Grand unified theory defined on higher-dimensional orbifolds provides a new way to solve the hierarchy problem. In gauge theory on an orbifold many different sets of boundary conditions imposed at orbifold fixed points (branes) are related by large gauge transformations, forming an equivalent class of boundary conditions. Thanks to the Hosotani mechanism the physics remains the same in all theories in a given equivalent class, though the symmetry of boundary conditions differs from each other. Quantum dynamics of Wilson line phases rearranges the gauge symmetry. In the nonsupersymmetric $SU(5)$ model the presence of bulk fermions leads to the spontaneous breaking of color $SU(3)$. In the supersymmetric model with Scherk-Schwarz SUSY breaking, color $SU(3)$ is spontaneously broken even in the absence of bulk fermions if there are Higgs multiplets.

1 Introduction

There are good reasons for investigating higher dimensional gauge theories. If the superstring theory is to describe the nature, we live in ten-dimensional spacetime. There must be hidden dimensions beyond the four-dimensional spacetime we see at the current energy scale. Dynamics in the string theory may result spacetime with the structure of an orbifold such that the four-dimensional spacetime we see be located at its fixed points (brane). The presence of D-branes in the string theory makes it quite probable that an effective gauge theory emerges in more than four dimensions.

Another important reason lies in the fact that many puzzles or problems difficult to be solved in four-dimensional theory can be naturally resolved thanks to the existence of extra dimensions and the special nature of orbifolds. Among the notable problems are the hierarchy problem in grand unified theories and the chiral fermion problem.

In formulating gauge theory on an orbifold, however, there appears, at a first look, arbitrary choice of boundary conditions imposed on fields at
orbifold fixed points (branes). This arbitrariness poses a serious obstacle to constructing unified theories in a convincing manner.

In this paper we shall show that the arbitrariness problem in the orbifold boundary conditions is partially solved at the quantum level by the Hosotani mechanism.\textsuperscript{3,4,5} Dynamics of Wilson line phases play a vital role in rearranging the gauge symmetry. Physical symmetry, in general, differs from the symmetry of the orbifold boundary conditions. Physical symmetry does not depend on the orbifold boundary conditions so long as the boundary conditions belong to the same equivalence class, whose qualification shall be detailed below.

2 Orbifold boundary conditions in gauge theory

We shall consider gauge theories on $\mathcal{M} = M^4 \times (S^1/Z_2)$ where $M^4$ is the four-dimensional Minkowski spacetime. Let $x^\mu$ and $y$ be coordinates of $M^4$ and $S^1$, respectively. $S^1$ has a radius $R$ so that a point $(x^\mu, y + 2\pi R)$ is identified with a point $(x^\mu, y)$. The orbifold $M^4 \times (S^1/Z_2)$ is obtained by further identifying $(x^\mu, -y)$ and $(x^\mu, y)$. Gauge fields and Higgs fields are defined in the five dimensional spacetime $\mathcal{M}$. We adopt the brane picture in which quarks and leptons are confined in one of the boundary branes, say, at $y = 0$. There arises no problem in having chiral fermions on the brane. It is possible to have fermions in the bulk five dimensional $M^5$, whose effect in the context of the Hosotani mechanism is also evaluated.

Fields are defined on the covering space $\mathcal{M}_{\text{cover}}$ of $M^4 \times S^1$. Physical quantities must be single-valued after a loop translation along $S^1$ and after $Z_2$ parity reflection around $y = 0$ or $y = \pi R$. This, however, does not imply the single-valuedness of the fields. In gauge theory the fields need to return to their original values up to a gauge transformation and a sign. Take the lower four-dimensional components of the gauge potentials, $A_\mu(x, y)$. They satisfy

\begin{align}
A_\mu(x, -y) &= P_0 P_0 \, A_\mu(x, y) P_0 \\
A_\mu(x, \pi R - y) &= P_1 P_0 \, A_\mu(x, \pi R + y) P_1 \\
A_\mu(x, y + 2\pi R) &= U A_\mu(x, y) U^\dagger
\end{align}

(1)

where $P_0^2 = P_1^2 = 1$, $P_0^\dagger = P_0$, $P_1^\dagger = P_1$, and $UU^\dagger = 1$. There holds a relation $U = P_1 P_0$.

It follows from (1) that $F_{\mu\nu}(x, -y) = P_0 F_{\mu\nu}(x, y) P_0^\dagger$. The gauge covariance can be maintained for the $\mu$-$y$ component only if $F_{\mu y}(x, -y) = -P_0 F_{\mu y}(x, y) P_0^\dagger$ etc., which implies

$A_y(x, -y) = -P_0 A_y(x, y) P_0^\dagger$
Notice the relative minus sign under $Z_2$ parity reflection.

Higgs fields and bulk fermion fields must satisfy similar relations. Take a bulk fermion field $\psi(x, y)$. Gauge covariance of a covariant derivative $D_\mu \psi$ demands that

$$\psi(x, -y) = \pm T_\psi [P_0] \gamma^5 \psi(x, y)$$
$$\psi(x, \pi R - y) = \pm e^{i \pi \beta_\psi} T_\psi [P_1] \gamma^5 \psi(x, \pi R + y)$$
$$\psi(x, y + 2\pi R) = e^{i \pi \beta_\psi} T_\psi [U] \psi(x, y)$$

where $T[P]$ represents an appropriate representation matrix. One immediate consequence is that a mass term $\bar{\psi} \psi$ is not allowed on $M$.

If $P_0$ or $P_1$ is not proportional to the identity, the original gauge symmetry is partially broken. It gives a genuine device to achieve gauge symmetry breaking without “Higgs scalar fields”. This feature has been successfully utilized to explain the triplet-doublet splitting problem in the $SU(5)$ model by Kawamura. However, at the same time it brings about arbitrariness or indeterminacy in the symmetry breaking pattern. This dilemma can be resolved by two distinct mechanisms. The first one is to ensure that different sets of boundary conditions ($P_0$, $P_1$) lead to the same physics, thus to make the choice of ($P_0$, $P_1$) irrelevant. The second one is to provide dynamics to select ($P_0$, $P_1$). In the final theory both mechanisms most likely will come into operation. In this article we show that the first mechanism is indeed in action.

3 Residual gauge invariance of the boundary conditions

It is necessary to pin down which parts of the original gauge symmetry are left unbroken by the orbifold boundary conditions. Under a general gauge transformation on the covering space $M_{\text{cover}}$

$$A_M \rightarrow A'_M = \Omega A_M \Omega^\dagger - \frac{i}{g} \Omega \partial_M \Omega^\dagger,$$

new gauge potentials satisfy, in place of (1) and (2),

$$\begin{bmatrix} A'_\mu(x, y) \\ A'_\nu(x, y) \end{bmatrix} = P_0' \begin{bmatrix} A'_\mu(x, y) \\ -A'_\nu(x, y) \end{bmatrix} P_0'^\dagger - \frac{i}{g} P_0' \left[ \partial_\mu - \partial_\nu \right] P_0'^\dagger$$

3
\[
\begin{align*}
\begin{bmatrix}
A_\mu'(x, \pi R - y) \\
A_y'(x, \pi R - y)
\end{bmatrix} &= P_1' \begin{bmatrix}
A_\mu'(x, \pi R + y) \\
-A_y'(x, \pi R + y)
\end{bmatrix} P_1'^\dagger - \frac{i}{g} P_1' \begin{bmatrix}
\partial_\mu \\
-\partial_y
\end{bmatrix} P_1'^\dagger \\
A_M'(x, y + 2\pi R) &= U' A_M'(x, y) U'^\dagger - \frac{i}{g} U' \partial_M U'^\dagger
\end{align*}
\]
(5)

where
\[
\begin{align*}
P_0' &= \Omega(x, -y) P_0 \Omega^\dagger(x, y) \\
P_1' &= \Omega(x, \pi R - y) P_1 \Omega^\dagger(x, \pi R + y) \\
U' &= \Omega(x, y + 2\pi R) U \Omega^\dagger(x, y) .
\end{align*}
\]
(6)

The theory, or more precisely speaking, the Hilbert space of the theory, is defined with the orbifold boundary conditions specified. The residual gauge symmetry in the theory consists of gauge transformations which preserve the boundary conditions so that
\[
\begin{align*}
\Omega(x, -y) P_0 &= P_0 \Omega(x, y) \\
\Omega(x, \pi R - y) P_1 &= P_1 \Omega(x, \pi R + y) \\
\Omega(x, y + 2\pi R) U &= U \Omega(x, y) .
\end{align*}
\]
(7)

The residual gauge symmetry is large. Although \((P_0, P_1, U)\) may not be invariant under global transformations of the gauge group, \(\Omega(x, y)\)'s satisfying \((P_0', P_1', U') = (P_0, P_1, U)\) extend over the whole group.

One example is in order. Take a SU(2) gauge theory with boundary conditions \((P_0, P_1) = (\tau_3, \tau_3 \cos 2\pi \alpha + \tau_1 \sin 2\pi \alpha)\). The residual global symmetry is \(U(1)\) for \(\alpha = 0, \pm \frac{1}{2}, \pm 1, \cdots\), and none left otherwise. The residual gauge symmetry is given by
\[
\Omega(x, y) = \exp \left\{ \frac{i}{4} \sum_{a=1}^{3} \omega_a(x, y) \tau_a \right\}
\]
\[
\omega_2(x, y) = \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{\infty} \omega_{2,n}(x) \sin \frac{ny}{R}
\]
\[
\begin{bmatrix}
\omega_1(x, y) \\
\omega_3(x, y)
\end{bmatrix} = \frac{1}{\sqrt{\pi R}} \sum_{n=-\infty}^{\infty} v_n(x) \begin{bmatrix}
\sin (n + 2\alpha)y/R \\
\cos (n + 2\alpha)y/R
\end{bmatrix} .
\]
(8)

These gauge transformations mix all Kaluza-Klein modes.

In many situations we are interested in physics at low energies, or symmetry seen at an energy scale much lower than \(1/R\), for which only \(y\)-independent gauge transformations are recognized. In the SU(2) example
presented above such symmetry survives for an integral $2\alpha$, i.e. the $U(1)$ gauge symmetry with $\omega_3 \sim v_{-2\alpha}(x)$ remains unbroken. In general cases such low energy gauge symmetry is given by $\Omega(x)$’s satisfying

$$
\Omega(x) P_0 = P_0 \Omega(x), \\
\Omega(x) P_1 = P_1 \Omega(x), \\
\Omega(x) U = U \Omega(x),
$$

that is, the symmetry is generated by generators which commute with $P_0$, $P_1$ and $U$. This symmetry is called the low energy symmetry of the boundary conditions.

4 Wilson line phases

Given the orbifold boundary conditions $(P_0, P_1, U)$ there appear new physical degrees of freedom which are absent in the Minkowski spacetime. Consider a path-ordered integral along a non-contractible loop on $S^1$:

$$
W(x, y) = P \exp \left\{ ig \int_y^{y+2\pi R} dy' A_y(x, y') \right\}.
$$

Under a gauge transformation $\Omega(x, y)$

$$
W(x, y) U \rightarrow \Omega(x, y) W(x, y) \Omega(x, y + 2\pi R)^\dagger U = \Omega(x, y) W(x, y) U \Omega(x, y)^\dagger
$$

where the last relation in (7) has been made use of. In other words the eigenvalues of $W(x, y) U$ are invariant under residual gauge transformations. Nontrivial $(x, y)$ dependence results when field strengths $F_{MN} \neq 0$. $F_{MN} = 0$ does not necessarily imply trivial $WU$, however.

Consider a configuration with constant $A_y$ and vanishing $A_\mu$, which certainly has $F_{MN} = 0$. To satisfy the orbifold boundary conditions (2), $A_y$ must anticommutes with $P_0$ and $P_1$. This configuration yields $WU = \exp\{2\pi i g \lambda A_y\} \cdot U$ which in general is gauge-inequivalent to $WU = 1$. Nontrivial phases are called Wilson line phases which are promoted to physical degrees of freedom. Let us write $A_M = \sum_a \frac{1}{2} A_M^a \lambda_a$ where $\text{Tr} \lambda^a \lambda^b = 2\delta^{ab}$. Wilson line phases on $M^4 \times (S^1/Z_2)$ are \( \{ \theta_a = g\pi RA_y^a, a \in \mathcal{H}_W \} \) where

$$
\mathcal{H}_W = \left\{ \frac{\lambda^a}{2} ; \{ \lambda^a, P_0 \} = \{ \lambda^a, P_1 \} = 0 \right\}. \tag{12}
$$

The presence of Wilson line phases as physical degrees of freedom reflects the degeneracy in classical vacua. The degenerate vacua are connected
by Wilson line phases. The degeneracy is lifted by quantum effects. It is at this place where dynamics of Wilson line phases induces rearrangement of gauge symmetry.

5 Equivalent classes of the orbifold boundary conditions

To further motivate investigating dynamics of Wilson lines we take a closer look at interrelations among different sets of boundary conditions. Recall that gauge-transformed potentials satisfy new boundary conditions given in Eqs. (5) and (6). If \((P'_0, P'_1, U')\) turns out constant in spacetime, i.e. \(\partial_M P'_0 = \partial_M P'_1 = \partial_M U' = 0\), then the new set of boundary conditions \((P'_0, P'_1, U')\) is of the allowed type. In general, \((P'_0, P'_1, U')\) is distinct from \((P_0, P_1, U)\). The low energy symmetry of the boundary conditions are different.

When two sets of boundary conditions are related by a boundary-condition-changing gauge transformation, the two sets are said to be in the same equivalent class:

\[
(U', P'_0, P'_1) \sim (U, P_0, P_1) .
\]

The relation is transitive. This defines equivalence classes of the boundary conditions.

It is easy to find nontrivial examples. Take

\[
\Omega(x, y) = e^{i(y+\alpha)\Lambda} \quad \text{where} \quad \{\Lambda, P_0\} = \{\Lambda, P_1\} = 0 ,
\]

which leads to

\[
P'_0 = e^{2i\alpha\Lambda} P_0 , \quad P'_1 = e^{2i(\alpha+\pi R)\Lambda} P_1 , \quad U' = e^{2i\pi RA} U .
\]

As the reader might recognize, a boundary-condition-changing gauge transformation has the correspondence to a Wilson line phase.

A boundary-condition-changing gauge transformation relates two different theories. There is one-to-one correspondence between these two theories. As they are related by a gauge transformation, physics of the two theories must be the same. Nevertheless, the two sets of the boundary conditions have different symmetry. How is it possible for such two theories to be equivalent? The equivalence of the two theories is guaranteed by the Hosotani mechanism.

6 The Hosotani mechanism and physical symmetry

Quantum dynamics of Wilson line phases controle the physical symmetry of the theory. The mechanism is called the Hosotani mechanism which
has originally been established in gauge theories on multiply-connected manifolds.\textsuperscript{3, 4} It applies to gauge theories on orbifolds as well.\textsuperscript{2, 5, 7, 8} The only change is that the degrees of freedom of Wilson line phases are restricted on orbifolds as explained in section 4. The mechanism can be applied to supersymmetric theories.\textsuperscript{9} It can induce spontaneous SUSY breaking in the gauged supergravity model.\textsuperscript{10}

**The Hosotani mechanism** consists of six parts.

(i) Wilson line phases along non-contractible loops become physical degrees of freedom which cannot be gauged away. They parametrize degenerate classical vacua.

(ii) The degeneracy is lifted by quantum effects, unless it is strictly forbidden by supersymmetry. The physical vacuum is given by the configuration of the Wilson line phases which minimizes the effective potential $V_{\text{eff}}$. (In two or three dimensions significant quantum fluctuations appear around the minimum of $V_{\text{eff}}$.\textsuperscript{11, 12})

(iii) If the effective potential $V_{\text{eff}}$ is minimized at a nontrivial configuration of Wilson line phases, then the gauge symmetry is spontaneously enhanced or broken by radiative corrections. This part of the mechanism is sometimes called the Wilson line symmetry breaking in the literature. Nonvanishing expectation values of the Wilson line phases give masses to those gauge fields in lower dimensions whose gauge symmetry is broken. Some of matter fields also acquire masses.

(iv) All zero-modes of extra-dimensional components of gauge fields in the broken sector of gauge group, which may exist at the classical level, become massive by quantum effects.

(v) The physical symmetry of the theory is determined by the combination of the boundary conditions and the expectation values of the Wilson line phases. Theories in the same equivalent class of the boundary conditions have the same physical symmetry and physics content.

(vi) The physical symmetry of the theory is mostly dictated by the matter content of the theory. It does not depend on the values of various coupling constants in the theory.

Part (v) of the mechanism is of the biggest relevance in our discussions. It tells us that the physics is independent of the orbifold boundary conditions so long as they belong to the same equivalent class of the boundary conditions.

The physical symmetry of the theory, $H_{\text{sym}}$, is determined as follows. Suppose that with the boundary conditions $(P_0, P_1, U)$ the effective potential is minimized at $\langle A_y \rangle$ such that $W = \exp (ig2\pi R \langle A_y \rangle) \neq 1$. One needs to know the symmetry around $\langle A_y \rangle$. Perform a boundary-condition-changing gauge transformation $\Omega(y) = \exp \{ ig(y + \beta)\langle A_y \rangle \}$, which brings
\[ \langle A_y \rangle \to \langle A'_y \rangle = 0. \] At the same time the orbifold boundary conditions change to
\[ (P^{\text{sym}}_0, P^{\text{sym}}_1, U^{\text{sym}}) = (e^{2i\beta\langle A_y \rangle} P_0, e^{2i\beta + \pi R\langle A_y \rangle} P_1, WU) \] (16)
where we have made use of \( \{ \langle A_y \rangle, P_0 \} = \{ \langle A_y \rangle, P_1 \} = 0. \) The physical symmetry is the symmetry of \( (P^{\text{sym}}_0, P^{\text{sym}}_1, U^{\text{sym}}) \) as the expectation values of \( A'_y \) vanish. In particular, the physical symmetry at low energies is spanned by the generators in
\[ \mathcal{H}^{\text{sym}} = \left\{ \frac{\lambda^a}{2} : [\lambda^a, P^{\text{sym}}_0] = [\lambda^a, P^{\text{sym}}_1] = 0 \right\}. \] (17)
The symmetry \( H^{\text{sym}} \) generated by \( \mathcal{H}^{\text{sym}} \) does not depend on the parameter \( \beta. \)

7 Effective potential

To find \( \langle A_y \rangle \) it is necessary to evaluate the effective potential for Wilson line phases. The effective potential is most elegantly evaluated in the background field gauge. The effective potential for a configuration \( A^M_0 \) is found by writing \( A_M = A^0_M + A^q_M, \) taking \( F[A] = D_M(A^0)A^q_M = \partial_M A^0 + ig[A^0, A^q_M] = 0 \) as a gauge fixing condition, and integrating over the quantum part \( A^q_M. \)

The effective potential in the background field gauge provides a natural link among theories with different sets of orbifold boundary conditions. Suppose that a gauge transformation (4) satisfies the relation
\[ \partial^M (\partial_M \Omega^\dagger \Omega) + ig[A^{0M}, \partial_M \Omega^\dagger \Omega] = 0. \] (18)
Then it is shown that
\[ V_{\text{eff}}[A^0; P_0, P_1, U] = V_{\text{eff}}[A'^0; P'_0, P'_1, U'] \] (19)
where \( (P'_0, P'_1, U') \) is given by (6). As observed in section 5, a Wilson line phase and a boundary-condition-changing gauge transformation have correspondence between them. For such \( A^{(0)} \) and \( \Omega \) the relation (18) is satisfied. The property (19) in turn implies that the minimum of the effective potential corresponds to the same symmetry as that of \( (P^{\text{sym}}_0, P^{\text{sym}}_1, U^{\text{sym}}) \) in the previous section. This establishes the part (v) of the Hosotani mechanism. We shall see it in more detail in the \( SU(5) \) models in sections 8 and 9.

The one-loop effective potential is given, in \( \mathcal{M}, \) by
\[ V_{\text{eff}}[A^0] = \sum \frac{1}{2} \text{Tr} \ln D_M(A^0)D^M(A^0) \] (20)
where the sum extends over all degrees of freedom of fields defined on the bulk $\mathcal{M}$. The sign is negative (positive) for bosons (ghosts and fermions). $D_M(A^0)$ denotes an appropriate covariant derivative with a background field $A^0_M$. $V_{\text{eff}}$ depends on $A^0_M$ and the boundary conditions $(P_0, P_1, U)$.

We are interested in the $A^0$-dependent part of $V_{\text{eff}}$. For a given $A^0$ and $(P_0, P_1, U)$, one can always take a basis of fields such that "Tr ln" in (20) decomposes into singlets and doublets of fields, among which only doublet fields yield $A^0$-dependence. This seems to result from the nature of the $\mathbb{Z}_2$-orbifolding.

A fundamental $\mathbb{Z}_2$-doublet $\phi^i = (\phi_1, \phi_2)$ satisfies the orbifold boundary conditions of the form

\[
\phi(x, -y) = \tau_3 \phi(x, y),
\]

\[
\phi(x, y + 2\pi R) = e^{-2\pi i \alpha_\tau^2} \phi(x, y),
\]

being expanded as

\[
\begin{bmatrix}
\phi_1(x, y) \\
\phi_2(x, y)
\end{bmatrix} = \frac{1}{\sqrt{\pi R}} \sum_{n = -\infty}^{\infty} \phi_n(x) \begin{bmatrix}
\cos (n + \alpha) y / R \\
\sin (n + \alpha) y / R
\end{bmatrix}. \tag{22}
\]

The coupling of $\phi$ to Wilson line phases is cast in the form

\[
\frac{1}{2} |D_y(A_y)\phi|^2 = \frac{1}{2} \left( \partial_y \phi_1 - \frac{\gamma}{R} \phi_2 \right)^2 + \frac{1}{2} \left( \partial_y \phi_2 + \frac{\gamma}{R} \phi_1 \right)^2 \tag{23}
\]

where $\gamma$ is a linear combination of Wilson line phases. Insertion of (22) into (23) yields

\[
\int_0^{\pi R} dy \frac{1}{2} |D_y(A_y)\phi|^2 = \frac{1}{2} \sum_{n = -\infty}^{\infty} \frac{(n + \alpha + \gamma)^2}{R^2} \phi_n(x)^2. \tag{24}
\]

Notice that the number of degrees of freedom is halved due to the $\mathbb{Z}_2$-orbifolding compared with that on $S^1$. Hence the contribution of a bosonic $\mathbb{Z}_2$-doublet $\phi$ to $V_{\text{eff}}$ is

\[
-\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{n = -\infty}^{\infty} \ln \left\{ -p^2 + \left( \frac{n + \alpha + \gamma}{R} \right)^2 \right\}
\]

\[
= -\frac{1}{64\pi^2 R^5} f_5 \left[ 2(\alpha + \gamma) \right] + \text{constant} \tag{25}
\]

where $f_D(x) = \sum_{n=1}^{\infty} n^{-D} \cos(n\pi x)$. 

9
Consider the non-supersymmetric $SU(5)$ gauge theory. We assume that the gauge fields and $N_h$ Higgs field in 5 live in the bulk five-dimensional spacetime $M$. Quarks and leptons are supposed to be confined on the boundary at $y = 0$. There may be additional $N^{5}_f$ and $N^{10}_f$ fermion multiplets in 5 and 10 defined in the bulk $M$.

Let us focus on the following boundary conditions.

(BC0) : $P_0 = \text{diag} \left( -1, -1, -1, 1, 1 \right)$, $P_1 = \text{diag} \left( 1, 1, 1, 1 \right)$
(BC1) : $P_0 = \text{diag} \left( -1, -1, -1, 1, 1 \right)$, $P_1 = \text{diag} \left( -1, -1, 1, 1 \right)$
(BC2) : $P_0 = \text{diag} \left( -1, -1, -1, 1, 1 \right)$, $P_1 = \text{diag} \left( -1, -1, 1, -1 \right)$
(BC3) : $P_0 = \text{diag} \left( -1, -1, -1, 1, 1 \right)$, $P_1 = \text{diag} \left( 1, 1, -1, -1 \right)$
(BC4) : $P_0 = \text{diag} \left( -1, -1, -1, 1, 1 \right)$, $P_1 = \begin{pmatrix}
-\cos \pi p & 0 & 0 & -i \sin \pi p & 0 \\
0 & -\cos \pi q & 0 & 0 & -i \sin \pi q \\
i \sin \pi p & 0 & 0 & \cos \pi p & 0 \\
0 & i \sin \pi q & 0 & 0 & \cos \pi q \\
\end{pmatrix}$.

(BC1), (BC2), (BC3) are special cases of (BC4). Their low energy symmetry of boundary conditions is

$G^{(0)}_{BC} = SU(3) \times SU(2) \times U(1)$
$G^{(1)}_{BC} = SU(3) \times SU(2) \times U(1)$
$G^{(2)}_{BC} = SU(2) \times U(1) \times U(1) \times U(1)$
$G^{(3)}_{BC} = SU(2) \times SU(2) \times U(1) \times U(1)$
$G^{(4)}_{BC} = U(1) \times U(1) \times U(1)$.

The boundary conditions (BC0) and (BC1), at a first look, seem natural to incorporate the standard model symmetry at low energies and to provide a solution to the triplet-doublet splitting problem. Indeed, (BC0) is the orbifold condition originally adopted by Kawamura.

One might ask why one should take (BC0) or (BC1)? Why can’t we adopt (BC2), (BC3), or even (BC4)? We shall demonstrate that, if (BC1) is a legitimate choice for the boundary conditions, then (BC2), (BC3), and (BC4) are as well. It does not matter which one to choose, as all of them lead to the same physics by the Hosotani mechanism.

First we note that the equivalence class of boundary conditions to which
(BC0) belongs consists of only one element, namely (BC0) itself. There is no Wilson line phase in the theory with (BC0), as \( P_1 \) is the identity.

(BC1), (BC2), (BC3) and (BC4) belong to the same equivalence class of boundary conditions. (BC1) and (BC4) are related to each other by a boundary-condition-changing gauge transformation

\[
\Omega(y) = \exp \left\{ -i(y/2R)T_{p,q} \right\}, \quad T_{p,q} = \begin{pmatrix} 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & q \\ p & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \end{pmatrix}.
\]

Hence all of (BC1) to (BC4) should have the same physics, which is confirmed by explicit computations of the effective potential for Wilson line phases.

In the theory with (BC1), Wilson line phases are given by

\[
2g\pi RA_y = \pi \begin{pmatrix} 0 & 0 & 0 & c_1 & c_4 \\ 0 & 0 & 0 & c_2 & c_5 \\ 0 & 0 & 0 & c_3 & c_6 \\ c_1^* & c_2^* & c_3^* & 0 & 0 \\ c_4^* & c_5^* & c_6^* & 0 & 0 \end{pmatrix}.
\]

There are twelve Wilson line phases. In the theory with (BC4), however, there survive only two phases;

\[
2g\pi RA_y = \pi T_{a,b}
\]

where \((a, b)\) are real. In evaluating the effective potential for (29) in the theory (BC1), one can utilize the residual \(SU(3) \times SU(2) \times U(1)\) invariance to reduce (29) to (30). Hence it is sufficient to evaluate the effective potential \(V_{\text{eff}}^{(p,q)}(a, b)\) for the configuration (30) in the theory (BC4) which includes (BC1) as a special case \((p, q) = (0, 0)\).

The evaluation is straightforward. It is reduced to identifying all \(Z_2\) singlets and doublets as described in section 7. The result is

\[
V_{\text{eff}}^{(p,q)}(a, b) = -\frac{3}{64\pi^2 R^5} \left\{ N_A \left[ f_5(a - p) + f_5(b - q) \right] + N_B \left[ f_5(a + b - p - q) + f_5(a - b - p + q) \right] + \frac{3}{2} \left[ f_5(2a - 2p) + f_5(2b - 2b) \right] \right\}
\]

where \(N_A = 3 + N_h - 2N_f^5 - 2N_f^{10}\) and \(N_B = 3 - 2N_f^{10}\). There are a few features to be noted: (i) \(V_{\text{eff}}^{(p,q)}(a, b) = V_{\text{eff}}^{(q,p)}(b, a)\). (ii) \(V_{\text{eff}}\) is periodic in
(a, b) with a period 2. (iii) $V_{\text{eff}}^{(p,q)}(a, b) = V_{\text{eff}}^{(0,0)}(a - p, b - q)$. (iv) The form of $V_{\text{eff}}$ and the location of its minimum are determined by $N_h$, $N_f^5$, and $N_f^{10}$, namely by the matter content.

The fact that $V_{\text{eff}}$ is a function of $a - p$ and $b - q$ is of critical importance. It manifests the relation (19), implying that the physical symmetry determined by the minimum of $V_{\text{eff}}$ is independent of $(p, q)$. The minimum is located at $(a - p, b - q) = (0,0), (1,1)$, or $(0,1) \sim (1,0)$, depending on the matter content.

Some examples are in order. In figure 1 $V_{\text{eff}}^{(0,0)}(a, b)$ is plotted for various $(N_h, N_f^5, N_f^{10})$: (a) $(1,0,0)$, (b) $(1,3,3)$, (c) $(1,1,1)$, and (d) $(1,0,2)$. The minimum is located at (a) $(a, b) = (0,0)$, (b) $(1,1)$, (c) $(0,0)$, and (d) $(0,1)$.

In the case (c) $(0,0)$ and $(1,1)$ are almost degenerate.

![Diagram](image_url)

**Figure 1:** The effective potential for various $(N_h, N_f^5, N_f^{10})$ in the non-supersymmetric models. $V_{\text{eff}}(a, b)/C$ ($C = 3/64\pi^7 R^2$) is plotted for $(p, q) = (0,0)$ in (31).

The physical symmetry at low energies in the case (a) is $H_{\text{sym}} = G^{(1)}_{BC} = SU(3) \times SU(2) \times U(1)$. In the case (b) we recall that $(a, b; p, q) = (1,1; 0,0)$ is equivalent to $(a, b; p, q) = (0,0; 1,1)$. Hence the physical symmetry is $H_{\text{sym}} = G^{(3)}_{BC} = [SU(2)]^2 \times [U(1)]^2$. In the case (d) $H_{\text{sym}} = G^{(2)}_{BC} =$
The resultant physical symmetry is independent of the values of \((p, q)\) in the boundary conditions. It is determined solely by the matter content in the bulk \(\mathcal{M}\). Dynamical rearrangement of gauge symmetry has taken place as a result of quantum dynamics of the Wilson line phases. Symmetry can be spontaneously enhanced or broken, depending on the boundary conditions.

9 Physical symmetry in the supersymmetric \(SU(5)\) model

If the theory has supersymmetry which remains unbroken, then the effective potential for Wilson lines stays flat due to the cancellation among contributions from bosons and fermions. Nontrivial dependence in \(V_{\text{eff}}\) appears if the supersymmetry is softly broken as the nature demands.

There is a natural way to introduce soft SUSY breaking on multiply connected manifolds and orbifolds. First note that \(N = 1\) SUSY in five dimensions induces \(N = 2\) SUSY in four dimensions. A five-dimensional (5-D) gauge multiplet \(V = (A_M, \lambda, \lambda', \sigma)\) is decomposed to a vector superfield \(V = (A_\mu, \lambda)\) and a chiral superfield \(\Sigma = (\sigma + iA_y, \lambda')\) in four dimensions. Similarly, 5-D fundamental Higgs hypermultiplets \(H = (h, h^c)^\dagger, \tilde{h}, \tilde{h}^c)^\dagger\) are decomposed into 4-D chiral superfields \(H = (h, \tilde{h}), H^c = (h^c, \tilde{h}),\) and \(\overline{H}^c = (\tilde{h}^c, \tilde{h})\). After a translation along a noncontractible loop, fields may have different twist, depending on their \(SU(2)_R\) charges. This is called the Scherk-Schwarz SUSY breaking,\(^{13}\)

On the orbifold \(\mathcal{M}\) this twisting is implemented for \(SU(2)_R\) doublets by generalizing (21),\(^{14,15}\) It reads, for the gauge multiplet \(V\), that

\[
\begin{align*}
\left(\begin{array}{c}
V \\
\Sigma
\end{array}\right)_{(x, -y)} &= P_0 \left(\begin{array}{c}
V \\
-\Sigma
\end{array}\right)_{(x, y)} P_0^T, \\
\left(\begin{array}{c}
A_M \\
\sigma
\end{array}\right)_{(x^\mu, y + 2\pi R)} &= U \left(\begin{array}{c}
A^M \\
\sigma
\end{array}\right)_{(x^\mu, y)} U^T, \\
\left(\begin{array}{c}
\lambda \\
\lambda'
\end{array}\right)_{(x, y + 2\pi R)} &= e^{-2\pi i \beta \sigma_2} U \left(\begin{array}{c}
\lambda \\
\lambda'
\end{array}\right) U^T.
\end{align*}
\]

Similarly nontrivial twisting is imposed on \((h, h^c)^\dagger\) and \((\overline{H}, \overline{H}^c)^\dagger\) doublets. The Scherk-Schwarz parameter \(\beta\) changes the spectrum, giving rise to the SUSY breaking scale \(M_{\text{SUSY}} \sim \beta/\pi R\).

The effective potential for Wilson line phases for the theory with \(N_h\)
sets of Higgs hypermultiplets $\mathcal{H} + \overline{\mathcal{H}}$ is

$$V_{\text{eff}}^{(0,0)}(a, b) = -\frac{3}{32\pi^2 R^6} \sum_{n=1}^{\infty} \frac{1 - \cos 2\pi n \beta}{n^5} \left\{ 2(1 - 2N_h)(\cos \pi na + \cos \pi nb) + 4 \cos \pi na \cos \pi nb + \cos 2\pi na + \cos 2\pi nb \right\} .$$

(33)

It vanishes at $\beta = 0$. The Higgs multiplets significantly affect the shape of the effective potential, and consequently the physical symmetry of the theory.

$V_{\text{eff}}$ is plotted in figure 2 with the boundary conditions (BC1) for (a) $N_h = 0$ and (b) $N_h = 1$. If there is no Higgs multiplet, then $V_{\text{eff}}$ is minimized at $(a, b) = (0, 0)$ so that the physical symmetry is $H_{\text{sym}} = G_{BC}^{(1)}$. For $N_h \geq 1$, $V_{\text{eff}}$ is minimized at $(a, b) = (1, 1)$ so that the physical symmetry is $H_{\text{sym}} = G_{BC}^{(3)}$. The presence of Higgs multiplets induces the breaking of color $SU(3)$ down to $SU(2) \times U(1)$.

As stated in part (iv) of the Hosotani mechanism in section 6, all extra-dimensional components of gauge fields in the broken sector of gauge group become massive by quantum effects. The magnitude of their masses is $g_4/R \sim g_4 M_{\text{GUT}}$ in the non-supersymmetric models, while $g_4 \beta/R \sim g_4 M_{\text{SUSY}}$ in the supersymmetric models.

10 Summary

In gauge theory on orbifolds boundary conditions have to be specified. The arbitrariness problem in the choice of the orbifold boundary conditions is
partially solved by the Hosotani mechanism. Various sets of boundary conditions are related by boundary-condition-changing gauge transformations, thus falling in one equivalent class of the boundary conditions. Theories in the same equivalent class, though they in general have different symmetry of boundary conditions at the classical level, have the same physics thanks to the dynamics of Wilson line phases. The physical symmetry is determined by the matter content of the theory.

The concept is schematically depicted in figure 3. In each theory with given boundary condition $BC_a$ there appear degrees of freedom of Wilson line phases. Their dynamics selects a particular configuration of the Wilson line phases which minimizes the effective potential and defines the physical vacuum. The selected configuration always has the same physics content, independent of the boundary condition $BC_a$. All of these have been confirmed in the various $SU(5)$ models.

There are several things to be investigated.
[1] We need to classify all equivalent classes of boundary condition. This poses an interesting mathematical exercise.
[2] We have shown that physics is the same in each equivalent class, but we have not so far explained which equivalent class one should start with. It is most welcome to have a dynamical mechanism to select an equivalent class of boundary conditions.
[3] The models discussed in this paper is not entirely realistic. For instance,
we have not implemented the electroweak symmetry breaking and quark-lepton masses.

[4] The fundamental Higgs field has not been unified with gauge fields. Their coupling to quarks and leptons remain arbitrary.

We shall come back to these points in due course.

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References