Interactions in Intersecting Brane Models

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Abstract

We discuss tree level three and four point scattering amplitudes in type II string models with matter fields localized at the intersections of D-brane wrapping cycles. Using conformal field theory techniques we calculate the four fermion amplitudes. These give "contact" interactions that can lead to flavour changing effects. We show how in the field theory limit the amplitudes can be interpreted as the exchange of Kaluza-Klein excitations, string oscillator states and stretched heavy string modes.

1 Introduction

The discovery of D-branes as non-perturbative objects in any theory of open and closed strings has led to great progress in our understanding of the structure of string theory. Furthermore, with the correspondence between D-branes and gauge theories their status as objects of fundamental importance to string phenomenology has been well established.

The requirement of chirality has led to a number of scenarios involving D-branes. These include D-branes on orbifold singularities [?], and more recently intersecting brane models [?, ?], which exploit the fact that chiral fermions can arise at brane intersections [?]. The spectrum is then determined by the intersection numbers of the D-branes which wrap some compact space. This gives a simple topological explanation of family replication which is rather attractive.

The success of the intersecting brane scenario in producing semi-realistic models has been well documented, for example refs.[?, ?, ?, ?, ?, ?, ?, ?]. It is possible to construct models similar to the standard model [?, ?, ?, ?, ?, ?] and also models with N=1 supersymmetry [?, ?, ?, ?], although this latter possibility is more difficult to achieve. Recently, attention has been diverted to a more detailed analysis of the phenomenology of such models [?, ?]. In particular, computation of Yukawa couplings [?, ?, ?] and flavour changing neutral currents [?]. This paper details the computation of the three point and four point functions of string states localised at intersecting branes. These results allow the estimation of other important constraints on viable configurations of intersecting branes.

Much of our analysis will be aided by the technology discussed in [?, ?] for closed strings on orbifolds. This is due to an analogy between twisted closed string states on orbifolds and open strings at brane intersections. This analogy is the subject of our first section. As a
warm up we then proceed to a determination of the classical part of the three point function, for which the computation runs along similar lines to the closed string cases in [?, ?]. The complete calculation of the four point function follows. We will consider the four point function on only 2 sets of intersecting branes, so that the classical part is actually somewhat easier to deal with than the three point function. The quantum part we evaluate using CFT techniques.

2 Remarks on closed and open string twisted states

In order to calculate the three point and four point functions, we will make use of the analogy between open string states that are stretched between branes at angles and closed string twisted states on orbifolds. This analogy allows us to make use of the CFT technology first discussed in ref.[?, ?]. Hence, this first section will be devoted to a description of this correspondence.

Let us first consider string states at an intersection with branes at right angles, in a pair of dimensions $X_1, X_2$. The set up is as shown in figure 1. We will refer to the complex world sheet coordinate as $\tilde{z}$ and map the space-time coordinates onto it directly, $X_1 + iX_2 = \tilde{z}$. Consider the Green function at a point in the plane for $X_1$. To calculate this we can use the method of images as shown in the figure. The boundary condition is Neumann on the $X_1$ axis and Dirichlet on the $X_2$ axis, so to take account of this we need 3 images with the Green functions of those at negative $X_2$ being added negatively. The total Green function is given by

$$G(\tilde{z}, \tilde{w}) = G_0(\tilde{z}, \tilde{w}) + G_0(\tilde{z}, \overline{\tilde{w}}) - G_0(\tilde{z}, -\overline{\tilde{w}}) - G_0(\tilde{z}, -\tilde{w})$$

(1)

$$= -\frac{\alpha'}{2} \ln |\tilde{z} - \tilde{w}| |\tilde{z} - \overline{\tilde{w}}|$$

(2)

Now map the world-sheet to the usual upper half plane by making the conformal transformation $z^2 = \tilde{z}$. In these coordinates the Green function becomes

$$G(z, w) = -\frac{\alpha'}{2} \ln \frac{\sqrt{z} - \sqrt{w}}{|\sqrt{z} + \sqrt{w}|} - \frac{\alpha'}{2} \ln \frac{|\sqrt{z} - \sqrt{w}|}{|\sqrt{z} + \sqrt{w}|},$$

(3)
which can be recognized as the Green function of the original point in the upper half plane, plus its image at \( \overline{w} \) in the lower half plane, in the presence of a \( Z_2 \) twist operator, \( \sigma_+ \), placed at the origin [7, 7, 7]. Consequently vertex operators involving states with ND boundary conditions must include such operators. Note that if the string has a Neumann condition at both ends (e.g. if only one end was attached to the \( X_1 \) brane with the second end free) we would add all the images and find the usual untwisted Green function

\[
G(z, w) = -\frac{\alpha'}{2} \ln |z - w| - \frac{\alpha'}{2} \ln |z - \overline{w}|. \tag{4}
\]

For more general angles one expects the appropriate twisted Greens functions to appear. Indeed, writing the complex boson as \( X = X_1 + iX_2 \), we can proceed in a similar manner, as shown in figure 2 for \( \pi/3 \). The total contribution to the Green function is given by the sum of the images;

\[
G(\hat{z}, \hat{w}) = \sum_{l=0}^{m-1} \rho^l G_0(\hat{z}, \rho^l \hat{w}) + \rho^l G_0(\hat{z}, \rho^l \overline{\hat{w}}) \tag{5}
\]

where \( \rho = e^{2\pi i/m} \) and the intersection angle is taken to be \( \pi/m \). Again transforming to the upper half plane with \( z = \hat{z}^m \) we find

\[
G(z, w) = \sum_{l=0}^{m-1} \rho^l G_0(z^{\frac{1}{m}}, \rho^l w^{\frac{1}{m}}) + \rho^l G_0(z^{\frac{1}{m}}, \rho^l \overline{w}^{\frac{1}{m}}) \tag{6}
\]

(where the \( \frac{1}{m} \) roots are the trivial ones) which we recognize as the Green function of the original point in the upper half plane, plus its image at \( \overline{w} \) in the lower half plane, in the presence of a \( Z_m \) twist operator, \( \sigma_+ \), at the origin [7].

This is of course entirely consistent with the fractional mode expansion of string states stretched between branes. In particular, consider a string stretched between two D-branes
intersecting at an angle \( \pi \hat{\theta} \), as depicted in Figure 3. The boundary conditions are,

\[
\begin{align*}
\partial_x X^2(0) &= \partial_x X^1(0) = 0, \\
\partial_x X^1(\pi) + \frac{\partial_x X^2(\pi)}{\cot(\pi \hat{\theta})} &= 0, \\
\partial_x X^2(\pi) - \frac{\partial_x X^1(\pi)}{\cot(\pi \hat{\theta})} &= 0,
\end{align*}
\]

which determine the holomorphic solutions to the string equation of motion to be,

\[
\begin{align*}
\partial X(z) &= \sum_k \alpha_{k-\theta} z^{-k+\theta-1}, \\
\partial \bar{X}(z) &= \sum_k \bar{\alpha}_{k+\theta} z^{-k-\theta-1}.
\end{align*}
\]

We can define the worldsheet coordinate, \( z = -e^{\tau - i \sigma} \), which has domain the upper-half complex plane. This can be extended to the entire complex plane using the ‘doubling trick’, i.e. we define,

\[
\partial X(z) = \left\{ \begin{array}{ll} \\
\partial X(z) & \text{Im}(z) \geq 0 \\
\partial \bar{X}(z) & \text{Im}(z) < 0,
\end{array} \right.
\]

and similarly for \( \partial \bar{X}(z) \).

The mode expansion in (8) is identical to that of a closed string state in the presence of a \( \mathbb{Z}_N \) orbifold twist field (with the replacement \( \frac{1}{2} = \hat{\theta} \)). Thus an open string stretched between intersecting D-branes is analogous to a twisted closed string state on an orbifold and to take account of this we introduce a twist field \( \sigma_\theta(w, \bar{w}) \) that changes the boundary conditions of \( X \) to those of eq.(7), where the intersection point of the two D-branes is at \( X(w, \bar{w}) \). Then, in an identical manner to the closed string case, we obtain the OPEs,

\[
\begin{align*}
\partial X(z) \sigma_\theta(w, \bar{w}) &\sim (z - w)^{-\langle 1 - \theta \rangle} \tau_\theta(w, \bar{w}), \\
\partial \bar{X}(z) \sigma_\theta(w, \bar{w}) &\sim (z - w)^{-\theta} \bar{\tau}_\theta(w, \bar{w}),
\end{align*}
\]

where \( \tau_\theta \) and \( \bar{\tau}_\theta \) are excited twists. Also, the local monodromy conditions for transport around \( \sigma_\theta(w, \bar{w}) \) are,

\[
\begin{align*}
\partial X(e^{2\pi i}(z - w)) &= e^{2\pi i \theta} \partial X(z - w), \\
\partial \bar{X}(e^{2\pi i}(z - w)) &= e^{-2\pi i \theta} \partial \bar{X}(z - w).
\end{align*}
\]

The mode expansion for \( X \) is then,

\[
X(z, \bar{z}) = \sqrt{\frac{\alpha'}{2}} \sum_k \left( \frac{\alpha_{k-\theta}}{k-\theta} z^{-k+\theta} + \frac{\bar{\alpha}_{k+\theta}}{k+\theta} \bar{z}^{-k-\theta} \right),
\]

Figure 3: A ‘twisted’ open string state
with the right and left moving modes being mapped into upper and lower half planes. A similar mode expansion is obtained for the fermions with the obvious addition of $\frac{1}{2}$ to the boundary conditions for NS sectors.

3 Three point functions

Now we proceed to calculate the classical part of the three point function, which in particular includes the yukawa interactions. We will consider the case of a compactified space which is factorizable into 2 tori, $T_2 \times T_2 \times T_2$, and in which branes $A, B, C$ wrap one cycles $L_{A,B,C}$. As shown in figure 3, the string states are localised at the vertices of a triangle whose boundary consists of the D-branes which wrap the internal space. One would expect the amplitude to be dominated by an instanton, and to be proportional to $e^{-\frac{A}{g_s} \sigma}$, where $A$ is the area of the triangle worldsheet (also, due to the toroidal geometry we would expect a contribution from wrapped triangles). This expectation is born out in the following calculation.

Denote the spacetime coordinates of our torus subfactor by $X = X^1 + iX^2$ and $\bar{X} = X^1 - iX^2$. The bosonic field $X$ can be split up into a classical piece, $X_{cl}$, and a quantum fluctuation, $X_{qu}$. The amplitude then factorises into classical and quantum contributions,

$$Z = \sum_{\langle X_{cl} \rangle} e^{-S_{cl} Z_{qu}}.$$  

(13)

$X_{cl}$ must satisfy the string equation of motion and possess the correct asymptotic behaviour near the triangle vertices.

The three point function requires 3 twist vertices, $\sigma_{\partial_i}(z_i, \bar{z}_i)$, corresponding to the three relevant intersections of the D-branes. These vertices are attached to the boundary of the tree-level disc diagram which can be conformally mapped to the upper-half plane. Then our classical solution $\partial X_{cl}$ is determined, up to a constant, by its holomorphicity and asymptotic behaviour at each D-brane intersection, which is given by the OPEs in eq.(10). Hence, mapping the spacetime coordinates $X$ and $\bar{X}$ into the worldsheet coordinate $z$, we obtain,

$$\frac{\partial X_{cl}}{\partial X_{cl}}(z) = a\omega(z), \quad \frac{\partial \bar{X}_{cl}}{\partial \bar{X}_{cl}}(z) = \bar{a}\omega'(z),$$

$$\frac{\partial \bar{X}_{cl}}{\partial \bar{X}_{cl}}(\bar{z}) = b\omega'(\bar{z}), \quad \frac{\partial \bar{X}_{cl}}{\partial \bar{X}_{cl}}(\bar{z}) = \bar{b}\omega(\bar{z}),$$  

(14)

where,

$$\omega(z) = \prod_{i=1}^{3}(z - z_i)^{-(1-\theta_i)}, \quad \omega'(z) = \prod_{i=1}^{3}(z - z_i)^{-\theta_i}.$$  

(15)
and $a, \bar{a}, b, \bar{b}$ are normalisation constants. Here we have used the doubling trick which also requires that $a^* = \bar{b}$ and $\bar{a} = b^*$ (upto a phase). We can therefore write,

$$S_{cl} = \frac{1}{4\pi\alpha} \left( |a|^2 \int d^2z |\omega(z)|^2 + |\bar{a}|^2 \int d^2z |\omega'(z)|^2 \right),$$

(16)

The contribution to $S_{cl}$ from $|\omega'(z)|$ diverges, hence we set $\bar{a} = 0$.

The normalisation constants are determined from the global monodromy conditions, i.e. the transformation of $X$ as it is transported around more than one twist operator, such that the net twist is zero. We determine this from the action of a single twist operator, $\sigma_\theta(w, \bar{w})$. In the closed string case this field acts to rotate and shift $X(z, \bar{z})$;

$$X(e^{2\pi i z}, e^{-2\pi i \bar{z}}) = \theta X(z, \bar{z}) + v$$

(17)

where $\theta$ is a complex phase factor with phase $\pi\theta$, and $v$ is over a coset of the toroidal compactification lattice $A$ which depends on the fixed point $f$;

$$v = (1 - \theta)(f + \Lambda).$$

(18)

Ignoring the lattice for the moment, the equivalent statement for open strings at intersections is that

$$X(e^{2\pi i z}, e^{-2\pi i \bar{z}}) = e^{2\pi i \theta} X + (1 - e^{2\pi i \theta})f,$$

(19)

where $f$ is the intersection point of the two D-branes. This can be seen from the local monodromy conditions (11) and the fact that $f$ must be left invariant. The global monodromy of $X$ is then simply a product of such actions. On integrating around 2 twist fields, the strings are embedded in the target space as shown in figure 5. The portion of integration around each vertex takes $X(z, \bar{z})$ from one brane to another, while integrating between two vertices introduces a shift along that particular brane.

To determine the shifts, we must consider triangles that wrap the torus and whose vertices are at the same intersection points. (Other shifts would contribute to other three point functions.) Keeping $f_1$ fixed and extending the triangle as shown in figure 6, we obtain a contribution to the same three point function provided,

$$x_A = k_AL_{AB}|I_{AB}|, \quad x_C = k_AL_{CB}|I_{CB}|,$$

(20)

where $k_A, k_B \in \mathbb{Z}$, $I_{ij}$ is the intersection number of branes $i$ and $j$ and $L_{ij}$ is the displacement between successive $i,j$ intersections along the $i^{th}$ brane. Using congruency of the triangle we obtain,

$$k_A = \frac{|I_{AB}|}{\gcd(|I_{AB}|, |I_{AB}|)}, \quad k_C = \frac{|I_{AB}|}{\gcd(|I_{AB}|, |I_{AB}|)},$$

(21)

where $l \in \mathbb{Z}$. On the other hand, if we instead keep $f_2$ fixed we obtain,

$$k_A = \frac{|I_{CB}|}{\gcd(|I_{CB}|, |I_{AC}|)}, \quad k_B = \frac{|I_{AC}|}{\gcd(|I_{CB}|, |I_{AC}|)},$$

(22)

where $l'$ is a different integer. Hence our lattice shifts must be of the form,

$$x_A(l) = \frac{|I_{CB}|L_A}{\gcd(|I_{CB}|, |I_{AB}|, |I_{AB}|)},$$

(23)

and similar for $x_{B,C}$ with the obvious replacement of indices, but with the same $l$. This is similar to the case discussed in ref.[?].

Considering all closed curves, $C_i$, with net twist zero, we obtain the global monodromy conditions,

$$\Delta_{C_i} X = \oint_{C_i} dz \partial X(z) + \oint_{C_i} \bar{z} \partial \bar{X}(z) = v_i,$$

(24)
Figure 5: Transporting $X(z, \bar{z})$ around two twist fields

Figure 6: Wrapping triangles
In our case there is only one independent, net twist zero, closed curve. This is shown in figure 7. We have set \( z_1 = 0, z_2 = 1 \) and \( z_3 \to \infty \) using \( SL(2, \mathbb{R}) \) invariance and the dashed lines denote branch cuts. Evaluating the contour integral we get,

\[
\oint_C dz \omega(z) = -z_\infty^{-\vartheta_1} \sin \pi \vartheta_2 \sin \pi \vartheta_1 e^{\pi i \vartheta_1} \frac{\Gamma(\vartheta_1) \Gamma(\vartheta_2)}{\Gamma(1 - \vartheta_3)},
\]

where phases from branch cuts are determined by starting at \( S \), which we assume is in the principal branch of each factor in the integrand. The global monodromy condition (24) is independent of this convention provided we order the transformations (19) according to the path taken around the curve \( C \), again starting at \( S \). This gives,

\[
\Delta C X_{cl} = 4 \sin \pi \vartheta_1 \sin \pi \vartheta_2 e^{-i(\vartheta_1 - \vartheta_2)}(f_1 - f_2 + x_A(l)),
\]

substituting (26) and (25) into (24) we then determine,

\[
|a|^2 = |z_\infty|^{2(1 - \vartheta_3)} |f_1 - f_2 + x_A(l)|^2 \frac{\Gamma(1 - \vartheta_3)}{\Gamma(\vartheta_1) \Gamma(\vartheta_2)}.
\]

We also require the integral,

\[
\int d^2 z |\omega(z)|^2 = |z_\infty|^{-2(1 - \vartheta_3)} \sin \pi \vartheta_2 (\Gamma(\vartheta_2))^2 \frac{\Gamma(\vartheta_3) \Gamma(\vartheta_1)}{\Gamma(1 - \vartheta_3) \Gamma(1 - \vartheta_1)},
\]

which can be performed using the method of \[?] to relate open and closed string amplitudes. Finally, substituting (28) and (27) into (16) we obtain,

\[
S_{cl} = \sum_{l \in \mathbb{Z}} \frac{1}{2 \pi \alpha'} \left( \sin \pi \vartheta_1 \sin \pi \vartheta_2 \left| f_1 - f_2 + x_A(l) \right|^2 \right).
\]

This is, as expected, the area of the triangle (and wrappings) defined by the intersecting D-branes that is swept out by the string worldsheet.

![Figure 7: Closed curve with net twist zero required for global monodromy condition.](image)

4 Four point functions

Having confirmed the efficacy of the conformal field theory techniques, let us now turn to the four point function on two sets of branes intersecting at angles. Processes involving contact interactions between 4 fermions have previously only been considered for orthogonal D-branes in ref.[?]. For theories with branes at angles these processes are particularly important...
because the sector of chiral fermions is rather independent of the general set up, whereas the scalars are more model dependent and may be tachyonic. As for the orthogonal case, we shall see that the processes get contributions from intermediate Kaluza-Klein excitations, winding modes and string excitations.

The massless fermions of interest appear in the Ramond sector with charges, \( q = 0, 3 \) for the 4 complex transverse fermionic degrees of freedom given by one of the following;

\[
q = (\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\]

\[
q = (\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\]

\[
q = (\pm \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\]

depending on the type of intersection. For example \( D_6 \) branes intersecting in \( (T_2)^3 \) are of the first kind. The GSO projection leaves only one 4D spinor, and the theory is chiral. For special values of angles (0 for example) supersymmetry may be restored, but generally supersymmetry is completely broken, and the scalars can be heavy or tachyonic. Fermions for \( D_5 \) branes intersecting in \( (T_2)^2 \times C/Z_N \) are of the second kind. Initially the GSO projection will leave only half the space time spinor degrees of freedom leaving a non-chiral theory. Hence a further orbifolding on the 1st complex dimension is required to get a chiral theory. Finally \( D_4 \) branes intersecting in \( T_2 \times C_2 / Z_N \) correspond to the last choice. Again the GSO projection leaves 4 states which need to be further projected out by orbifolding in the \( C_2 \) dimensions. The particular orbifolding do not effect the Ramond charges above so the quantum part of the amplitude will be unaffected by it. The classical part of the amplitude depends purely on the world sheet areas. However as the orbifolding is orthogonal to the space in which the branes are wrapping, it will not affect the classical part either. The only affect of the orbifolding is therefore in projecting out the chiralities above. In addition we shall not consider orientifolding which does not effect the quantum part of the intraction, except in so far as it determines the gauge groups. For the classical part, one would expect complication due to the presence of image branes, but these could be treated as essentially separate branes for the present discussion. Hence our results can be easily adapted to those cases as well.

The four fermion scattering amplitude is given by a disc diagram with 4 fermionic vertex operators in the \(-1/2\) picture, \( V^{(a)} \) on the boundary. The diagram is then mapped to the upper half plane with vertices on the boundary. The positions of the vertices \( (x_1 \ldots x_4) \) will eventually be fixed by \( SL(2, \mathbb{R}) \) invariance to \( 0, x, 1, \infty \) (where \( x \) is real), so that the 4 point ordered amplitude can be written

\[
(2\pi)^4 \delta^4 \left( \sum a \right) A(1, 2, 3, 4) = -i \frac{1}{2s^4} \int_0^1 dx (V^{(1)}(0, k_1) V^{(2)}(x, k_2) V^{(3)}(1, k_3) V^{(4)}(\infty, k_4)).
\]

(31)

To get the total amplitude we sum over all possible orderings;

\[
A_{total}(1, 2, 3, 4) = A(1, 2, 3, 4) + A(1, 3, 2, 4) + A(1, 2, 4, 3)
+ \ A(4, 3, 2, 1) + A(4, 2, 3, 1) + A(4, 3, 1, 2).
\]

(32)

The vertex operators for the fermions are of the form

\[
V^{(a)}(x_a, k_a) = const \lambda^a u_a S^\alpha \prod_i \sigma_\alpha e^{-\phi/2} e^{ik_a \cdot x_a}(x_a).
\]

(33)

Here \( u_a \) is the space time spinor polarization, and \( S^\alpha \) is the so called spin-twist operator of the form

\[
S^\alpha = \prod_i : \exp(\bar{q}_i H_i) :
\]

(34)
with conformal dimension
\[ h = \sum_i \frac{q_i^2}{2}, \]
and \(\sigma_{\vartheta_i}\) is the \(\vartheta\) twist operator acting on the \(\vartheta\)'th complex dimension, with conformal dimension
\[ h_i = \frac{1}{2} \vartheta_i (1 - \vartheta_i). \]

The calculation of the 4-point function of the bosonic twist operators can be done analogously to the closed string case \[?\], with only minor modifications to take account of the image Green function and the fact that the vertices are on the real axis.

For completeness we now outline the derivation. Consider the contribution from a single complex dimension in which the branes intersect with angle \(\vartheta \pi\). We begin with the asymptotic behaviour of the Green function in the vicinity of the twist operators. As we saw earlier, we take account of the world-sheet boundary by adding an image charge. The Green function can then be written
\[ G(z, w; z_i) = g(z, w; z_i) + g(z, \overline{w}; z_i) \]
where \(g(z, w; z_i)\) is the usual Green function for the closed string. It has the following asymptotics
\[ g(z, w; z_i) \sim \begin{cases} \frac{1}{(z-w)^2} + \text{finite} & z \to w \\ \frac{1}{(z-x_{1,3})^{1-\vartheta}} & z \to x_{1,3} \\ \frac{1}{(z-x_{2,4})^{1-\vartheta}} & z \to x_{2,4} \\ \frac{1}{w-x_{1,3}} & w \to x_{1,3} \\ \frac{1}{w-x_{2,4}} & w \to x_{2,4} \end{cases} \]

and as we have seen the holomorphic fields are
\[ \partial X(z) \equiv \omega_\vartheta(z) = [(z-x_1)(z-x_3)]^{-\vartheta}[(z-x_2)(z-x_4)]^{-(1-\vartheta)} \]
\[ \partial \overline{X}(z) \equiv \omega_{1-\vartheta}(z) = [(z-x_1)(z-x_3)]^{-(1-\vartheta)}[(z-x_2)(z-x_4)]^{-\vartheta}. \]

This half of the Green function may now be determined up to an additional term usually denoted \(A\) by inspection;
\[ g(z, w; z_i) = \omega_\vartheta(z)\omega_{1-\vartheta}(w) \left\{ \frac{\vartheta(z-x_1)(z-x_3)(w-x_2)(w-x_4)}{(z-w)^2} + (1-\vartheta) \frac{(z-x_2)(z-x_4)(w-x_1)(w-x_3)}{(z-w)^2} + A \right\}. \]

Next one considers
\[ \frac{\langle T(z)\sigma_\vartheta\sigma_\vartheta\sigma_\vartheta \rangle}{\langle \sigma_\vartheta\sigma_\vartheta\sigma_\vartheta \rangle} = \lim_{w \to z} [g(z, w) - \frac{1}{(z-w)^2}] \]
\[ = \frac{1}{2} \vartheta(1-\vartheta) \left( (z-x_1)^{-1} + (z-x_3)^{-1} - (z-x_2)^{-1} - (z-x_4)^{-1} \right)^2 \]
\[ + \frac{A}{(z-x_1)(z-x_2)(z-x_3)(z-x_4)} \]
where for shorthand we denote $\sigma_\pm \equiv \sigma_{\vartheta \pm 1}$, and compares it to the OPE of $T(z)$ with the twist operator

$$T(z) \sigma_+(x_2) \sim \frac{h_1}{(z - x_2)^2} + \frac{\partial_x \sigma_+(x_2)}{(z - x_2)} + \ldots.$$  (42)

Equating coefficients of $(z - x_2)^{-1}$, and then using $SL(2, R)$ invariance to fix $(x_1, x_2, x_3, x_4) = (0, x, 1, x_\infty)$ yields a differential equation for $\langle \sigma_- \sigma_+ \sigma_- \rangle$ of the form

$$\partial_x \ln \left[ x_\infty^{\vartheta(1-\vartheta)} \langle \sigma_- \sigma_+ \sigma_- \rangle \right] = \partial_x \ln \left[ (x(1-x))^{-\vartheta(1-\vartheta)} \right] - \frac{A(x)}{x(1-x)}$$  (43)

where

$$A(x) = -x_\infty^{-1} A(0, x, 1, x_\infty).$$  (44)

All that remains is to determine $A(x)$ which can be done using monodromy conditions for $\partial_x X \partial_\vartheta X$. We proceed along the same lines as in the three point calculation and consider the two independent loops $C_i$ around combinations of twists that add up to zero (again using the doubling trick), applying the global monodromy conditions (24). On encircling a twist $\sigma_f^\pm$, associated with states at intersection $f_1$, plus antitwist, $\sigma_{f_i}^-$, at intersection $f_2$ the quantum part of $\partial X$ should be invariant. There are two independent pairs of twists and integrating around the the corresponding closed loops yields

$$A(x) = \frac{x(1-x)}{2} \partial_x \ln [F(x) F(1-x)]$$  (45)

where $F(x)$ is the hypergeometric function

$$F(x) = F(\vartheta, 1 - \vartheta; 1; x) = \frac{1}{\pi} \sin(\vartheta \pi) \int_0^1 dy y^{-\vartheta} (1 - y)^{-(1-\vartheta)} (1 - xy)^{-\vartheta}.$$  (46)

Inserting this into eq.(43), we find a contribution from the product of $\vartheta_i$ twisted bosons of

$$\prod_i (\sigma_+(x_\infty) \sigma_-(1) \sigma_+(x) \sigma_-(0)) = \text{const} \left[ x_\infty x(1-x) \right]^{-\vartheta(1-\vartheta)} [F(1-x), F(x)]^{1/2}.$$  (47)

In this expression we use a dot to indicate the product of the contributions from each complex dimension. When we collect all the contribution together the dependence on $\vartheta_i$ cancels between the bosonic twist fields and the spin-twist fields giving

$$A(1, 2, 3, 4) = -g_{\alpha'}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 + \lambda^4 \lambda^3 \lambda^1) \int_0^1 dx x^{-1-\alpha'(1-x)^{-1} - \alpha'} \times \left[ \Pi(1) \Pi(4) \Pi(3) \right] \sum e^{-S_{\vartheta}(x)} \frac{1}{[F(1-x), F(x)]^{1/2}}$$  (48)

where $s = -(k_1 + k_2)^2$, $t = -(k_2 + k_3)^2$, $u = -(k_1 + k_3)^2$ are the usual Mandelstam variables. We can check this expression against that in ref[?] where the branes considered were orthogonal D7 branes and D3 branes. The quantum part of the above expression with intersections in two complex subplanes should agree with that from four ND fermions attached to two sets of coincident D3 branes and a stack of D7 branes. We find agreement upto a factor of $F(1-x)/F(x)$ in the integrand.

We now turn to the classical contribution to the four point function which can be calculated as for the three point function. On integrating around the closed loops we again can have arbitrary shifts $v_i$ along the D-branes. Consider encircling a twist $\sigma_{f_i}^+$ associated with states at intersection $f_1$, plus antitwist, $\sigma_{f_i}^-$, at intersection $f_2$. The corresponding open strings can encircle one of the branes appearing at the intersection determined by the relative positions of $\sigma_{f_i}^+$ and $\sigma_{f_i}^-$ with respect to the branch cuts in the field $X$. The two
Figure 8: Joined pairs of twisted/antitwisted states.

possibilities are shown in figure 8. A twisted state at $f_2$ followed by an anti-twisted state at $f_1$ corresponds to a shift $(1 - \theta)\left(f_1 - f_2 + \Lambda_A\right)$ where $\Lambda_A$ is an integer number of shifts along the $A$-brane. Note that $f_1 - f_2$ is in this case a vector along the $A$ brane as well. An anti-twisted state at $f_1$ followed by twisted state at $f_2$ causes a shift $(1 - \theta)(f_1 - f_2 + \Lambda_B)$ where now $\Lambda_B$ is an integer shift along the $B$ brane and $f_1 - f_2$ is now the displacement between the points along the $B$-brane. The vertex ordering in the disc diagram means that in the amplitude the factor $\sum e^{-S_{cl}}$ now splits into separate sums over $\Lambda_A$ or $\Lambda_B$, $\sum \equiv \sum_{\Lambda_A,\Lambda_B}$. The classical action is then found to be

$$S_{cl} = \frac{\sin \vartheta}{4\pi \alpha' \tau} \left(|v_A'|^2 + \tau^2|v_B'|^2\right),$$

where in the sums over $\Lambda_A$ we have

$$v_{A,B}' = \Delta_{A,B}^i f + nL_{A,B},$$

Here $n \in \mathbb{Z}$, and $L_{A,B}$ are vectors in the two torus describing the wrapped D-branes. We have defined $\Delta_{A,B}^i$ to be the displacement between the intersections involved in the $C_i$ loop taken along the $A, B$ brane, and we include both combinations (i.e. $i = 1, 2$ going respectively with $A, B$ or $B, A$). In addition we have defined $\tau(x) = \frac{F(1-x)}{F(x)}$ for convenience. (For $Z_2$ twists, i.e. intersections at right-angles, this would be the modular parameter of a “fake” annulus but it has no such interpretation for more general intersections.)

To illustrate, consider the generic open string four point diagram, shown in fig 9. In this case one expects the action to go as the area. To show this we can use a saddle point approximation for the $x$ integral in $A(1, 2, 3, 4)$. This gives $\tau(x) = |v_A'|/|v_B'|$ where, for strings stretched between $f_1$ and $f_2$ propagating along the $A$-brane direction we choose $\Delta_A f = f_3 - f_2$ and $\Delta_B f = f_1 - f_2$. This is the leading contribution, but there is an additional contribution to the amplitude where the displacement is $\Delta_A f = f_1 - f_2$ measured along the
$A$-brane and $\Delta_B f = f_2 - f_3$ measured along the $B$-brane. This corresponds to diagrams where a string stretched between $f_2$ and $f_3$ propagates in the $f_1 - f_2$ direction, as shown in fig.10.

The accuracy of the saddle approximation is a function of the width of the saddle, given by $\sqrt{4\pi \alpha'/ R_c} \sim \frac{1}{R_c}$, where $R_c$ is the compactification scale ($R_c \sim L_A, L_B$). Thus, as expected the approximation of world-sheet instanton suppression breaks down when the size of the world sheet is comparable to the D-brane thickness. Substituting back into the action we find

$$A(1, 2, 3, 4) \sim \sum_{\Lambda_A, \Lambda_B} \text{fermion factors} \times \left( \frac{4\pi \alpha'}{R_c} \right)^2 \exp \left( -\frac{1}{2\pi \alpha'} \sin \vartheta \pi |v'_A||v'_B| \right). \quad (51)$$

We find the expected suppression from the mass of the intermediate stretched string state, times by an instanton suppression factor again given by the area of the world sheet. As pointed out in ref.[?] diagrams such as this can lead to large flavour changing processes such as $\tau \mu \rightarrow \mu e$ if the compactification scale is too small.

Diagrams which do not explicitly violate flavour, such as $\mu^+ \mu^- \rightarrow e^+ e^-$ cannot be completely approximated this way. For such processes the intersection separation in the leading term $v'_B = \Delta_B f_{ee}$ for one pair of twist operators is zero, and so this term cannot be treated as above. Consider the summation over $\Lambda_A$ in $v'_A$ for the other pair, whose separation is $v'_A = \Delta_A f_{ee} + n L_A$. Poisson resumming we find

$$\sum e^{-S_{cl}} \equiv \sum_{\Lambda_A \in \Lambda_A^*} \sqrt{\frac{4\pi^2 \alpha' \tau}{L_A^2 \sin \vartheta \pi}} \exp \left[ -\frac{4\pi^2 \alpha' \tau}{\sin \vartheta \pi} p_A^2 \right] \exp \left[ 2\pi i \Delta f_{ee} - p_A \right] + \text{subleading} \quad (52)$$

where $p_A \in \Lambda_A^*$ is summed over the dual lattice;

$$p_A = \frac{n_A}{|L_A|^2} L_A. \quad (53)$$
Figure 10: The first subleading contribution to the same amplitude.

This expression describes the leading exchange of gauge bosons plus their KK modes along the A brane. (The subleading terms (those with $v_B' = n_B L_B$ with integer $n_B \neq 0$) can still be treated using the saddle point approximation above.) To obtain the field theory result, we take the limit of coincident vertices; $x \to 0$ or $x \to 1$. For example the former contribution can be evaluated using the asymptotics

$$F(x) \sim 1, \quad \tau \sim F(1-x) \sim \frac{1}{\pi} \sin \vartheta \pi \ln \frac{\delta}{x}$$

(54)

where $\delta$ is given by the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$;

$$\delta = \exp(2\psi(1) - \psi(\vartheta) - \psi(1 - \vartheta)).$$

(55)

We find

$$A(1, 2, 3, 4) = g_s \left[ \pi(1) \gamma_{\mu} u(2) \pi(4) \gamma_{\nu} u(3) \right] \frac{2\pi \sqrt{\alpha'}}{L_A \sqrt{\sin \vartheta \pi}}$$

$$\times \left( \frac{1}{s} + 2 \sum_{n=1}^{\infty} \frac{\cos (2\pi \Delta f_{\nu} p_n) \delta^{-\alpha'M_n^2}}{s - M_n^2} \right),$$

(56)

(57)

where $M_n = n/2\pi L_A$, and $p_n = nL_A/L_A$. This result agrees with a naive field theory calculation as shown in ref[?], provided that $\alpha'M_n^2 \ll 1$. (That is, the brane separation should again be larger than the brane thickness.) This can lead to severe flavour changing effects due to the KK mode contribution which is flavour non-universal. We should note here that this is not in contradiction with the fact that there is an overall translational $U(1)$ symmetry on the torus, since the result is a function of the relative displacement. The leading term corresponds to massless gauge boson exchange so that we can normalize the coupling to the measured Yang-Mills couplings in the obvious way.
The extension to higher dimensional intersecting branes follows straightforwardly, and we now find that the form factor $\delta^{-\alpha' M_{\text{R}}^2}$ naturally provides the UV cut-off which in the field theory has to be added by hand. Physically the cut-off arises because the intersection itself has thickness $\sim \sqrt{\alpha'}$, and thus cannot emit modes with a shorter wavelength.

5 Conclusion

In summary, we have discussed the calculation of three point amplitudes for open strings localised at D-brane intersections. We computed the four point “contact interaction” at tree level for diagrams involving fermions living at two sets of intersecting D-branes. For the four point functions we were able to adapt the techniques of ref.[?] to the open string case, so that by considering the conformal field theory we obtained the required information to compute the entire contact interaction. It receives contributions from both KK modes and heavy string modes. The obvious extension of this work would be to calculate the four point functions on three sets of intersecting branes. This would allow us to factorize the four point functions on the three point Yukawa couplings yielding additional information about the latter. In addition we considered only the simplest kind of toroidal compactification, and it would be interesting to contemplate the same process in more complicated set-ups, and also to apply it to a “realistic” SM-like model.

More generally these calculations may be used to discuss or support more generic field theoretic ideas in set-ups with fermions localized in extra dimensions. For example, in a parallel work [?], it has been demonstrated in that very low (TeV) string scales are incompatible with the experimental absence of FCNCs. When the compactification scale is small we encounter subleading flavour changing diagrams involving stretched strings. When it is large it is instead the KK modes which contribute to FCNCs. Taken together these two contributions constrain the string scale. There are many conceptual problems when discussing that type of effect that cannot be addressed in field theory. For example, sums over Kaluza-Klein modes can diverge, so that it is generally necessary to provide a UV cut-off. As is well known, in the string calculation such a divergence should always be naturally regulated at the string scale, and this indeed happens automatically for the interactions discussed here. This has already been seen in the literature in for example [?].

Note Added

Whilst this paper was in the final stages of preparation we received ref.[?] which also discusses the calculation of both 3 and 4 point functions.

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