into two (chiral and anti-chiral) multiplets.

5.2. Physical states in the long string sector

In the long string sector, we examine only the ground state with $\Delta = Q = 0$. The $U(1)$ charge condition (2.23c) leads to $l = 0$, and therefore the state is given by

$$|V\rangle = |p, q, \theta, \tilde{\theta}, 0, 0\rangle V^{(0)}(p, q, \theta, \tilde{\theta}).$$

The physical state conditions become

$$\left( D \bar{D} \bar{D} \bar{D} - \bar{D} \bar{D} \bar{D} \bar{D} \right) V^{(0)} = 0,$$

$$\delta V^{(0)} = \bar{D} D A^{(-1)} + \bar{D} \bar{D} A^{(1)},$$

20
where the supercovariant derivatives in this sector have the forms
\[
\begin{align*}
D &= \frac{\partial}{\partial \theta} + \frac{1}{2} j \bar{\theta} - \frac{1}{2} q^* \bar{\theta}, \\
\tilde{D} &= \frac{\partial}{\partial \bar{\theta}} + \frac{1}{2} j \theta - \frac{1}{2} q^* \theta.
\end{align*}
\]

The conditions (5.24) are essentially the same as those for the four-dimensional vector multiplet. In the WZ gauge,
\[
V = \frac{1}{2} \bar{\theta} \theta A_+ - \frac{1}{2} \bar{\theta} \theta A_- + \frac{1}{2} \bar{\theta} \theta A^* + \frac{1}{2} \bar{\theta} \theta A,
\]
\[
+ \bar{\theta} \theta \lambda - \bar{\theta} \theta \lambda - \bar{\theta} \theta \lambda + \theta \bar{\theta} \theta \lambda + \theta \bar{\theta} \theta \lambda, \tag{5.26}
\]
the equations of motion (5.24a) lead to the Maxwell equations
\[
\begin{align*}
\frac{1}{4} j(pA_- + jA_+ + qA^* + q^* A) - \frac{1}{2}(pj + qq^*)A_- &= 0, \\
\frac{1}{4} p(pA_- + jA_+ + qA^* + q^* A) - \frac{1}{2}(pj + qq^*)A_+ &= 0, \\
\frac{1}{4} q^*(pA_- + jA_+ + qA^* + q^* A) - \frac{1}{2}(pj + qq^*)A^* &= 0, \\
\frac{1}{4} q(pA_- + jA_+ + qA^* + q^* A) - \frac{1}{2}(pj + qq^*)A &= 0, \tag{5.27}
\end{align*}
\]
the massless Dirac equations
\[
\begin{align*}
(q^* \bar{\lambda} - j \bar{\lambda}) &= 0, & (p \bar{\lambda} + q \lambda) &= 0, \\
(q \bar{\lambda} - j \lambda) &= 0, & (p \bar{\lambda} + q^* \lambda) &= 0, \tag{5.28}
\end{align*}
\]
and $D = 0$ for the auxiliary field. The massless spectrum of the long string is thus the vector multiplet in the four-dimensional free-field space $(X^\pm, Z, Z^*)$.

\section*{6. Summary and Discussion}

In this paper, we have studied four-dimensional superstrings in NS-NS plane-wave backgrounds using the hybrid formalism. This description of the superstring has been obtained through a field redefinition of the worldsheet fields in the super-NW model.\footnote{Because we have adopted a weak GSO projection that restricts only the total $U(1)_R$ charge to integer values, the model has enhanced supersymmetry, which is manifest in the hybrid formalism. The Hilbert space consists of two sectors, describing short and long strings, and including all the spectrally flowed representations of types II and I, respectively.\footnote{Then, we studied physical states to find boson-fermion asymmetry in the massless spectrum of the short string.}} Then, we studied physical states to find boson-fermion asymmetry in the massless spectrum of the short string.
string. There are two massless bosons, called tachyon-like and graviton-like in Ref. 5), but no fermionic partners. We have also identified massive physical states at the levels $N = \eta$ and $1 - \eta$ in the short string sector and massless physical states in the long string sector. The massless physical spectrum of the long string is the vector multiplet freely propagating in the four-dimensional space $(X^\pm, Z, Z^*)$.

The massive physical states obtained by solving the physical state conditions are also created by acting with the DDF operators

$$\mathcal{P}_n = \oint \frac{dz}{2\pi i} e^{i (\frac{n + \eta}{p + \eta}) z} \left( i \partial Z - \left( \frac{n + \eta}{p + \eta} \right) \theta^{+-} p_- \right),$$
$$\mathcal{P}_n^* = \oint \frac{dz}{2\pi i} e^{i (\frac{n - \eta}{p + \eta}) z} \left( i \partial Z^* - \left( \frac{n - \eta}{p + \eta} \right) \theta^{-+} p_+ \right),$$
$$\mathcal{Q}_n^{++} = \oint \frac{dz}{2\pi i} e^{i (\frac{n + \eta}{p + \eta}) z} \left( p_- + \frac{1}{2} i \partial X + \theta^{+-} \right)$$
$$\quad + \frac{1}{2} \left( i \partial Z - \left( \frac{n + \eta}{p + \eta} \right) \theta^{+-} p_- \right) \theta^{+-} + \frac{1}{8} \partial (\theta^{-+} \theta^{+-}) \theta^{+-},$$
$$\mathcal{Q}_n^{+-} = \oint \frac{dz}{2\pi i} e^{i (\frac{n - \eta}{p + \eta}) z} \left( p_+ + \frac{1}{2} i \partial X + \theta^{--} \right)$$
$$\quad + \frac{1}{2} \left( i \partial Z - \left( \frac{n - \eta}{p + \eta} \right) \theta^{-+} p_+ \right) \theta^{+-} - \frac{1}{8} \partial (\theta^{-+} \theta^{+-}) \theta^{+-}, \quad (6.1)$$
on the massless physical states. These operators include $\mathcal{P} = \mathcal{P}_n$, $\mathcal{P}^* = \mathcal{P}_n^*$, $\mathcal{Q}^{++} = \mathcal{Q}_n^{++}$ and generate an affine extension of the supersymmetry algebra (2.18):

$$[\mathcal{J}, \mathcal{P}_n] = \left( \frac{n + \eta}{p + \eta} \right) \mathcal{P}_n,$$
$$[\mathcal{J}, \mathcal{P}_n^*] = \left( \frac{n - \eta}{p + \eta} \right) \mathcal{P}_n^*,$$
$$[\mathcal{P}_n, \mathcal{P}_m] = \left( \frac{n + \eta}{p + \eta} \right) \mathcal{F} \delta_{n+m,0},$$
$$[\mathcal{J}, \mathcal{Q}_n^{++}] = \left( \frac{n + \eta}{p + \eta} \right) \mathcal{Q}_n^{++},$$
$$[\mathcal{J}, \mathcal{Q}_n^{+-}] = \left( \frac{n - \eta}{p + \eta} \right) \mathcal{Q}_n^{+-},$$
$$[\mathcal{Q}^{--}, \mathcal{P}_n] = - \left( \frac{n + \eta}{p + \eta} \right) \mathcal{Q}_n^{++},$$
$$[\mathcal{Q}^{+-}, \mathcal{P}_n^*] = - \left( \frac{n - \eta}{p + \eta} \right) \mathcal{Q}_n^{--},$$
$$\{ \mathcal{Q}_n^{++}, \mathcal{Q}_m^{+-} \} = \mathcal{F} \delta_{n+m,0},$$
$$\{ \mathcal{Q}^{+-}, \mathcal{Q}^{-+} \} = \mathcal{J},$$
$$\{ \mathcal{Q}^{--}, \mathcal{Q}_n^{++} \} = \mathcal{P}_n,$$
$$\{ \mathcal{Q}^{+-}, \mathcal{Q}_n^{--} \} = \mathcal{P}_n^*. \quad (6.2)$$

This extended supersymmetry algebra provides an enhanced space-time symmetry. This symmetry is obtained by taking the Penrose limit of $\mathcal{N} = 2$ superconformal symmetry, being the isometry of the $AdS_3 \times S^1$ background. It is interesting to trace this limit by
studying hybrid superstrings in $AdS_3 \times S^1$. Such an investigation is currently underway, and it will be reported elsewhere.\textsuperscript{15}

The above-mentioned boson-fermion asymmetry in the short-string spectrum does not exist in the RNS formalism, because there are two additional physical states in the Ramond sector.\textsuperscript{*} However, this nonequivalence of the physical spectrum does not give rise to a tree-level inconsistency, because the extra RNS fermions are also supersymmetry singlets. It would be interesting to give an interpretation of this new spectrum, although we have to further check the modular invariance of the one-loop partition function for quantum consistency.

It would also be interesting to understand general aspects of the holographic duality in plane wave backgrounds.\textsuperscript{5,11} This will require further investigation. We hope that the manifest supersymmetry in the hybrid formalism will shed new light on such problems.

Acknowledgements

The author would like to thank Y. Hikida and Y. Sugawara for valuable discussions. This work was supported in part by Grants-in-Aid for Scientific Research from the Japan Society for the Promotion of Science (No. 11640276) and from the Ministry of Education, Culture, Sports, Science and Technology of Japan (No.13135213).

References


\textsuperscript{*} The author would like to thank the referee for pointing out this fact.


15) H. Kunitomo, work in progress.