Three particle states and $t$-exchanges

Norbert E. Ligterink
Department of Physics and Astronomy, University of Pittsburgh,
3941 O’Hara Street, Pittsburgh PA 15260, USA
E-mail: nol1@pitt.edu
March 12, 2003

Abstract

The multiple particle final states in modern data from $4\pi$ detector experiments offer a wealth of information about higher hadronic resonances and hadron interactions. However, it requires careful analysis to extract model independent results from those data.

1 Introduction

The handling of $t$-exchanges in scattering experiments requires serious consideration. It is the first and dominant process involving multi-particle states, which is relevant for the study of hadronic resonances at energies of the second resonance region (1.5 GeV and above). In most approaches three and more particle states are handled as effective two-particle states. The state with one nucleon and two pions is often modeled as a rho-nucleon or a sigma-nucleon state. However, if the higher resonances are to be understood in a model-independent way, the three-particle states, and the non-resonant production processes, such as the $t$-exchange processes should be handled in their full glory. Fundamentally, this is not a big problem; the theory is well understood. However, practically it is a different story. Central to the computational problems lie the singularities associated with the three-particle states going on-shell. It should be noted that in a simplistic Yukawa treatment of the $t$-exchange, ignoring the energy dependence of the exchange diagram, such a problem does not occur. However, if the three-particle states and the singularities are treated seriously, they should be regulated and integrated over which requires three scales: the regulation scale $\epsilon \ll$ the integration scale $\Delta E \ll$ the physical scale $\mu$. Every problem is big in numerical sense, because the handling of triple scales. Furthermore, if one works with momentum variables, the singularities form curves in the kinematical domain, so extra care is required to locate the singularities and extrema on the curve with respect to the chosen grid. In terms of energy variables the position of the singularity is obvious; when the scattering energy equals the three-particle energy, however, the numerical integration still requires great care. In recent
years some methods, such as the Lorentz transform, have been developed, which essentially takes one away from the real axis and the singularities, however, they are rather complicated.

In this paper another approach is investigated. Central is the expansion of the amplitudes and relevant function in an orthogonal basis with known dispersion integrals, which solves one from doing any singular integration. In terms of bookkeeping it does require a serious effort, as one should keep track of the real and imaginary parts of both the two-particle and the three-particle thresholds and amplitudes. The threshold of the three-particle state depends on the two-particle states from which it originates; if the two-particle state has zero momentum, i.e., \( E = m_a + m_b \), the three-particle state it produces can have zero momentum for all particles too, and its threshold lies at \( E = m_a + m_b + \mu \), while if the two-particle state has a higher momentum, the threshold of the three-particle states it can produce will lie appropriately higher. Furthermore, given the energies of the initial and final two-particle state \( \omega_i \) and \( \omega_f \), there is an upper bound to the energy of the three-particle state it can produce in the \( t \)-exchange where the exchange particle is emitted from particle \( a \):

\[
E_{\text{max/min}} = \sqrt{(k_i \pm k_f)^2 + \mu^2} + \sqrt{k_i^2 + m_b^2} + \sqrt{k_f^2 + m_a^2},
\]

where \( k \) is the momentum associated with the energy \( \omega \):

\[
k = \frac{\sqrt{(\omega_i^2 - m_a^2 - m_b^2)^2 - 4m_a^2m_b^2}}{2\omega}.
\]

Eventually, the kinematical restrictions on a two-particle state in a given partial wave will lead to a set of three-particle states, which can be labeled by the angle between the momentum of the exchange particle and the momentum of the two-particle state. This integration can be performed analytically and leads in the case of a two-particle \( s \)-wave to a second-order transition amplitude:

\[
T^{(0)}(E, \omega_i, \omega_f) = \frac{g^2}{k_i k_f} \log \left[ \frac{E - E_{\text{min}}}{E - E_{\text{max}}} \right],
\]

where \( g \) is the coupling constant, and numerical factors are ignored. The three-particle state in now implicitly defined through the imaginary part of the two-particle to two-particle transition, which saves one from constructing a basis for the three-particle states. As every three-particle state follows from the emission of an exchange particle from the two-particle state, the three-particle amplitude is given by the two-particle amplitude times the imaginary part of the elementary transition amplitude \( T^{(0)} \). The two-particle amplitude is determined from solving the Lipmann-Schwinger equation with the elementary transition amplitudes like \( T^{(0)} \) in the kernel.

The real part of the amplitude is dominant. The leading order imaginary part is the first order process given by the imaginary part of \( T^{(0)} \). However, as with most problems, the qualitative changes in the results through the imaginary parts, i.e., on-shell three-particle states, can only be understood properly if they are included in the calculation.
Making a separable expansion of the amplitude lies at the basis of an effective method to sum the processes to all order to yield a unitary and analytic transition amplitude:

\[ T^{(0)}(E, \omega_i, \omega_f) = \sum_{lm} \alpha_{lm} \phi_l^{(E)}(\omega_i) \phi_m^{(E)}(\omega_f) \]  

(4)

where one should take care the the basis \( \phi_a^{(E)}(\omega) \) is orthogonal (or band-diagonal, such as splines) for a well defined expansion, and it has known dispersion integrals:

\[ \tilde{\phi}_{lm}^{(E)}(E') = \frac{1}{\pi} \int d\omega \frac{\phi_l^{(E)}(\omega) \phi_m^{(E)}(\omega)}{E' - \omega} \]  

(5)

The real part of the amplitude spans the full kinematical domain \( m_a + m_b < \omega \), hence an expansion in \( \xi = (\omega^2 - (m_a + m_b)^2)/\omega^2 \in [0, 1] \) can serve as a basis \( \phi_l(\omega) \):

\[ \phi_l(\omega) = \sqrt{4l + 2(m_a + m_b)/\omega^{3/2}} P_l(2\xi - 1) \]  

(6)

for products of which a closed form dispersion integral can be calculated. Therefore, for the expansion of the real part of \( T^{(0)} \) the suffix “\( E \)” is not necessary.

However, the imaginary part of \( T^{(0)} \) is only non-zero in a restricted but infinite domain of \( E \otimes \omega_i \otimes \omega_f \); when the scattering energy \( E \) lies in between \( E_{\text{min}}(\omega_i, \omega_f) \) and \( E_{\text{max}}(\omega_i, \omega_f) \). The domain in the \( \omega_i \otimes \omega_f \) space given a scattering energy \( E \) is the energy equivalent of the smooth triangular kinematical region of a Dalitz plot, where the each of the corners is sharper if the energy is larger compared to the mass \( m_a, m_b \), or \( \mu \). As the energies \( E, \omega_i, \) and \( \omega_f \) increase, the domain approaches the triangular area:

\[ E < \omega_i + \omega_f \text{ and } E > \omega_i \text{ and } E > \omega_f \]  

(7)

This correlation between the variables does not allow for a separable expansion of the imaginary part of \( T^{(0)} \).

It seems that a useful parametrization of Dalitz plot amplitudes and it generalization to more particles and more dimensions does not yet exist. In the massless case, the domain is a simplex on which Appell polynomials and generalized mappings of orthogonal polynomials from the \( n \)-ball to the \( n \)-simplex form a multi-dimensional basis. In the massive case a system of a weight function that serves also as domain definition times polynomials \( P_I \) in the single particle energies \( \epsilon \) of the three or more particle state might serve many needs:

\[ \phi_I = \theta(E - E_{\text{edge}})(E - E_{\text{edge}})^{\beta} P_I^{(\beta)}(\epsilon_1, \cdots, \epsilon_{n-1}) \]  

(8)

such that

\[ \int_{\text{domain } E} d\epsilon_1 \cdots d\epsilon_{n-1} \phi_I \phi_J = \delta_{IJ} \]  

(9)

For each case of different masses (\( \pi \pi N, \pi \rho N, \) etc) the coefficients of the orthogonal polynomials \( P_I \) must be determined. Furthermore, the dispersion integrals
must be calculated for both:

\[
\frac{1}{\pi} \int d\omega \frac{\phi_I \phi_J}{E - \omega} \quad \text{and} \quad \frac{1}{\pi} \int d\omega \frac{\phi_I}{E - \omega}.
\]  (10)

However, after this preliminary work, the calculation of analytic and unitarity transition amplitudes that incorporate resonant and $t$-exchange contributions, and three-particle final states will be a straightforward problem. This work is under investigation.

**References**