Closed String Tachyon Condensation and Worldsheet Inflation

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Abstract

Closed string tachyon condensation in spacetime generates potentials on the worldsheet that model two-dimensional inflationary cosmology. These models illustrate and elucidate a variety of aspects of inflation, in particular the generation of quantum fluctuations and their back-reaction on geometry. We exhibit a class of Liouville gravity models coupled to matter that can exhibit, for example: (a) pure de Sitter gravity; (b) slow-roll inflation; (c) topological inflation; and (d) graceful exit into an FRW phase. The models also provide a quantitative testing ground for ideas about the origin of inflation, such as the various ‘no-boundary/tunnelling’ proposals, and the ‘eternal/chaotic’ inflationary scenario. We suggest an alternative mechanism for quantum creation of cosmological spacetimes which, in the context of the model, provides a natural explanation for why the typical FRW cosmology at large scales underwent a period of inflation at small scale.

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1 Introduction

The development of precision cosmology (c.f. [1, 2, 3] for recent reviews) has provided a stunning confirmation of the basic predictions of inflationary theory (c.f. [4, 5] for reviews). Nevertheless, many questions yet remain: What is the origin of inflation? What is the inflaton? Is the standard linearized analysis of fluctuations correct? Is inflation a generic phenomenon? Why is the current value of the cosmological constant so small? Are there reasonable alternative explanations of the data? There is much still to be understood.

Insight can often be gained by the development of simple models that capture key features of the phenomenon under study. Two-dimensional quantum field theory models have a rich history of mirroring key aspects of their higher-dimensional siblings; for example, dimensional transmutation (the $\mathbb{CP}^n$ model), spontaneous symmetry breaking (the Gross-Neveu model), confinement (the Schwinger and ’t Hooft models), anomalies and topology (the Wess-Zumino-Witten model among others), instanton effects (many examples), and supersymmetry (many more examples), to name but a few.

Two-dimensional models have also been a fruitful laboratory for the investigation of quantum gravity and its interaction with matter, for example in the context of string theory in two or less target space dimensions (c.f. [6] for a review); and in the investigation of quantum black holes (c.f. [7, 8, 9] for reviews). Our interest here lies in the development of two-dimensional models which may shed light on some of the issues surrounding inflationary cosmology mentioned above.

As we review below in section 2, two-dimensional (anti) de Sitter cosmology is the Liouville gravity model. It arises as the worldsheet description of a test string propagating in the presence of a homogeneous closed string tachyon condensate, in string theory above the critical dimension [10, 11, 12]. In this setting, the target space time coordinate has an interpretation as the scale factor of the 2d worldsheet metric (the Liouville field $\phi$), with the spatial coordinates of the target playing the role of (conformal) matter fields.

Interesting models of inflation arise upon coupling the gravity and matter degrees of freedom. The example we focus on primarily is the Liouville-Sine-Gordon theory, where the matter potential is $\cos(kX)$ for some scalar field $X$. When gravitationally dressed, this potential describes an inhomogenous condensate of the closed string tachyon. For sufficiently small $k$, this model satisfies the criteria of slow-roll inflation. For a particular value of $k$, the model turns out to be exactly solvable – it is simply the complexification of Liouville theory. We exhibit a number of elementary solutions, describing universes that inflate, reach a maximum scale size, and then recollapse. We also discuss an amusing example of topological (domain wall) inflation.

A shortcoming of these models is that the target space background is not fully believable as a consistent string background; the tachyon condensate that is responsible for eternal worldsheet de Sitter space cannot persist indefinitely, rather
it must saturate at some point and allow the target space geometry to back-react. This would lead to a transition to some other phase of the worldsheet cosmology; however, we do not have sufficient understanding of the target space dynamics to determine what worldsheet dynamics (if any) emerges. However, condensation of a closed string tachyon localized on a defect does not suffer this problem – there is strong evidence that the tachyon eventually relaxes to a smooth time-dependent solution of the low-energy string equations \cite{13, 14, 15, 16}. Models of this sort provide an intriguing analogue of our universe, exhibiting an early inflationary phase followed by relaxation of the cosmological constant into an FRW phase. We give an example of such a model in section 4.

We then begin, in section 5, a study of quantum effects in these models, calculating the quantum stress tensors of matter as well as the minisuperspace wavefunctions of some of the solutions. We also compare several approaches to 2d de Sitter thermodynamics.

One of our motivations for the construction of these two-dimensional models was to provide a concrete testing ground for various scenarios regarding the origin of inflation, such as the ‘no-boundary’ or ‘tunnelling’ proposals \cite{17, 18, 19, 20, 21, 22, 23}, as well as ‘eternal’ or ‘chaotic’ inflation \cite{24, 19, 25, 26, 27}. We comment in section 6 on what our models might have to say about these proposals, and add a proposal of our own. The implicit equivalence of the worldsheet metric scale factor and the target time coordinate in string theory means that worldsheet inflation is a region of strong scale/time dependence; quantum processes such as string (universal) creation in this time-dependent background generate an ensemble of 2d universes where none were present before, and thus a probability measure on cosmologies at late times/large scales.

We thus anoint ourselves as Gedankengöttter of the two-dimensional cosmos, and proceed to create the universe.

2 Review of Liouville theory

The ordinary bosonic Liouville action is (c.f. \cite{6}, whose conventions we follow)

\[ S_{\text{Liouville}} = \frac{1}{4\pi} \int d\tau d\sigma \sqrt{-g} \left( -\frac{1}{2}(\nabla \varphi)^2 - \frac{1}{2}QR(\hat{g})\varphi - \frac{\mu}{2\gamma^2} e^{\gamma\varphi} \right). \]  \hspace{1cm} (1)

The field \( \varphi \) is to be thought of as the conformal factor for the 2d metric

\[ ds^2 = e^{\gamma\varphi}(-d\tau^2 + d\sigma^2). \]  \hspace{1cm} (2)

In the classical theory, \( Q = 2/\gamma \), and the classical equations of motion describe metrics of constant curvature

\[ R(e^{\gamma\varphi}\hat{g}) = -\frac{\mu}{2}. \]  \hspace{1cm} (3)
Thus one can interpret $\mu$ as (minus) the cosmological constant. The general classical solution for $\varphi$ can be expressed locally as

$$e^{\gamma \varphi} = -\frac{16}{\mu} \frac{\partial A(x^+) \partial B(x^-)}{[A(x^+) - B(x^-)]^2},$$

(4)

where $x^\pm = \tau \pm \sigma$. For $\mu > 0$, one has two-dimensional AdS spacetime; the only spatially homogeneous solution is

$$e^{\gamma \varphi} = \frac{4}{\mu} \frac{\varepsilon^2}{\cosh^2(\varepsilon \tau)},$$

(5)

describing an expansion out of a big bang and into a big crunch singularity. The usual static AdS geometries

$$e^{\gamma \varphi} = \frac{4}{\mu} \frac{\varepsilon^2}{\sinh^2(\varepsilon \sigma)}$$

(6)

$$= \frac{4}{\mu} \frac{1}{\sigma^2}$$

(7)

$$= \frac{4}{\mu} \frac{\varepsilon^2}{\sinh^2(\varepsilon \sigma)}.$$  

(8)

(corresponding to global, Poincaré, and Rindler slices), with a timelike boundary at spatial infinity, are not compatible with the compact spatial topology we want to impose to make contact with string theory; thus we find only a patch of global AdS for spacetimes of cylindrical topology.

For $\mu < 0$, there are three spatially homogeneous solutions:

$$e^{\gamma \varphi} = \frac{4}{\mu} \frac{\varepsilon^2}{-\mu \sinh^2(\varepsilon \tau)}$$

(9)

$$= \frac{4}{\mu} \frac{1}{\tau^2}$$

(10)

$$= \frac{4}{\mu} \frac{\varepsilon^2}{-\mu \sin^2(\varepsilon \tau)}.$$  

(11)

Evaluating the stress-energy tensor

$$T_{++} = \frac{1}{2} (\partial_+ \varphi)^2 - \frac{1}{2} Q \partial_+^2 \varphi$$

$$T_{--} = \frac{1}{2} (\partial_- \varphi)^2 - \frac{1}{2} Q \partial_-^2 \varphi$$

(12)

of these solutions, one finds the energy $E_L - \frac{1}{8} Q^2 = \frac{\varepsilon^2}{2\gamma^2} > 0$ for the first, ‘hyperbolic’ solution (9); $E_L - \frac{1}{8} Q^2 = 0$ for the second, ‘parabolic’ solution (10); and $E_L - \frac{1}{8} Q^2 = -\frac{\varepsilon^2}{2\gamma^2} < 0$ for the third, ‘elliptic’ solution (11), see figure 1.

3Rather, they are compatible with a timelike identification, a fact we will make use of below.
Figure 1: The three classes of solution of de Sitter Liouville theory, corresponding to positive, zero, and negative energy (relative to the Casimir energy on the cylinder $-\frac{1}{8}Q^2$). Large positive $\phi$ corresponds to large scale factor. The positive energy solution describes gravity coupled to matter energy density; it has an FRW-like 'big bang' in the past and dS asymptotics in the far future. The zero energy solution describes de Sitter geometry in flat coordinates, while the negative energy solution describes the 'bounce' geometry of global de Sitter space.

To interpret these solutions as two-dimensional cosmologies, it is useful to transform the conformal time $\tau$ to proper time $t$ via $dt = \sqrt{-\mu} e^{\gamma \phi/2} d\tau$; in these coordinates the geometries become

$$ds^2 = \frac{4}{-\mu} (-dt^2 + \varepsilon^2 \sinh^2 t d\sigma^2)$$

$$= \frac{4}{-\mu} (-dt^2 + e^{2t} d\sigma^2)$$

$$= \frac{4}{-\mu} (-dt^2 + \epsilon^2 \cosh^2 t d\sigma^2).$$

The first cosmology describes a universe which has a past Milne-type singularity at $t = 0$ ($\tau = -\infty$) and expands into an asymptotically de Sitter future as $t \to \infty$ ($\tau \to 0$); the second cosmology describes eternal de Sitter space in 'flat' coordinates, while the third describes eternal de Sitter space in 'global' coordinates. Note that, because we are identifying $\sigma \sim \sigma + 2\pi$, the latter two metrics are not related by analytic continuation. The solutions to the equation of motion (3) are always locally constant curvature geometries, and so must cover a portion of the global conformal diagram of (anti)de Sitter space; the relevant domains for the three metrics (9), (10), (11) are shown in figure 2.

4Of course, if $\sigma$ were a noncompact coordinate, the surface $t = -\infty$ in the flat coordinates is just a past horizon; continuing past it reveals a second flat coordinate patch needed to cover the entire global de Sitter space.
2.1 The spacetime interpretation

Quantum consistency of two-dimensional gravity requires cancellation of the conformal anomaly
\[ c_{\text{tot}} = 1 + 3Q^2 + c_{\text{matter}} - 26 = 0 \]  
(14)
as well as the quantum scale invariance of the exponential interaction, which amounts to the relation
\[ Q = \frac{2}{\gamma} + \gamma, \]  
(15)
a slight modification of the classical value \( Q = 2/\gamma \). The semiclassical limit of the Liouville theory is the large \( c_{\text{matter}} \) limit; rescaling \( \varphi \rightarrow \frac{1}{\gamma} \varphi \) in (1), one sees that \( \gamma \) is the coupling constant of the theory. There are actually two such limits. First, one can take \( c_{\text{matter}} \rightarrow -\infty \), in which case the Liouville field is a standard positive metric scalar field. Alternatively, one can take \( c_{\text{matter}} \rightarrow \infty \), but then (14) requires that \( Q \) is pure imaginary; to maintain reality of the action, one Wick rotates \( \varphi = i\phi \) so that \( \phi \) is a timelike (negative metric) scalar field. We also remove the factors of \( i \) in the definition of \( Q \) and \( \gamma \), so that for instance \( Q = \sqrt{(d - 25)/3} \). Note also that the Liouville energy (12) changes sign.

We will be interested primarily in matter systems with a string theoretic interpretation as target space geometries, for which \( c_{\text{matter}} \) is the spatial dimension \( d \). These theories were considered as cosmological models quite some time ago [10, 11, 12]. We thus take the matter system to be \( d \) free fields \( Y_i, i = 1, ..., d \); then the spacetime interpretation of the combined Liouville plus matter system is that of a test string in the background of a spatially homogeneous closed string tachyon condensate in the bosonic string, in the supercritical spacetime dimension \( d + 1 > 26 \). A central theme of our investigation surrounds the dual interpretations (introduced in the works just mentioned) of timelike Liouville theory as describing on the one hand propagation in target spacetime of a test string in the presence of a closed string.
tachyon condensate, and on the other hand as describing a de Sitter phase of the worldsheet geometry.\(^5\)

In the supercritical dimension \(d > 25\), the dilaton

\[
g_{\text{str}} = e^D = e^{-Q\phi/2}
\]

is such that strings are strongly coupled as \(\phi \to -\infty\), which corresponds to small scale factor in the 2d cosmology [10, 11, 12]. Thus we do not really have consistent control over the ‘big bang’ region of the two-dimensional theory—topological fluctuations of the worldsheet are unsuppressed. Passing to the Einstein frame in spacetime

\[
(ds_{(ST)}^E)^2 = e^{-4D/d-1}(ds_{\text{str}}^{(ST)})^2,
\]

one sees that the source of the problem is that the spacetime geometry is itself a big bang cosmology [32, 33]

\[
(ds_{(ST)}^E)^2 = -dT^2 + (\frac{Q}{d-1})^2 T^2 dY_i^2
\]

(where \(T = \frac{d-1}{\sqrt{2Q}} e^{\frac{Q\phi}{d-1}}\)), so that early time \(\phi \to -\infty\) corresponds to a big bang in the target space. Perhaps this problem could be avoided by having the target space cosmology emerge from an inflationary phase of the sort described in [34], but this is a higher order speculation.

Even so, it is not clear to us what pathology in the two-dimensional physics ensues if by hand we restrict the worldsheet topology to be a cylinder. Alternatively, we could work in the critical dimension \(d = 25\) where the dilaton can be kept constant and small at early times (although the 2d gravity theory is strongly coupled in this case). Therefore let us forge ahead with the worldsheet dynamics. The BRST constraints demand a vanishing expectation value of the total worldsheet stress tensor. At the semi-classical level, the zero mode of this condition is

\[
E_L + E_{\text{matter}} - 1 = -\frac{\epsilon^2}{2\gamma^2} + E_{\text{matter}} - \frac{1}{8} Q^2 - 1 = 0.
\]

Note in particular that for standard positive energy matter, there is only a finite range of matter energies for which the global de Sitter solution \(-\epsilon^2 = \epsilon^2 > 0\) is allowed, although that range is large for large \(d\) (where \(Q^2 \sim d/3\)). Since both ends \(\tau \to \pm \infty\) of the cosmology (11) correspond to \(\phi = +\infty\), these solutions represent classical production on-shell of pairs of strings (which are thus themselves tachyonic) in the time-dependent tachyon field.\(^6\) The string theory interpretation gives a completely different perspective on the global de Sitter solution— one does not

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\(^5\)This idea has resurfaced more recently in the context of brane worlds, via ‘mirage’ cosmologies: [28, 29, 30, 31], to name a few.

\(^6\)Quantum mechanically, there is an nonzero amplitude to produce any string state, however for high mass strings this proceeds by a tunnelling process and is exponentially suppressed. See for instance [35] and section 5 below.
view it as a contracting phase followed by a bounce into an expanding phase, rather there is a pair of expanding universes produced at a time $\phi \sim -\frac{1}{\gamma} \log\left[\frac{4\epsilon}{1-\mu}\right]$.

Similarly, the solution (10) is the dressing of a string state that is effectively ‘massless’ in the tachyon background, while the solution (9) dresses a ‘massive’ string mode. At large $d$, there are actually many ‘tachyons’ and ‘massless modes’ and hence there are many options for the state of the spatial string coordinates $Y_i$.

It is interesting to note that the two possible signs of the $2d$ cosmological constant correspond to the two inequivalent directions in which one can push the bosonic string tachyon. The effective potential of the bosonic string has the structure

$$V_{\text{eff}} = - c_2 T^2 + c_3 T^3 + \ldots$$

(with $c_2$ and $c_3$ positive); negative $2d$ cosmological constant $\mu > 0$ corresponds to a tachyon field $T$ exponentially growing in time in the direction of what might be a metastable minimum at positive $T$, while positive $2d$ cosmological constant $\mu < 0$ has the tachyon growing in what naively looks like an unbounded direction. Of course, the higher order corrections to $V_{\text{eff}}$ are order one, so one cannot say reliably that there is a local minimum, or an unbounded direction.

In the worldsheet theory, the positive tachyon condensate acts as a barrier that prevents test strings from exploring the region of large positive $T$ – the classical solution (5) suggests that strings try to annihilate one another, since $\tau = \pm \infty$ both correspond to early time $\phi \rightarrow -\infty$ in target space. On the other hand, a negative tachyon condensate wants to push test strings toward the asymptotic future in finite worldsheet conformal time. Note that this type of tachyon condensate is absent in the fermionic string (i.e. the type 0 theory), where the two signs of $\mu$ are actually equivalent – changing the sign of $\mu$ is undone by a chiral R-parity transformation on the worldsheet. The physics there is always that of AdS gravity since only this sign of cosmological term is compatible with worldsheet supersymmetry.

In the analogous open string problem, one has similar issues, and the full theory does have a local minimum of the tachyon effective potential – the closed string vacuum. In this regard, note also that the two signs of open string tachyon condensate also have very different physics [36, 37], with an interpretation similar to the closed string dynamics above. Some aspects of the worldsheet physics of the $\mu < 0$ boundary cosmological term in the bosonic string were explored in [38] (for a particular inhomogeneous condensate, see [37] and section 3 below). One may be able to ascribe some of the pathologies of their results to the fact that the open string condensate is heading in the bottomless direction of the spacetime effective potential. It would be interesting to make a similar analysis of the $\mu > 0$ theory, especially since this is the case relevant to the superstring. With this sign of boundary cosmological term, the boundary potential ‘repels’ the boundary from the future $\phi \rightarrow \infty$, which is a way of seeing that open strings are absent from the endpoint of the condensation process. Of course, absent a bulk tachyon the dynamics in the interior of the worldsheet is free to propagate to the future, so the most likely outcome of the dynamics is that the two ends of the string ‘annihilate’ one another and
make a closed string, see figure 3.

Figure 3: *The boundary of a test open string propagating in an open string tachyon background is ‘repelled from the future’. The ends of the string find each other and annihilate, leaving a free closed string whose propagation to the future is unhindered.*

In the case of open string tachyon condensation (say in the superstring, where we don’t have to worry about issues of bulk closed string tachyons), there is by now a fairly well-developed picture of the physics of the condensate at late times from a variety of points of view [39, 40, 41, 42, 36]. In a sense, open string boundaries are confined by the boundary potential, as we have seen above, and at weak string coupling one expects to find the closed string vacuum plus radiation. If the initial unstable brane was space-filling, one has a finite energy density of closed strings, leading to an FRW cosmology; if the initial unstable brane is localized, its decay emits a spherical pulse of closed string radiation that travels outward to spatial infinity.

For closed string tachyons, the picture is less clear. The endpoint of bulk closed string tachyon condensation is unknown, although there have been speculations [43, 16] that condensation of the type 0 tachyon in the critical dimension $d = 9$ is related to type II string theory; if so, one would find a cosmological solution of type II, not the type II vacuum, as the condensate and the strings that are produced by it source the Einstein equations. In any event, there is no known limit of such bulk tachyon condensation where there is a controlled understanding of the late-time behavior.\(^7\) Certainly the initial exponential growth of the tachyon field does not carry on forever. Thus the Liouville action above is not valid for all $\phi$, rather it breaks down whenever strings explore the region of large $\phi$.

\(^7\)Although the parallels with the open string case, and the fact that perturbative strings are repelled from the region of forming condensate, suggest a phase where closed fundamental strings are confined.

In the fermionic string, the worldsheet cosmological constant is always negative,
thus giving rise to AdS cosmologies; as in the open string case, the worldsheet boundary is repelled from the region of large positive $\phi$. There might be a self-consistent treatment of some aspects of string propagation in the developing condensate, because strings do not explore the region where we do not trust the condensate. However, the homogeneous tachyon condensate describes AdS geometries and thus does not give rise to inflation. What we need is a nontrivial matter potential, suitably chosen to give slow-roll inflation.

3 Inhomogeneous tachyon condensates

Tachyon condensates that are inhomogeneous in target space describe gravitationally dressed matter potentials; suitable choices of matter potential give rise to interesting two dimensional models of inflation. Consider, for instance, the gravitational dressing of a cosine matter potential – the Liouville-Sine-Gordon model

$$S_{\text{LSG}} = \frac{1}{4\pi} \int d\tau d\sigma \sqrt{-\hat{g}} \left( \frac{1}{2} (\hat{\nabla} \phi)^2 + \frac{1}{2} Q R(\hat{g}) \phi - \frac{1}{2} (\hat{\nabla} X)^2 + \frac{\mu}{2\alpha^2} e^{\alpha \phi} \cos kX \right),$$  \hspace{1cm} (21)

where we have suppressed a further $d_\perp$ scalar fields $Y$ in which the tachyon condensate is homogeneous. The conditions of conformal invariance are now

$$Q = \sqrt{(d_\perp - 24)/3}$$

$$\alpha = -\frac{1}{2} Q + \sqrt{\frac{1}{4} Q^2 + 2 - k^2}.$$  \hspace{1cm} (22)

For sufficiently small $k$, the matter potential is very flat and the conditions of slow-roll inflation are well satisfied. Note also that $\alpha \sim \gamma \sim \sqrt{12/d_\perp}$ in the semiclassical limit of large $d_\perp$.

To compare the dynamics of this model with that of inflation, it is useful to write the equations of motion in synchronous gauge. Let us define the ‘proper time’

$$dt = e^{\alpha \phi/2} d\tau \equiv a \, d\tau,$$  \hspace{1cm} (23)

the local ‘Hubble parameter’\(^8\)

$$H = \dot{a}/a = \frac{1}{2} \alpha \dot{\phi}$$  \hspace{1cm} (24)

(where dot denotes $t$ derivative), and the matter potential

$$V(X) = \frac{-\mu}{2\alpha^2} \cos(kX).$$  \hspace{1cm} (25)

\(^8\)Note that this is not quite the expansion of the scale factor $a = \exp[\gamma \phi/2]$; rather, one has $a = a^\gamma/\alpha$. The use of $a$ simplifies the equations of motion.
Then the equations of motion and the Hamiltonian constraint become (for spatially homogeneous fields, and setting $\dot{R} = 0$)

$$\begin{align*}
H^2 &= \frac{\alpha^2}{2n} \left( \frac{1}{2} \dot{X}^2 + V(X) - a^{-2}(\frac{1}{8}Q^2 + 1) + \rho_\perp \right) \\
\dot{H} &= -\frac{\alpha^2}{2} \left( \frac{1}{2} \dot{X}^2 - a^{-2}(\frac{1}{8}Q^2 + 1) + \rho_\perp + P_\perp \right) \\
0 &= \ddot{X} + nH \dot{X} + V'(X)
\end{align*}$$

(26)

with $n = 1$ the number of spatial dimensions. For $n$ spatial dimensions, these are precisely the Friedmann equations of a spatially homogeneous cosmology if we substitute $\frac{1}{2} \alpha^2 \rightarrow \frac{8}{\pi} G_{n+1}$, where $G_{n+1}$ is Newton’s constant.\(^9\)

The conditions for slow-roll inflation are then that the dimensionless parameters

$$\begin{align*}
\epsilon &\equiv \frac{1}{2} \frac{2}{\alpha^2} \left( \frac{V'}{V} \right)^2 \\
\eta &\equiv \frac{2}{\alpha^2} \left( \frac{V''}{V} \right)
\end{align*}$$

(27)

have magnitude much smaller than one. For the cosine matter potential, this amounts to

$$\begin{align*}
\epsilon &= (k/\alpha)^2 \tan^2 kX \ll 1 \\
-\eta &= 2(k/\alpha)^2 \ll 1 ;
\end{align*}$$

(28)

both are well satisfied for $k \ll \alpha$ and an initial matter distribution that starts near the top of the matter potential.\(^10\)

For the special value $k = \alpha = \frac{1}{4}(-Q + \sqrt{Q^2 + 16})$, the classical theory can be solved exactly.\(^11\) Unfortunately, the second of the slow roll conditions (28) is not satisfied for this value of $k$; nevertheless, our hope is that one may still extract useful lessons from the analysis of this case. While the condition $\epsilon \ll 1$ is the requirement that the cosmological expansion is dominated by the inflaton potential rather than its kinetic energy, the second condition $\eta \ll 1$ is to some extent a convenience; it prescribes the circumstance in which the inflaton motion is friction dominated, so that $\ddot{X} \ll H\dot{X}$.\(^12\) Precisely this situation, namely $\epsilon \ll 1$ but $|\eta| \sim 1$, was studied in [44, 45], where it was termed ‘fast-roll’ inflation. Estimates there indicated that one

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\(^9\)Note that the zero-point energy $-\frac{1}{8}Q^2 - 1$ plays the role of spatial curvature. The sign is appropriate to a closed spatial universe in higher dimensions.

\(^10\)The field $X$ can be localized on scales larger than the string scale (which we are setting equal to one). For $k \ll \alpha$ this will always be the case, especially in the weak coupling regime of gravity $\alpha \sim \sqrt{3/d_\perp}$.

\(^11\)While only this value allows exact solution to the classical equations of motion, we expect the behavior for other (sufficiently small) values of $k$ to be similar.

\(^12\)More precisely, $\eta$ not small does not preclude the inflationary phase; however, it does affect observables such as the spectral tilt. This level of detail is beyond the scope of the present investigation.
can achieve a large expansion of the scale factor without undue fine tuning; indeed, explicit exact solutions below will support that analysis.

Writing $\Phi = \phi + iX$, the action separates for $k = \alpha$ into a Liouville theory for $\Phi$ plus that of its complex conjugate:

$$S = \frac{1}{4\pi} \int d\tau d\sigma \sqrt{g} \left( \frac{1}{4} (\nabla \Phi)^2 + \frac{1}{4} Q R (\dot{g}) \Phi + \frac{\mu}{4\alpha^2} e^{\alpha \Phi} + \text{c.c.} \right). \quad (29)$$

The classical equations of motion thus have the general solution (4), where now we allow the functions $A(x^+)$ and $B(x^-)$ to be generically complex.

For example, the zero energy solution

$$e^{\alpha \phi} = \frac{4}{-\mu} \frac{1}{(\tau + ib)^2} \quad (30)$$

describes a scalar field $X$ that starts off at the top of its cosine potential at $\tau = -\infty$, rolls down to the potential minimum at $\tau = 0$, and then climbs back up to the top of the potential as $\tau \to \infty$. As this occurs, the geometry is nearly an expanding de Sitter space for large negative $\tau$; it reaches a maximum scale factor $e^{\alpha \phi} = \frac{4}{\mu} b^{-2}$ at $\tau = 0$, at which point it enters a contracting phase and returns to zero scale factor and asymptotically de Sitter geometry as $\tau \to \pm \infty$.\(^{13}\) Solving for the scale factor and matter field, we have

$$e^{\alpha \phi / 2} = a(\tau) = \frac{1}{\sqrt{\tau^2 + b^2}},$$

$$\tan(\alpha X / 2) = -b / \tau. \quad (31)$$

If the lower bound on localizing the matter field $X$ is the string scale $\Delta X \sim 1$, then the total amount of inflation is

$$\frac{a_{\text{final}}}{a_{\text{initial}}} \sim \frac{2}{\alpha b^2}. \quad (32)$$

The choice of $b$ amounts to the choice of the scale factor $a$ at which one chooses to tune $X$ to lie within a string length of the top of its potential. By tuning $b$ to be small we can make this ratio of scales as large as we like.

The analogues of the other two solutions exhibited in equation (13) have similar interpretations. The hyperbolic solution is

$$e^{\alpha \phi} = \frac{4}{-\mu} \frac{(\varepsilon_\phi + i \varepsilon_X)^2}{\sinh^2[(\varepsilon_\phi + i \varepsilon_X)\tau + ib]}$$

$$e^{\alpha \phi} = \frac{8}{-\mu} \frac{(\varepsilon_\phi^2 + \varepsilon_X^2)}{[\cosh[2\varepsilon_\phi \tau] - \cos[2\varepsilon_X \tau + b]]} ; \quad (33)$$

\(^{13}\)Asymptotically de Sitter in the sense of constant curvature; one is not approaching conformal infinity of the global de Sitter geometry.
again the geometry expands to a maximum scale, which in the limit of small $\varepsilon_\phi, \varepsilon_X,$ and $b$ reduces to

$$e^{\alpha \phi_{\text{max}}} \sim \frac{4}{-\mu} \frac{(\varepsilon_\phi^2 + \varepsilon_X^2)^2}{b^2 \varepsilon_\phi^2}.$$  \hfill (34)

Here both $\tau \to +\infty$ and $\tau \to -\infty$ give 2d Milne singularities. The maximum scale factor achieved is finite unless we fine tune $b \to 0$, so that the field reaches the top of the cosine potential just as the scale factor starts inflating. The BRST constraint

$$-\frac{\varepsilon_\phi^2}{2\alpha^2} + \frac{\varepsilon_X^2}{2\alpha^2} + E_\perp - \frac{Q^2}{8} - 1 = 0$$  \hfill (35)

relates the two zero mode energies.

Clearly there are a wide variety of tachyon profiles in target space that one could investigate, yielding a corresponding variety of inflaton potentials; one could imagine modelling any of the popular inflationary scenarios (c.f. [4] for a review), limited only by one’s ability to solve the dynamics.

### 3.1 The fermionic string

Suitable actions for the fermionic string are much the same as (1), (21); one simply substitutes the superspace integral $\int d^2x d^2\theta$ and derivative $\hat{D}$ for their bosonic counterparts, and promotes $\phi$ and $X$ to worldsheet superfields. Conformal invariance requires

$$Q = \sqrt{(d_{\perp} - 8)/2}$$

$$\alpha = -\frac{i}{2} Q + \sqrt{\frac{1}{4} Q^2 + 1 - k^2}.$$  \hfill (36)

Elimination of auxiliary fields leads to a bosonic potential

$$\mathcal{V} = G^{\phi\phi}(\partial_\phi \mathcal{W})^2 + G^{XX}(\partial_X \mathcal{W})^2$$

$$= \frac{\mu^2}{4\alpha^4} e^{2\alpha \phi} (-\alpha^2 \cos^2 kX + k^2 \sin^2 kX)$$  \hfill (37)

where $G$ is the target space metric. The minimum in $X$ yields an effective AdS cosmological constant $-(\mu/4\alpha)^2$ (as mentioned above in section 2.1, this is required by supersymmetry), while the maximum corresponds to a dS cosmological constant $+(\mu k/4\alpha^2)^2$. Again the choice $k = \alpha = \frac{1}{4}(-Q + \sqrt{Q^2 + 8})$ leads to a classically integrable theory – the complexified super-Liouville model. Its classical solutions differ little from those discussed above.

### 3.2 Winding modes and domain wall inflation

Another scenario for inflation uses topological defects as seeds for ‘eternal’ inflation [46, 47]. Since a scalar field at the core of such a defect is pinned at the maximum
of its potential, if the characteristic size of the defect exceeds the Hubble scale, the interior of the defect will inflate. The Sine-Gordon-Liouville theory provides a simple model of this sort as well – we simply compactify the scalar $X$ on a circle of radius $2/k$ (in string units) and consider winding strings. A prototypical solution of this sort in the $\alpha = k$ model has

$$A(x^+) = \exp[(\varepsilon - i w) x^+] , \quad B(x^-) = \exp[(-\varepsilon - i w) x^-] ,$$

leading to

$$\exp[\alpha \Phi] = \frac{4}{-\mu \sinh^2(\varepsilon \tau - i w \sigma)} \frac{\varepsilon^2 + w^2}{\varepsilon^2 + w^2}.$$ 

At large $\tau$, one sees that the imaginary part $X = \text{Im} \Phi$ winds $w$ times. For the component Liouville and matter fields $\phi = \text{Re} \Phi$ and $X$, one finds

$$e^{\alpha \phi} = \frac{8}{-\mu \cosh(2\varepsilon \tau) - \cos(2w \sigma)} \frac{\varepsilon^2 + w^2}{\sinh(2\varepsilon \tau) \sin(2w \sigma)} \sinh(2\varepsilon \tau) \sin(2w \sigma)$$

$$\sin(\alpha X) = \frac{\cosh(2\varepsilon \tau) - \cos(2w \sigma)}{\cosh(2\varepsilon \tau) + \cos(2w \sigma)}$$

Figure 4: Plots of the scale factor $\phi$ and matter field $X$ during topological inflation. At early and late worldsheet time $\tau$, the matter field $X$ has one unit of winding. The spacetime picture, however, consists of a pair of oppositely wound strings annihilating and leaving behind an unwound string perched at a maximum of the cosine matter potential.

From the plots of $\phi$ and $X$ in figure 4, one sees that this solution describes a pair of oppositely wound strings which annihilate, leaving an unwound string perched
at the top of the cosine potential. Of course, in the quantum theory one does not expect this late-time (large \( \phi \)) behavior to be stable; rather, one expects that the field \( X \) falls off the hill in either direction. It is interesting to note that, even though we naively start off with a coordinate domain \( \mathbb{R} \times S^1 \) that we wish to interpret as a single closed universe cosmology, the actual dynamics grows a second ‘child universe’ that absorbs the domain wall core and allows it to recollapse after fluctuations drive \( X \) off its potential maximum.\(^{14}\) It would be interesting to explore whether such a mechanism disallows eternal topological inflation in higher dimensions; even so, such topological structures might be a natural way of providing the initial conditions for inflation, since the core of the defect is placed on the top of the inflaton potential at the point in time when the topological charge is shed.

### 4 Localized tachyon condensates

While the above 2\( d \) models of inflation are simple and appealing, there are several concerns that need to be addressed arising from the target space interpretation in terms of closed string tachyon condensation. The unlimited exponential growth in time of the tachyon condensate is unrealistic; at some point, the tachyon condensate grows large enough that one cannot ignore its back-reaction on the target space geometry. Ultimately, one will need to know the late-time state that the condensate evolves toward and whether it even has a perturbative string interpretation. The fate of bulk closed string tachyon condensates in both bosonic and fermionic string theory is an open question. In addition, unless one works in the critical dimension, one should worry about the consistency of propagation of a single string (our 2\( d \) cosmos) in a background where the string coupling increases linearly in target space time as we go to the past; the ‘in’ region is not under control.

An alternative to this admittedly murky situation is provided by localized closed string tachyon condensates, in the critical dimension. The above difficulties are associated to the instability of the target spacetime, which occur everywhere in space with little understanding of the outcome. However, the test string which is our two-dimensional cosmology can be localized in the target space in a region of localized instability; away from this region, string theory is stable. In a sense, one can view the two-dimensional cosmology as a kind of scattering of a test string off of the decaying localized defect.\(^{15}\) String theory in the supercritical dimension always has bulk tachyons, even with a chiral GSO projection, \(c.f.\) [49]. Therefore, we must work in the critical dimension \( d + 1 = 10 \) of the type II string, in a background with a localized closed string tachyon. Such backgrounds have been explored in various contexts in [13, 14, 15, 16], as well as many further works. In these examples, the target space contains an unstable defect, and the closed string tachyon is a

\(^{14}\)Modulo issues of ‘eternal’ inflation, which we defer to section 6.1.

\(^{15}\)L. Susskind [48] has expressed a similar point of view on de Sitter cosmology in a related context.
description of its initial stages of decay. Of course, the critical dimension is the limit \( Q \to 0, \gamma \to 1 \), which is a ‘strong coupling limit’ from the point of view of the worldsheet gravity theory. What this means in practice is that fluctuations of the timelike Liouville field \( \phi \) are of the same order as those of the matter fields, of order one at the string scale. We will also need to take care that slow-roll conditions can be satisfied.

As argued first in [13], localized closed string tachyon condensation leads to a stable remnant, or flat spacetime, with an outgoing pulse of radiation. As viewed by a test string that remains near the origin, this process would appear as an exponential growth of the timelike Liouville field coupled to a localized matter perturbation, which then saturates and relaxes to a smooth final configuration. In other words, the 2d cosmological constant dynamically relaxes to zero at large scale factor \( \phi \). We thus find an explicit and self-consistent realization of a ‘wormhole’ style mechanism [50] for relaxing the cosmological constant.

Consider, for example, the throat models introduced in section 5 of [15]. The CFT background is

\[
\mathbb{R}^{5,1} \times \left( \frac{SL(2)}{U(1)} \times \frac{SU(2)}{U(1)} \right)/\mathbb{Z}_k .
\]  

(41)

The chiral GSO projection demands \( n \in 2\mathbb{Z} + 1 \). The \( SU(2)/U(1) \) factor in the background (41) is quantum equivalent to an \( \mathcal{N} = 2 \) supersymmetric Landau-Ginzburg model for a chiral superfield \( X \) [51, 52], with a superpotential

\[
\mathcal{W}_X = X^{kn}.
\]  

(42)

The cigar CFT \( SL(2)/U(1) \) is quantum equivalent [53] to a \textit{spacelike} \( \mathcal{N} = 2 \) supersymmetric Liouville theory, not to be confused with the \textit{timelike} \( \mathcal{N} = 1 \) Liouville theory we will use to describe the time evolution of the tachyon condensate. We describe this spacelike Liouville theory using a chiral \( \mathcal{N} = 2 \) superfield \( Y \) with the superpotential

\[
\mathcal{W}_Y = \tilde{\mu} e^{-Y/\tilde{Q}}.
\]  

(43)

where \( \tilde{Q}^2 = 2/\text{kn} \) is the linear dilaton slope in the spatial Liouville direction \( \varphi = \text{Re} \ Y \), and \( \tilde{\mu} \) determines the string coupling at the tip of the cigar.

This theory has a moduli space, related to the moduli space of the throat of \( k \) fivebranes that remains after tachyon condensation [15]. This moduli space is explored by deforming the \( \mathcal{N} = 2 \) superpotential (42) plus (43) to

\[
\mathcal{W} = X^{kn} + \tilde{\mu} e^{-Y/\tilde{Q}} + \sum_{j=2}^{k-1} \lambda_j X^{n(j-k)} e^{-\tilde{Q}jnY}.
\]  

(44)

\footnote{The idea that the 2d cosmological constant is relaxed by ‘wormholes’ or ‘baby universes’ was explored already in the work of [11, 12]. At that time, no definite conclusions could be drawn due to the state of our understanding of bulk closed string tachyon condensation. Our addition to the discussion is to consider localized tachyons in an otherwise stable closed string background.}
Note that all terms in (44) are invariant under the $\mathbb{Z}_k$ symmetry in (41), which acts as

$$X \to e^{2\pi i k} X \ ; \ Y \to Y - 2\pi i \tilde{Q} n \ ; \ \theta \to e^{2\pi i n} \theta \ ; \ \bar{\theta} \to \bar{\theta}$$

The parameters $\lambda_j$ in (44) determine the locations of the underlying fivebranes; for $\lambda_j = 0$, they are located at the origin in $X$ and arranged in a $\mathbb{Z}_k$ symmetric fashion on the $\text{Im} Y$ circle.

The modes corresponding to these deformations have normalizable wavefunctions concentrated down the throat (at large negative $\text{Re} Y$). When the $\lambda_j$ are turned on, they are localized away from the origin in $X$: The $\mathcal{N} = 2$ superpotential generically has $k$ distinct $n$-fold degenerate extrema where $\mathcal{N} = 2$ supersymmetric ground states are localized. There are also tachyonic deformations surviving the $\mathbb{Z}_k$ projection (45). These are perturbations $\delta W = \mu_\ell X^{k\ell}$; again only $\ell \in 2\mathbb{Z} + 1$ survives the GSO projection. In order to satisfy the BRST constraints, we dress these perturbations with time dependence to put them on-shell. The time dependence breaks the global worldsheet $\mathcal{N} = 2$ supersymmetry; we can only write the tachyon background in terms of an $\mathcal{N} = 1$ worldsheet superpotential

$$\delta (W + \bar{W}) = \sum_{\ell=1}^{n-1} (\mu_\ell X^{k\ell} + \text{c.c.}) e^{-\alpha_\ell \phi}.$$  

A nontrivial potential is generated for the test string due to the competition between the moduli-induced potential (44), which wants to localize the string near one of the fivebrane sources, and the tachyon-induced potential (46), which wants to localize the string elsewhere (e.g. the origin if we only turn on $\mu_1$). By suitable tuning of $\tilde{\mu}$, the $\mu_\ell$, and the $\lambda_j$, one can arrange that the test string is initially trapped in a metastable minimum of the potential at large $|X|$ for an extended period of time, generating inflation. Eventually, the tachyon potential (46) grows at large $\phi$ and dominates the dynamics – the scalar $X$ rolls down from this large value to a minimum at $X = 0$. The conditions of slow roll will be satisfied for sufficiently large initial values of $X$ (large relative to the string scale), for which the ‘inflaton’ rolls down from large values and starts off with a wavefunction that is localized on the string scale. After this period of inflation, at late time the test string settles down with some value of the worldsheet energy $E_\phi$ corresponding to an expanding FRW-like cosmology at late times, since the growth of the tachyon condensate saturates and blows out to spatial infinity (far from $X \sim 0$) leaving a vanishing potential for $\phi$, when $\phi$ is large and $X$ is near zero.\(^{17}\)

\(^{17}\)It is possible that the condensation process excites a motion of the fivebranes on their moduli space that causes them to collide; coincident fivebranes generate large string coupling and the perturbative approximation breaks down. It will be interesting to see if this happens; the analysis is left to future work. Regardless of the result, we can arrange for the fivebranes to be initially well-separated, so that the collision occurs a long time after the exponential growth of the tachyon condensate, and therefore the period of worldsheet inflation, shuts off.
We believe that one can also form interesting inflationary models using the non-supersymmetric orbifold constructions of [13, 14, 15], although we have not investigated them in much detail. The key feature allowing for an inflationary epoch in both these models and the fivebrane throat models discussed above, is the ability to describe string scale geometry through the use of an effective scalar potential rather than a nonlinear sigma model. Prime examples of this are the equivalence of $SL(2)/U(1)$ to $N = 2$ Liouville, and of $SU(2)/U(1)$ to a Landau-Ginsburg model – geometry is entirely converted into a scalar potential. It would be interesting if this feature extended to higher dimensions, so that the effective fields that govern the theory of the early universe are best described in terms of dynamics in an effective potential, which depends on the epoch in spacetime of interest and may be absent at late times.

5 Quantum effects

In this section, we explore three commonly considered semiclassical phenomena: The quantum stress tensor of matter in a time-dependent gravitational background; the minisuperspace approximation to the wavefunction of the universe; and the gravitational thermodynamics of de Sitter space. In addition, we discuss string pair creation in closed string tachyon backgrounds.

5.1 The stress tensor and particle production

A standard method of assessing the effects of quantum fluctuations on inflation is to compute the semiclassical fluctuations around the classical slow-roll inflation solution. In the complexified Liouville model of section 3, we are in a considerably better situation – it may be possible to analyze the full quantum dynamics, given the fact that the theory is classically integrable. We leave this exercise to future work. As a first step, let us perform the standard semiclassical analysis on this model (c.f. [54] and references therein). This approach should be valid in the limit of large transverse dimension $d_{\perp}$, where the fluctuations of the geometry are parametrically suppressed by $1/d_{\perp}$.

For a conformal field theory of central charge $c$, the conformal anomaly relates the expectation value $\langle T_{\alpha\beta} \rangle$ of the stress tensor $T_{\alpha\beta} = 2\pi \delta S / \delta g^{\alpha\beta}$ in the metric $e^{\gamma\phi} \hat{g}$ to its expectation value $\langle \hat{T}_{\alpha\beta} \rangle$ in the metric $\hat{g}$:

$$
\langle T_{+ -} \rangle = \langle \hat{T}_{+ -} \rangle - \frac{c}{24} g_{+ -} R
$$

$$
\langle T_{+ +} \rangle = \langle \hat{T}_{+ +} \rangle - \frac{c\gamma^2}{12} \left( \frac{1}{2} (\partial_+ \phi)^2 - \frac{1}{2} Q \partial_+^2 \phi \right)
$$

$$
\langle T_{- -} \rangle = \langle \hat{T}_{- -} \rangle - \frac{c\gamma^2}{12} \left( \frac{1}{2} (\partial_- \phi)^2 - \frac{1}{2} Q \partial_-^2 \phi \right).
$$

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Thus, begin with the theory on 2d Minkowski spacetime and conformally transform to the (flat) cylinder; this gives a stress tensor on the cylinder

\[ \hat{T}_{++} = 0 \quad , \quad \hat{T}_{+-} = \hat{T}_{-+} = -\frac{c}{24} ; \]  

(48)

then perform the Weyl transformation to the metric (2). For instance, the global de Sitter metric (11) gives

\[ T_{+-} = -\frac{c}{24} \frac{\varepsilon^2}{\sin^2(\varepsilon \tau)} , \quad T_{++} = T_{--} = -\frac{c}{24} (1 - \varepsilon^2) . \]  

(49)

In the semiclassical approximation, this agrees with the analysis of section 2, since \( \frac{1}{24}c = \frac{1}{8}Q^2 \). Similarly, the stress tensor for the geometry (9) (similarly (10)) is obtained by the analytic continuation \( \varepsilon = i \epsilon \) (similarly \( \epsilon \to 0 \))

\[ T_{+-} = -\frac{c}{24} \frac{\varepsilon^2}{\sinh^2(\varepsilon \tau)} , \quad T_{++} = T_{--} = -\frac{c}{24} (1 + \varepsilon^2) . \]  

(50)

The shrinking proper size of the spatial circle in the far past leads to a diverging Casimir energy for conformal matter, as for instance the scalar quantity \( T^\alpha_\beta T_\alpha^\beta \) diverges as \( \tau \to -\infty \). Thus, even though the total stress tensor of matter plus ghosts plus Liouville gravity vanishes, individual ingredients have diverging stress-energy in this sense. Of course, if we do not interpret the Liouville field as the scale factor of the two-dimensional metric, then we are free to use the standard CFT vacuum on the flat cylinder (48); it is only the attempt to interpret the target space time as the 2d metric that causes difficulty; even then, it is the choice of observable such as \( T^\alpha_\beta T_\alpha^\beta \) that is problematic, and not necessarily the CFT state that we have defined.

The complexified Liouville theory of section 3 leads to a more intricate geometry, and the corresponding quantum stress tensor is now time-dependent. For example, the zero energy solution (30) leads to a stress tensor

\[ T_{+-} = -\frac{c}{24} \frac{\tau^2 - b^2}{(\tau^2 + b^2)^2} , \quad T_{++} = T_{--} = -\frac{c}{24} \left(1 + \frac{b^2}{(\tau^2 + b^2)^2}\right) . \]  

(51)

The source of this time-dependence is the dynamical evolution of the inflaton \( X \), which trades energy with the gravitational field through the gravitational dressing of the cosine potential.

In the positive energy case, we specialize for simplicity to \( \varepsilon_X = 0 \). The Liouville stress tensor is

\[ T_{+-} = -\frac{c}{24} \frac{\varepsilon_\phi^2 [\cos(2b) \cosh(2\varepsilon_\phi \tau) - 1]}{[\cosh(2\varepsilon_\phi \tau) - \cos(2b)]^2} \]

\[ T_{++} = T_{--} = -\frac{c}{24} \left(1 + \frac{\varepsilon_\phi^2 [\sinh^2(2\varepsilon_\phi \tau) - 2 \cos(2b) \cosh(2\varepsilon_\phi \tau) - 1]}{[\cosh(2\varepsilon_\phi \tau) - \cos(2b)]^2}\right) \]

(52)
which has the same qualitative features as the zero energy solution.

Linearized analysis of the analogous four-dimensional problem also leads to an apparent negative stress-energy [55, 56] along the lines of (50), (51). These works interpreted this result as a possible back-reaction that ‘slows’ inflation, or ‘screens’ the cosmological constant. The Liouville-Sine-Gordon model throws some doubt on this interpretation. The worldsheet stress-energy ‘induced’ by the Weyl anomaly (47) is in fact just the expectation value of the quantum Virasoro (or BRST) constraints on the state of the string. The quantum matter stress-energy simply determines the class of de Sitter solution that the cosmology follows. There is no dynamical ‘screening’ of the tachyon condensate at late times; the relations (47) are related to the characterization of the string state, and do not alter couplings in the worldsheet action such as the value of the cosmological constant.

Another important aspect of quantum field theory in curved spacetime concerns the fact that, generically, a time-dependent metric will cause the production of field quanta (c.f. [54] and references therein). The generation of such fluctuations during inflation and their subsequent back-reaction on the geometry provides the seeds of large-scale structure, one of the landmark successes of inflationary theory (for reviews, see e.g. [4, 5]). The present 2d models provide a laboratory to study concretely and quantitatively the mechanisms of generation and back-reaction of quantum fluctuations, as well as related issues in quantum cosmology.

The great advantage of two-dimensional models is the simple nature of 2d gravity. However, the Liouville-Sine-Gordon model is in this regard perhaps too simple – apart from the inflaton field, all other matter is massless and conformally coupled, and therefore only sensitive to the geometry through the Weyl anomaly. In particular, there will be no particle creation – all transverse fields remain happily in their conformal vacuum [54] (or whatever initial state we decide to put them in). There is nobody present to disturb the state and observe the generation of fluctuations! In order to recover the standard inflationary paradigm, one needs either to introduce massive matter (appropriately gravitationally dressed), or some class of observables such as a collection of comoving particle detectors. These will then register the presence of fluctuations in the inflaton field, which decohere into classical fluctuations of the geometry, and lead to the standard picture of structure formation. This lack of decoherence, together with the exact integrability of the model, may explain why the solutions (30),(33) of the complexified Liouville theory are time-symmetric and recollapse in precisely the same way that they expand. One would expect that anything that disturbs the state, such as interaction of the inflaton with massive matter, or the insertion by hand of some observable to measure the state, will destroy the time symmetry and lead to a recollapsing state rather different in character from the quiescent initial state.\textsuperscript{18}

\textsuperscript{18}See related comments in [57, 58].
5.2 The wavefunction of the universe

Another common approximation scheme is the so-called minisuperspace approximation, which suppresses the non-zero mode fluctuations of the fields. While perhaps less well-justified than the semiclassical approximation above, it often yields fruitful insights into the dynamics. In particular, it captures many of the qualitative features of Liouville theory in subcritical dimensions $d < 25$, c.f. [6]. We begin with an examination of the Liouville model in the supercritical dimension $d > 25$, and then turn to the complexified Liouville model of inflaton dynamics.

The Schrödinger (Wheeler-DeWitt) equation of Liouville quantum mechanics is

$$\frac{1}{2}(L_0 + \bar{L}_0)\Psi_\omega(\phi) = \left[\frac{1}{2} \left(\frac{\partial}{\partial \phi}\right)^2 - \frac{\mu}{8\gamma^2} e^{\gamma \phi} - \frac{Q^2}{8}\right] \Psi_\omega(\phi) = (1 - E_{\text{matter}})\Psi_\omega(\phi). \quad (53)$$

The solutions to this equation are Bessel functions, whose asymptotics are

$$J_\nu(z) \sim \begin{cases} \frac{1}{\Gamma(\nu+1)} \left(\frac{1}{2}z\right)^\nu & (z \to 0) \\ \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) & (z \to \infty) \end{cases} \quad (54)$$

$$N_\nu(z) \sim \begin{cases} -\frac{\Gamma(\nu)}{\pi} \left(\frac{1}{2}z\right)^{-\nu} & (z \to 0) \\ \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) & (z \to \infty) \end{cases} \quad (55)$$

The de Sitter solutions which are pure positive frequency plane waves in the far past,

$$\Psi_\omega^{in}(\phi) \sim \frac{1}{\sqrt{2\omega}} e^{-i\omega\gamma\phi}, \quad (56)$$

correspond to the classical trajectories (9). They have the minisuperspace wavefunction (see e.g. [35, 38])

$$\Psi_\omega^{in}(\phi) = \frac{1}{\sqrt{2\omega}} \left(\frac{-\mu}{4\gamma^2}\right)^{i\omega} \Gamma(1 - 2i\omega) J_{-2i\omega}\left(\sqrt{\frac{\gamma}{4}} e^{\gamma\phi/2}\right)$$

$$\omega^2 = \frac{2}{\gamma^2} (E_{\text{matter}} - 1 - \frac{1}{8}Q^2) = \left(\frac{\varepsilon}{\gamma^2}\right)^2. \quad (57)$$

The classical trajectory is of course the WKB trajectory, which is a good approximation for sufficiently large $\omega$. For example, at small scale factor the solution (57) behaves as (56), a plane wave of momentum $\omega\gamma$, and the corresponding classical solution (9) of the same Liouville energy $E_L$ has the same momentum, $\varepsilon = \omega\gamma^2$. Similarly, the solutions (11) describing classical pair production are given by the analytic continuation of $\omega$ to imaginary values, so that the wavefunction vanishes exponentially in $\phi$ at small scale factor, and has exponentially large momentum $\Pi_\phi \sim \pm \exp[\gamma\phi/2]$ at large scale factor.\(^{19}\)

\(^{19}\)Both signs appear because a given scale factor appears twice in the classical solution – once during the contracting phase, and again in the expanding phase.
Although the wavefunction is analytic in $\mu$, the analytic continuation to AdS cosmological constant $\mu > 0$ does not naively give the wavefunction corresponding to the AdS solution (5). The analytic continuation sends $J_\nu(z) \rightarrow J_\nu(iz) = iI_\nu(z)$, which blows up exponentially at large $z = \sqrt{\frac{\sqrt{\mu}}{\gamma^2} e^{\gamma \phi/2}}$, and has only an expanding and not a contracting phase at small $\phi$ – certainly not what we expect on the basis of the classical solution (5), which expands to a maximum value and then recontracts. Rather, the AdS solution (5) corresponds to the Bessel function

$$K_\nu(z) = \frac{i\pi}{2} e^{i\nu\pi/2} [J_\nu(iz) + iN_\nu(iz)] \sim \begin{cases} \frac{\pi}{2 \sin \pi \nu} \left( (z/2)^{-\nu} - (z/2)^{\nu} \right) & (z \rightarrow 0) \\ \sqrt{\frac{\pi}{2z}} e^{-z} & (z \rightarrow \infty) \end{cases}$$

This solution has both positive and negative frequency components at small scales, matching the fact that a given scale factor reached classically appears both during expansion out of the big bang (when $\phi$ has positive momentum) and contraction toward the big crunch (when $\phi$ has negative momentum); and it also vanishes exponentially for large scales, corresponding to the fact that, for a given $\varepsilon = \omega \gamma^2$, there is a maximum scale that is obtained before the universe recollapses.

The analytic continuation in $\mu$ of the wavefunction (57) appears instead to be related to the static solutions (6), (7), and (8). Each of these admits a formal periodic identification along their timelike Killing vector to yield a spacelike cylindrical worldsheet, and is the solution directly related by analytic continuation of (11), (10), and (9) under $\mu \rightarrow -\mu$, together with $\tau \leftrightarrow \sigma$. These classical AdS solutions have support at large scale factor $\phi \rightarrow +\infty$, but now it is at spatial infinity of the classical solution, and the metric is identified along a timelike isometry. This aspect of the wavefunctions – having parts of the wavefunction associated to closed universe cosmologies, and parts of the wavefunction associated to spatially noncompact regions with timelike identification – is in some respects reminiscent of the Nappi-Witten geometries studied in [59].

To summarize, there are two linearly independent solutions to the Wheeler-DeWitt equation, and each can be given an interpretation in terms of classical solutions in the WKB approximation. The boundary conditions to be imposed on the solution amount to the choice of a preferred linear combination of these solutions, and it is not clear that Feynman boundary conditions (which are most natural to the string theoretic interpretation) are the most natural choice. The imposition of Feynman boundary conditions leads to exponentially growing wavefunctions in the far future. We have interpreted this behavior as being related to static AdS geometries, but clearly this feature needs to be better understood.

The complexified Liouville theory of section 3 gives a trivial modification of (53). Take as the relevant Schrödinger equation for the holomorphic coordinate $\Phi$

$$\frac{1}{2} \left( L_0 + \bar{L}_0 \right) \Psi_\omega(\Phi) = \left[ \frac{1}{4} \left( \frac{\partial}{\partial \Phi} \right)^2 - \frac{\mu}{16\alpha^2} e^{\alpha \Phi} + \text{c.c.} \right] \Psi_\omega(\Phi) = -\frac{\alpha^2 \omega^2}{4} \Psi_\omega(\Phi)$$

The worldsheet coordinates are of course invisible to the Wheeler-DeWitt formalism.
for complex $\omega$. For example, the tensor product of the wavefunctions (57)

$$
\Psi^{(n)}_\omega(\Phi)\Psi^{(n)}_{\bar{\omega}}(\bar{\Phi})
$$

then satisfies the Schrödinger equation

$$
\frac{1}{2}(L_0 + \bar{L}_0)\Psi_{\omega}(\Phi)\Psi_{\bar{\omega}}(\bar{\Phi}) = -\frac{\alpha^2}{4}(\omega^2 + \bar{\omega}^2)\Psi_{\omega}(\Phi)\Psi_{\bar{\omega}}(\bar{\Phi}) .
$$

In terms of the classical solutions of section 3, one has $\varepsilon_{\phi} + i\varepsilon_{X} = \omega\alpha^2$.

An important issue surrounds the choice of boundary conditions on the wavefunction. The particular choice (60) does not satisfy the condition of periodicity under $X \rightarrow X + 2\pi/\alpha$. Here it proves useful to employ Hankel functions

$$
H^{(1)}_\nu(z) = J_\nu(z) + iN_\nu(z)
$$

$$
H^{(2)}_\nu(z) = J_\nu(z) - iN_\nu(z),
$$

for any integer $m$. For simplicity, we will restrict our considerations to $\text{Im}\omega = \varepsilon_{X}/\alpha^2 = 0$; this is in any case the situation of most interest, where the zero mode of the matter field is initially stationary. Using these transformation properties, one may show that

$$
\Psi_\nu(z, \bar{z}) = C_\nu \left( e^{\nu\pi i} H^{(1)}_\nu(z)H^{(1)}_\nu(\bar{z}) - e^{-\nu\pi i} H^{(2)}_\nu(z)H^{(2)}_\nu(\bar{z}) \right)
$$

(64)

(where $C_\nu$ is a normalization) is invariant under $z \rightarrow e^{m\pi i}z$. Appropriate wavefunctions for the complexified Liouville theory are then $\Psi_\nu(e^{\alpha\Phi/2}, e^{\alpha\Phi/2})$, where we have taken the liberty to absorb the factor $\sqrt{-\mu/\alpha^4}$ into a shift of $\Phi$. The asymptotics of $\Psi_\nu$ are

$$
C^{-1}_\nu \Psi_\nu(z, \bar{z}) \sim \begin{cases} 
-\frac{i}{2} \frac{\pi}{\nu} \left[ (z\bar{z})^\nu \frac{(z\bar{z})^{-\nu}}{(1+\nu)^2} + \frac{(z\bar{z})^{-\nu}}{(1-\nu)^2} \right] & |z| \rightarrow 0 \\
-\frac{4i}{\pi |z|} \cos(z + \bar{z}) & |z| \rightarrow \infty .
\end{cases}
$$

(65)

For $\nu = i\omega$, $z = \exp[\alpha\Phi/2]$, we see that there is a unit reflection amplitude, as in (58) for geometries expanding from small scale factor. However, there is also the behavior (54) appropriate to a global de Sitter geometry at large scale factor. What is going on? The scalar field $X$ in the geometry expanding out of small scale factor falls down its potential and recollapses with probability one, in accordance with the classical solution; but also, regions of the scalar potential corresponding to a de Sitter geometry give rise to string pair production, which is seen in the large scale factor
behavior of the wavefunction. As we saw in section 2.1, this pair production occurs classically for tachyonic modes; it also occurs quantum mechanically for massive modes, as we discuss in the next subsection following the analysis of [35]. It is interesting that the negative energy de Sitter solutions analogous to \( J_\nu \) for real \( \nu \) have disappeared from the sensible spectrum; namely, if we consider (64) for real \( \nu \), the wavefunction has a nice oscillatory behavior in the far future but necessarily blows up in the far past.

There is of course a second solution of the Schrödinger equation satisfying the periodicity requirement:

\[
\Upsilon_\nu(z, \bar{z}) = \tilde{C}_\nu \left[ 2 \cos \nu \pi H^{(1)}_\nu(z) H^{(1)}_\nu(\bar{z}) - e^{-\nu \pi i} \left( H^{(1)}_\nu(z) H^{(2)}_\nu(\bar{z}) + H^{(2)}_\nu(z) H^{(1)}_\nu(\bar{z}) \right) \right]
\]

(66)

where again \( \tilde{C}_\nu \) is a normalization. These functions have the asymptotic behavior

\[
\tilde{C}_\nu^{-1} \Upsilon_\nu(z, \bar{z}) \sim \begin{cases} 
\frac{2ie^{-2\nu \pi}}{\sin \pi \nu} [(z \bar{z})^\nu + (z \bar{z})^{-\nu}] & |z| \to 0 \\
\frac{\sinh[i(z - \bar{z})]}{|z|^{\nu+1}} & |z| \to \infty 
\end{cases}
\]

(67)

These wavefunctions thus blow up as \( \cosh[2\exp(\frac{d}{2}) \sin(\frac{d}{2} X)] \) for large scale (and also at small scale for real \( \nu \)), and so have the same large scale asymptotics as the de Sitter wavefunctions (57) when they are analytically continued to the AdS sign of \( \mu \). We would like to interpret the wavefunctions (67) similarly as being related to spatially noncompact solutions with AdS asymptotics. There are indeed spatially noncompact solutions of the complexified Liouville model, which are just the complexification of the static AdS geometries (6), (7), and (8). We might choose to discard these wavefunctions for the same reason, namely that the exponential growth at large scale is related to static AdS geometries, and not some ‘proliferation of de Sitter space’. Note again that the imposition of Feynman boundary conditions in the far past will necessarily involve these solutions. We will return to a discussion of boundary conditions in the Wheeler-DeWitt equation in section 6.

### 5.3 String production

In the semiclassical regime of large \( d \), there are many tachyons in the string spectrum. They will all be generated at the quantum level, and once generated, they will grow as \( T \sim \exp[(|m^2| - p^2)^{\frac{1}{2}} t] \) for a mode of spatial momentum \( \vec{p} \). If we regard the coefficient of the exponent as a measure of the decay rate, we can estimate the total decay rate per unit volume from all the tachyons as

\[
\Gamma_{\text{total}} = \int_0^{\sqrt{m^2}} dm \int d^4 p \rho(m) \sqrt{|m^2| - p^2} \\
\sim \text{const.} \times d^{\frac{d+1}{2}} \exp[4\pi d/12],
\]

(68)
where we have used the asymptotic density of states $\rho(m) \sim \exp[4\pi \sqrt{d/12} m]$. The result (68) can be regarded as an estimate of the overall rate of growth of instabilities in the target space background.

We saw in the classical solutions (11) and quantum minisuperspace wavefunctions (57) that there is unsuppressed classical production of strings at scale factors up to $e^{\gamma \phi} \sim 4/|\mu|$ (the minimum scale factor of global de Sitter space). This is to be expected for the tachyonic modes of the string field, simply reflecting the instability of the target space background. However, pair production also occurs for massive modes of the string field, as we now discuss.

Here, we are implicitly assuming the equivalence of $2d$ cosmologies and closed strings, with the naturally associated particle/string interpretation of in and out states at $\phi \to \pm \infty$, as e.g. in [54]. Thus, for instance, a de Sitter cosmology that is usually thought of as contracting and then re-expanding, is rather thought of as the creation of a 'universe–anti-universe pair'; these differ only in the relation between the direction of the expansion of the scale factor and coordinate time. Since what we physically detect is not coordinate time but some thermodynamic arrow of time that agrees with the direction of cosmic expansion, this is not expected to be an observable distinction. In other words, there is no 'collapsing de Sitter universe', a de Sitter universe propagating backwards in 'scale factor time', rather there is a de Sitter universe\(^{21}\) propagating forward in scale factor time.

The pair production rate of massive open strings in an exponential tachyon background was estimated in the minisuperspace approximation in [35, 38]. The closed string calculation is essentially the same in the minisuperspace approximation, since the zero mode feels the same exponential potential. For de Sitter space $\mu < 0$, the pure incoming plane wave solutions are the $J$ Bessel functions (57), whereas the pure outgoing plane wave solutions $\Psi_{out}(\phi)$ are the Hankel functions (62). The result for the appropriate ratio of Bogoliubov coefficients giving the transformation between in and out bases is\(^{22}\)

$$|\gamma_\omega| = \exp[-2\pi \omega \sqrt{\alpha'}]$$  \hspace{1cm} (69)

(recall $\omega \sim m/\sqrt{2\gamma}$). This ratio gives the probability amplitude to pair produce any individual heavy ($\omega \gg 1$) string mode. The typical string is produced at around $\phi = 0$, where the scale factor is of order one.

The pair production probability $|\gamma_\omega|^2$ is exponentially suppressed in the mass of the string state; however, this is compensated by the exponentially large density of states at energy $\omega$,

$$\rho(\omega) \sim \exp \left[+4\pi \omega \sqrt{\alpha'} \cdot \gamma \sqrt{\frac{d-1}{12}}\right].$$  \hspace{1cm} (70)

\(^{21}\)Unfortunately, we cannot call it an anti-de Sitter universe.

\(^{22}\)We thank A. Strominger and T. Takayanagi for pointing out an error in this and the following expression in an earlier version of this paper. Also, in order to facilitate comparison to [35, 38], we restore the factors of $\alpha'$ in this and the following expressions. In the rest of the text, we have set $\alpha' = 2$ in accordance with the conventions of [6].
Since $\gamma\sqrt{\frac{d-1}{12}} > 1$, the exponential suppression of the Bogoliubov coefficients is always outweighed by a faster growth in the density of states; the total rate of production of massive string modes diverges. Once produced, such strings gain energy exponentially rapidly from the unbounded Liouville potential. Their back reaction becomes important at some finite time. The overall picture is of a Hagedorn gas of strings being formed by the exponential time dependence of the target space background.

For $\mu > 0$, the tachyon condensate seems to have the opposite effect; classically, strings are ‘repelled from the future’. As mentioned above, the wavefunction that describes an expanding and recollapsing AdS solution is exponentially damped at large scale factor – there are no strings at late time.

The wavefunction (64) of complexified Liouville theory exhibits the large scale factor asymptotics of Hankel functions (62) appropriate to pair production in de Sitter space, even though the reflection probability at small scale factor is one. We interpret this result as an indication that the incoming cosmology follows its classical trajectory and recollapses; yet at the same time, for any Liouville energy, there is pair production of strings from the region in $X$ where the matter potential generates the de Sitter sign of the cosmological constant. This leads to a nonzero amplitude to find a 2$d$ universe at large scale factor.

5.4 de Sitter thermodynamics

While the statistical mechanics underlying the entropy of gravitating systems in anti de Sitter space has by now achieved a rather firm footing (c.f. [61] for a review), gravitational entropy in de Sitter space has remained a bit more mysterious. There have been a number of analyses of the entropy of two-dimensional de Sitter space, for example using ideas of quantum entanglement [62], or considering topological gravity and its asymptotic symmetry algebra [63, 64, 65] (following the ideas of [66]).

One should be careful in drawing conclusions about de Sitter thermodynamics, and in comparing different approaches to it, since the thermodynamic quantities of energy, entropy, and temperature are observer dependent. For example, the entanglement approach of [62] takes the point of view of a static observer, considering the entanglement of their observable degrees of freedom with those behind their horizon; energy and temperature are those measured by the static observer. On the other hand, the approach of [63, 64, 65] uses asymptotic symmetries on the spacelike slice at conformal infinity, and is thus necessarily observing the entire state. In particular, it is not clear how to relate the energies of the two approaches, nor is it clear what interpretation to give the entropy computed in [63, 64, 65].

23In the semiclassical limit, $\gamma\sqrt{\frac{d-1}{12}} \sim 1 + \frac{6}{d-1} + ...$, and in the critical dimension, $\gamma\sqrt{\frac{d-1}{12}} = 2$.

24It seems most closely related to the ‘observable entropy’ of the $N$-bound [67] discussed below.
In more than two dimensions, the de Sitter gravitational entropy is the canonical
\[ S = \frac{A}{4G}, \]  
(71)
where \( A \) is the area of the horizon of a static observer, and \( G \) is Newton’s constant.

In two dimensions, this formula is rather ambiguous – the horizon of the static patch is a zero-dimensional sphere, which is only a pair of points; and one must identify the quantity playing the role of \( G \), since there is no Einstein gravity per se. The two approaches [62] and [63, 64, 65] make differing assumptions, but both arrive in the end at expressions of the form
\[ S_{2d} = \text{const.} \times c \]  
(72)
where the constant depends on the assumptions made, and \( c \) is the relevant conformal central charge of the system. Note that we obtain this sort of answer if we treat (as in [62]) both the horizon area and the gravitational coupling as dimensionless numbers. The horizon area is 2, for instance if we evaluate the general formula \( \text{Vol}(S^{n-1}) = 2\pi^{n/2}\Gamma(\frac{1}{2}n) \) for \( n = 1 \), or equivalently since the zero-dimensional transverse sphere consists of two points; and in section 3, we saw that \( 8\pi G = \frac{1}{2}a^2 = 6/c \).

These values yield
\[ S_{\text{BH} 2d} = \frac{2\pi}{3} c \]  
(73)
as the Bekenstein-Hawking entropy of two-dimensional de Sitter space.

Let us apply yet another heuristic argument for this form of the de Sitter entropy. A route to the de Sitter entropy in higher dimensions is through the Schwarzschild de Sitter geometry [68]. In \( n + 1 \) dimensions, the Schwarzschild de Sitter geometry is
\[ ds^2 = -\left(1 - \frac{r_0^{n-2}}{r^{n-2}} - \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{r_0^{n-2}}{r^{n-2}} - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega^{2}_{n-1} \]
\[ r_0^{n-2} = \frac{16\pi G_{n+1} m}{(n-1)\text{Vol}(S^{n-1})}. \]  
(74)
Blithely continuing to \( n = 1 \), after the shift \( r \rightarrow r - \frac{4\pi G m \ell^2}{n-1} \) one finds
\[ ds^2 = -(a^2 - r^2/\ell^2) dt^2 + \frac{dr^2}{(a^2 - r^2/\ell^2)} \]
\[ a^2 = [1 + (\frac{4\pi G m \ell^2}{n-1})^2] \]  
(75)
with \( \ell^2 = -4/\mu \). This metric is appropriate to the static patch of an inertial observer in the global geometry (11), where the time \( t/\ell \) along that observer’s worldline in (75) is identical to the proper time in (13). As a result, we may identify \( \epsilon = a \). Demanding regularity of the Euclidean continuation \( t_E = it \) of (75) requires \( t_E \approx t_{\text{E}} + 2\pi \ell/a \), and determines the temperature \( T = a/2\pi \ell \). Note that one has \( T \sim E \) at large \( E \),
appropriate to a $0 + 1$ dimensional gas. This suggests that there might be a model for the entropy in terms of a reservoir of quantum mechanical degrees of freedom on the de Sitter horizon.

For geometries that are asymptotically de Sitter in both past and future, $m^2 < 0$; so after continuing $M = im$ one has

$$dS_{2d} = -\frac{dE}{T} = -\frac{2\pi \ell dM}{\sqrt{1 - (4\pi G \ell^2 M/n^2)^2}}$$

(76)

(the sign is due to the fact that the energy under discussion is being extracted from the horizon, not added to it). The integral should run from the point of zero entropy $\epsilon = 0$ to the value of interest; one finds

$$S_{2d}(M) = \frac{n-1}{2G_2} \cos^{-1}\left(\frac{4\pi G \ell^2 M}{n-1}\right).$$

(77)

One thus finds the entropy of global de Sitter space by this reasoning to be

$$S_{dS,2d} = \frac{\pi(n-1)}{4G_2}.$$ (78)

The factor $(n-1)/G_2$ is best understood from the dimensional continuation of the Einstein action. Consider the standard gravitational action $\frac{1}{16\pi G_{n+1}} \int \sqrt{g} R$ in a metric which is a Weyl rescaling $g = e^{\rho} \hat{g}$ of a fiducial metric $\hat{g}$, near $n + 1 = 2$ dimensions. One finds

$$\frac{1}{16\pi G_{n+1}} \int \sqrt{g} R = \frac{1}{16\pi G_{n+1}} \int e^{\frac{n-1}{2} \rho} \sqrt{\hat{g}} \left[ \hat{R} - n \hat{\nabla}^2 \rho - \frac{n(n-1)}{4} (\hat{\nabla} \rho)^2 \right] \sim \frac{1}{16\pi G_{n+1}} \int \sqrt{\hat{g}} \left[ \hat{R} + \frac{n-1}{4} (\hat{\nabla} \rho)^2 + \frac{n-1}{2} \rho \hat{R} + \ldots \right].$$ (79)

Thus the Liouville action arises as the leading non-topological term in the Einstein action expanded around two dimensions. Comparing to (1), we identify

$$\frac{n-1}{G_2} = \frac{8}{\gamma^2} = \frac{2c}{3}.$$ (80)

Plugging into (78), we find

$$S_{dS,2d} = \frac{\pi}{6} c$$ (81)

which is $1/4$ of the answer (73). But we should be suspicious of the naive Bekenstein-Hawking answer (73); it suggests that the entropy is independent of the energy, and thus does not decrease if we add some particles to the de Sitter vacuum. On the other hand, it is reasonable to expect that $dS = -dE/T$; extracting particles from the de Sitter horizon is only consistent with constant $S$ for infinite Hawking temperature, which is incompatible with the finite periodicity of the Euclidean continuation of the geometry.
One might question some of the assumptions made in the derivations that lead to (73) and (81), for instance the evaluation of the Newton constant is different in the two cases. Also, in the analysis of the first law, we dimensionally continued away from two dimensions \( n = 1 \). Let us do the same for the area law. Consider the cosmological horizon of the Schwarzschild de Sitter solution (74); as \( n \to 1 \), it is located at
\[
\frac{r_{\text{hor}}}{\ell} = -\frac{\ell}{2r_0} + \sqrt{\left(\frac{\ell}{2r_0}\right)^2 + 1} ,
\]
(82)
where
\[
\frac{\ell}{r_0} = \frac{8\pi G_2 m \ell}{n - 1} ,
\]
(83)
The \( S^{n-1} \) horizon sphere has area \( \text{Vol}(S^{n-1}) (r_{\text{hor}})^{n-1} \), so as \( n \to 1 \)
\[
dA = 2(n - 1) d \log \left[ -\frac{4\pi G_2 m \ell}{n - 1} + \sqrt{\left(\frac{4\pi G_2 m \ell}{n - 1}\right)^2 + 1} \right]
\]
(84)
Again continuing \( M = im \), we find
\[
\frac{dA}{4G_2} = \frac{n - 1}{2G_2} d \cos^{-1}\left(\frac{4\pi G_2 M \ell}{n - 1}\right) ,
\]
(85)
in agreement with (77). Thus, the dimensional continuation of the area law agrees with the corresponding continuation of the first law.

Finally, it has been argued in [67] that the entropy of de Sitter space should bound the ‘observable’ entropy of matter in a space that is asymptotically de Sitter in both the past and future. The basic idea is that extracting too much entropy from the de Sitter horizon of a static observer requires putting too much energy into the spacetime, and hence a cosmological singularity develops either in the past or future. Let us see how this argument works in two dimensions.

Global de Sitter space is the vacuum \( \epsilon = 1 \) of the solution (11), where the matter fields are also unexcited. As we populate the levels of the matter system, \( \epsilon \) decreases due to the constraint (19), until it reaches zero at \( E_{\text{matter}} = c/24 \). At this point the geometry is on the cusp of having a singularity, and indeed there will be a big crunch (Milne singularity) if we add more matter energy, since the solutions will cross over into the hyperbolic class (9). Note that the addition of matter makes the de Sitter space ‘taller’, see figure 2; higher-dimensional gravity also exhibits this property, as shown in [69] and applied to holographic bounds in [70, 71].

Due to the exponential growth in the density of levels, the total number of matter states in the class of geometries which is asymptotically de Sitter both in the past and in the future is bounded by the density of levels at \( L_0 = L_0 = c/24 \); in this way, one reaches an analogue of the ‘\( N \)-bound’ of [67]
\[
S_{\text{matter}} = 2\pi \sqrt{\frac{\pi}{6} L_0} + 2\pi \sqrt{\frac{\pi}{6} L_0} \leq \frac{\pi}{3} c = 2S_{dS,2d} ,
\]
(86)
One sees from the previous considerations that twice the de Sitter entropy given by
(81) bounds the ‘observable’ entropy of matter. The bound argued for in [67, 70, 71]
is actually $S_{\text{matter}} \leq S_{\text{dS}}$ rather than twice $S_{\text{dS}}$. It would be interesting to understand
the source of this discrepancy, and more generally, to understand why the entropy
(77) differs in form from (86).

The above results point the way toward an understanding of de Sitter quantum
gravity and its thermodynamics. Global 2d de Sitter space is the vacuum of matter
coupled to the ground state of Liouville theory, and is thus a unique state. One
might then view the de Sitter entropy of a static observer as entirely an effect of
quantum entanglement. Alternatively, global de Sitter space only lives for a finite
amount of conformal time; one might also explain the entropy as an inability to
decide precisely the number of quanta in the global CFT state, since field modes do
not undergo a full period of oscillation in the time available to observe the state.
However, de Sitter entropy seems rather different from black hole entropy, which
for instance in the AdS/CFT correspondence is a count of distinct microstates of
quantum gravity.

6 Discussion

6.1 The beginning of the beginning

One of the outstanding issues facing inflationary cosmology is the origin of the
inflationary phase. A number of suggestions have been put forward in this regard:

- The ‘old’ [72] and ‘new’ [73, 74] inflationary scenarios, whereby the universe
is assumed to start off in a metastable or slightly unstable inflationary phase;

- The various ‘no-boundary’ or ‘tunneling’ proposals [17, 18, 19, 20, 21, 22, 23]
whereby the initial inflationary state is determined by a tunnelling event;

- ‘Chaotic’ inflation [24], in which the initial value of the inflaton field is taken to
be a random variable, so that some fraction of the ensemble of initial conditions
leads to inflation;

- ‘Eternal/chaotic’ inflation [19, 25, 26, 27], whereby quantum fluctuations of the
inflaton drive the volume-weighted average of the field to climb its potential.
The ‘typical’ FRW domain is surrounded by, and came from, an inflationary
phase.

Each of these scenarios has its drawbacks. Old inflation does not end gracefully,
producing unacceptably large fluctuations; new inflation begins by fiat with a spe-
cially tuned initial condition – a quiescent scalar field at a very flat local maximum

25 We are of course assuming that quantum Liouville theory with this sign of the cosmological
constant exists as a well-defined Lorentzian CFT.
of its potential – and so to some extent begs the question of how the universe became large, flat, and isotropic. One must find some measure on the space of initial conditions for cosmology, explain why the inflating solutions have a non-negligible probability, and so on (c.f. [58] for a recent discussion).

The no-boundary/tunnelling proposals are an attempt to provide such a measure which makes the inflating state a reasonable starting point. However, one could also regard it as an arbitrary selection criterion on states, based on a notion of simplicity – typically only ‘nice’ (simple, relatively symmetric) states have a ‘nice’ (nonsingular) Euclidean continuation. In quantum field theory any nontrivial S-matrix process does not have such a ‘nice’ Euclidean continuation; there, the proposal would throw out essentially all the interesting structure! In the context of 2d models, any state (other than the CFT vacuum) with a truly nonsingular Euclidean continuation would amount to a tadpole amplitude for the string background, which would then not satisfy the classical string equations of motion.

Chaotic inflation is to some extent a similar selection criterion. The inflaton field is assumed to lie in a chaotic distribution of initial conditions, or in other words its quantum wavefunction is broadly distributed. However the wavefunction should not be so chaotic that the inflaton kinetic energy dominates over its potential energy; at least, some regions of space should be sufficiently quiescent for a sufficiently long period that sufficient inflation takes place. For instance, a hot big bang is not the right kind of initial chaos if the potential is \( V(X) = m^2 X^2 \). Nevertheless, there will be a relatively large phase space of acceptable initial conditions. Note that inflating wavefunctions of our two-dimensional models have somewhat the character of chaotic inflation – the quantum wavefunction can be taken to be more or less uniformly smeared over all values of the inflaton field \( X \) (quantum fluctuations in two dimensions do this naturally). Some fraction of the distribution will inflate along trajectories of the sort described in section 3.

In the context of minisuperspace models, the discussion of appropriate boundary conditions to be imposed on the Wheeler-DeWitt wavefunction has a rich and varied history (for reviews, see for instance [75, 76]). In pure Liouville gravity, the minisuperspace ‘no-boundary’ proposal [18] amounts to the selection of wavefunctions \( \Psi = J_\nu \) for real \( \nu \) (whose classical geometries are the global de Sitter geometries (11)). These real-valued functions behave as the cosine of the scale factor \( \cos(a) \) at large scales, and as a power law of the scale which vanishes as \( a^\nu \) at small scales. The tunneling proposal of [21] selects purely expanding universes at large scale, which are the Hankel functions \( H_\nu^{(2)} \). Yet another choice suggested in [77] is to consider only expanding universes at small scale, which selects \( \Psi = J_\nu \) for imaginary \( \nu \). The latter is particularly related to the implementation of Feynman boundary conditions and the associated interpretation of 2d cosmologies as quanta in a string field theory. This interpretation arose some time ago in the context of minisuperspace models of baby universes (c.f. [78] for a review, and also [79, 80, 81, 82, 83]).

Finally, let us turn to the chaotic/eternal inflationary scenario. Its logic rests on a number of assumptions that are open to question. First of all, given the apparent
success of the notion of holography in black hole physics, one might wonder whether the appropriate probability measure of a geometry in quantum gravity is related to its proper volume, rather than for instance the area of its horizon or some other measure.\textsuperscript{26} Second, as in black hole physics it is dangerous to ascribe objective reality to properties of spacetime that lie beyond the horizon of any observer in the spacetime. Any given observer sees a finite probability per unit proper time for inflation to terminate, thus there is no observable that detects an eternal proliferation of inflating domains.\textsuperscript{27} Third, the notion of eternal inflation has a Zeno-like quality – one takes the tiny fraction of the probability density where the field jumps up the potential, counter to the dictates of classical dynamics; that region wins the competition of volume growth as the field slowly rolls back down; then one takes the further tiny fraction of the probability density where the field jumps up the potential; and so on. It would seem that at some point one will run out of probability density. Fourth, the domains of large proper volume so obtained occupy a vanishingly small (conformal) coordinate domain, which is ostensibly compensated by a huge conformal factor in that region, leading to the typical fractal conformal diagram of the late-time structure of eternal/chaotic inflation (see figure 5). But how are we to distinguish these violent fluctuations of the conformal factor from the standard fluctuations of a quantum field on all scales, that we are accustomed to regularizing and renormalizing away?

The essential driving force behind this violent oscillation of the scale factor is the negative metric on the kinetic energy of the scale factor. In the context of two-dimensional models of the sort we have been considering, many if not all of the fluctuations of the scale factor (Liouville mode) are gauge artifacts, which are eliminated by the BRST constraints on the Wheeler-DeWitt wavefunction. The physical state space has positive metric. In fact, in the initial state at $\phi \to -\infty$, the matter potential turns off and all the fields become free. Then, up to BRST trivial shifts $|\Psi\rangle \to |\Psi\rangle + |Q_{\text{BRST}}\Upsilon\rangle$, we can consider states having only the zero mode of the scale factor excited. It is hard to see how a fractal structure of the scale factor is going to emerge, given that the Virasoro constraints dictate the future evolution of the scale factor in terms of the state of the matter fields; it would seem that such violent fluctuations would have to be built in by the choice of particular highly excited matter states, which is hardly the appropriate starting point for inflation.

One way that the notion of chaotic inflation might be consistent with the analysis of the preceding sections would be if the fluctuations of the scale factor had an interpretation in terms of universe production. We saw an example of this in the context of topological inflation, wherein a singularity in the scale factor found an interpretation in terms of the branching off of a child universe that carried off the

\textsuperscript{26}Replacing volume by area would not however seriously affect the estimates of chaotic inflation \textit{e.g.} in \cite{27}.

\textsuperscript{27}At least, no local observable. There might be nonlocal observables; they would have to be stitched together out of holographic data on the horizons of all local observers, taking proper account of the entanglement of this data (\textit{c.f.} \cite{84, 85, 86}).
The fractal structure of the eternally/chaotically inflating universe. The shaded domains represent the ordinary FRW cosmological regions that arise when the chaotic inflaton sporadically falls out of the inflationary regime. In a two-dimensional model, such configurations represent violent UV fluctuations of the Liouville mode, which might not have physical significance.

Perhaps when one is able to calculate reliably, the fluctuations of the scale factor in chaotic inflation will be seen to actually reflect the creation of child universes. However, note that in the minisuperspace approximation, the appearance in the wavefunction of strings having large scale factor occurred by mechanism separate from the dynamics of the expanding universe at early time; that universe returned to zero scale factor with probability one, and the universes appearing at late time were the result of pair creation processes.

6.2 An alternative: associated production of universes

The string dynamics that results from tachyon condensation has an amusing 2d cosmological interpretation, that provides a new explanation for the origin of inflation. Back reaction of the target space geometry eventually becomes important, and determines the actual late time (large scale factor) behavior of the theory. The late-time behavior of bulk tachyon condensation is not well understood, but it is conceivable that it will have the effect of saturating the exponential growth of the worldsheet potential, and thus relaxing the 2d cosmological constant. Moreover, strings are pair produced by the condensate and eventually become important in determining the late time (large scale factor) behavior. The production of large numbers of strings
(2d cosmologies) in association with a tachyon condensate, which becomes an ordinary supergraviton wave at late times, provides a new perspective on the origin of inflation.

Localized tachyon condensation provides a controlled laboratory where precisely these mechanisms occur. As we argued in section 4, the late time behavior of the condensate in the throat models is a pulse of radiation travelling up the throat, leaving behind the throat of \( k \) fivebranes. The exponential growth in time of the tachyon \( T \) saturates due to backreaction and evolves into an ordinary sigma model background (albeit time-dependent). At late time, the worldsheet of a given string left behind by the outgoing pulse sees a flat potential, and the classical solution of the Liouville field will be approximately linear in worldsheet conformal time \( \tau \), \( \phi \sim \varepsilon \tau \), corresponding to an expanding Milne geometry. This is an FRW cosmology with scale factor \( a(t) \sim t \) – the cosmological constant relaxes to zero at late time. Strings initially trapped in AdS regions (large \( |\partial_\phi T| \), small \( |\partial_\chi T| \)) do not reach large scale factor \( \phi \); strings in the initial state finding themselves in an inflating region (large \( |\partial_\chi T| \), small \( |\partial_\phi T| \)) do reach large scale factor, and in addition there will be string pair creation by the time-dependent background in these de Sitter regions. The latter effect generates a probability distribution on 2d cosmologies.

We can then ask the question: What is the character of the typical 2d universe at large scale factor \( \phi \)? The answer will be that it is an expanding, radiation dominated FRW universe! The estimates of [35] quoted in section 5.3 indicate that the pair production rate is substantial; there is of course also the ‘classical’ production of the tachyon modes themselves, which become the coherent pulse of supergravitons propagating to infinity. The latter is essentially the analogue of the exponential production of (cold, empty) ‘baby universes’ observed in [79, 80, 81, 82, 83]. There are also many strings produced in excited states, as we saw in section 5.3, and it will be an important task to determine their distribution. If the typical 2d universe left behind by the pulse arises from such a pair production process, we have a situation where observers living in the typical late-time universe would see the structure of the sort we want to arise from inflationary theory, namely that their universe experienced an early period of inflation which exited into an expanding FRW phase. The problem of how inflation got started is neatly solved by the fact that most of the (non-empty) 2d universes at late time were created by a quantum process, and the attempt to trace their geometries backward classically to a big bang at \( \phi = -\infty \) is not a valid procedure. Note that this mechanism is distinct from the ‘tunnelling/no-boundary’ proposals of [17, 18, 19, 20, 21, 22, 23, 87] and others. There, the entire interpretation of the wavefunction is in terms of single universes. Here, the mechanism is the quantum mechanical Bogoliubov transformation between ‘in’ and ‘out’ states of the

\[ \text{In the 2d energy constraint (26), the energy density evolves as } \rho \sim \text{const.} \times a^{-2}, \text{ which is the same as the ‘spatial curvature’ term } -\left( \frac{1}{8}Q^2 + 1 \right)a^{-2}. \text{ Thus, once radiation dominated, always radiation dominated – there will never be a critical scale factor at which the latter term takes over and causes recollapse.} \]

\[ \text{Which appear from ‘nothing’ by tunnelling; however, as mentioned above, the embedding of this idea in 2d models takes on the character of an aesthetic criterion for the selection of an ‘in’} \]
string field. Such effects will be present even if the string coupling is tuned to be small.

Sadly, these late-time cosmologies have unrealistic aspects (besides the fact that they are two-dimensional). Being strings, they will always have moduli (that locate the string in the smooth target space), and the matter energy density of the late-time cosmology is radiation dominated, residing in the fluctuations of these moduli. Intrinsic to our model is the problem of moduli production at the end of inflation.\(^{30}\)

It would certainly be interesting if such a scenario could be exhibited in more realistic models. There are certain features of string theory that parallel what we have found here. Known constructions of de Sitter space in string theory \([34, 89]\) have the property that the de Sitter phase has finite duration, eventually relaxing to an FRW phase (\(e.g.\) as described in \([33]\) for the models of \([34]\)). Thus the asymptotic geometry at large scale factor is always FRW. Related to this, the qualitative structure of the configuration space of string theory is the same as in our two-dimensional model of section 4—the ‘tachyon’ (the cosmological constant) is ‘localized’ in field space; there always seem to be one or more ‘flat directions to infinity’ \([90]\), with the nontrivial potential only in some finite region of the low-energy field space. In other words, the instability to exponential growth of the scale factor only exists in a localized region of field space, and persists only for finite time, much as in the 2d model. Topology change is expected to occur in string theory (\([91, 92, 93, 94]\), to name a few); could topology change result in ‘universal pair production’? Does this idea explain why the typical universe at large scale emerged from an inflationary phase?\(^{31}\)

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\(^{30}\)And of course we may try to search for solutions. For example, we can try to limit the number of flat, runaway directions in the potential to one by constructing higher-dimensional throats of the sort in \([88]\), which have fewer noncompact ‘runaway’ directions.

\(^{31}\)If so, one might imagine generalizations in various directions. For example, the number of macroscopic dimensions of spacetime might be explained as a consequence of the genericity of an effective potential in four as opposed to other dimensions, together with considerations of the available phase space for universe creation.
References


