Dynamical analysis of the nonlinear growth of the $m = n = 1$ resistive internal mode

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(Dated: June 6, 2005)

A dynamical analysis is presented that self-consistently takes into account the motion of the critical layer, in which the magnetic field reconnects, to describe how the $m = n = 1$ resistive internal kink mode develops in the nonlinear regime. The amplitude threshold marking the onset of strong nonlinearities due to a balance between convective and mode coupling terms is identified. We predict quantitatively the early nonlinear growth rate of the $m = n = 1$ mode below this threshold.

PACS numbers: 52.30.Cv, 52.35.Py, 52.35.Mw, 52.55.Tn

The large scale dynamics and confinement properties of tokamak plasmas depend intimately on the behavior of $m = n = 1$ magnetohydrodynamic (MHD) internal kink modes. This has motivated an intense, long-lasting, experimental and theoretical research, notably devoted to study their implication in magnetic reconnection or as triggers of the sawtooth oscillations and crashes. These phenomena typically proceed beyond the linear regime, that is now rather well understood but assumes very small amplitudes of the modes. To offer a quantitative, predictive description of their nonlinear manifestations remains a difficult objective of both academic interest and very practical importance. This is especially relevant for the design of fusion burn experiments in which the fulfillment of linear stability constraints is challenged by the search for ignition. Such devices are thus expected to operate at best close to marginal stability for the $m = n = 1$ ideal mode so that nonlinear effects come into play for fairly small values of the mode amplitude $\eta$. In this Letter, we focus on the $m = n = 1$ resistive mode in which a finite resistivity $\eta$ destabilizes the otherwise marginally stable ideal MHD internal kink mode. Since Kadomtsev’s scenario predicting the complete reconnection of the helical flux within the $q = 1$ surface on a timescale of order $\eta^{-1/2}$, that later appeared too large to account for observations, the nonlinear behavior of the $m = n = 1$ mode has become a somewhat controversial issue. Some numerical simulations suggested that the mode still grows exponentially into the nonlinear regime which was supported by a theoretical model. Later some analytic studies, supported by numerical simulations, rather predicted a transition to an algebraic growth early in the nonlinear stage. This result was challenged by Aydemir’s recent simulations using a dynamical mesh. These did show the linear exponential stage evolving towards an algebraic stage, yet this was brutally interrupted by a second nonlinear exponential growth. A modified Sweet-Parker model was able to fit continuously both stages of evolution and the transition related to a change in the geometry of the current sheet. However, some fundamental questions remain unanswered or unclear. Among them, how to relate the transition threshold with $\eta$? or what is the role of the $q$-profile? The aim of this Letter is to describe analytically how the $m = n = 1$ resistive mode develops in the nonlinear regime, by focusing on the equations controlling plasma dynamics.

We consider the low-$\beta$ reduced MHD equations

$$\frac{\partial U}{\partial t} = [\phi, U] + [J, \psi]$$

$$\frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta(J - J_0)$$

assuming helical symmetry. Only a single angular variable is then involved in the problem, namely the helical angle $\alpha = \varphi - \theta$, with $\varphi$ the toroidal and $\theta$ the poloidal angles. $U = \nabla_r^2 \phi$ is the vorticity and $J = \nabla_r^2 \psi$ the helical current density, with $\nabla_r^2 = -r^{-1}\partial_r r \partial_r + r^{-2}\partial_\theta^2$. Time is normalized by the poloidal Alfvén time ($t \rightarrow t/\tau_{H_P}$), the radial variable by the minor radius ($r \rightarrow r/a$) and $\eta$ is the dimensionless resistivity, inverse of the magnetic Reynolds number $S (\eta \equiv S^{-1} = \tau_{H_P}/\tau_R)$ with the poloidal Alfvén time $\tau_{H_P} = (\mu_0 \rho_0)^{1/2} R/B_0^2$ and resistive time $\tau_R = \mu_0 a^2/\eta_0$. The Poisson brackets are defined by $[\phi, U] = -\dot{\phi} \cdot (\nabla_{\perp} \phi \times \nabla_{\perp} U) = r^{-1} (\partial_r \phi \partial_\theta U - \partial_r U \partial_\theta \phi)$, $\phi$ and $\psi$ are the plasma velocity and helical magnetic field potentials expressed in cylindrical coordinates, so that the velocity is $\mathbf{v} = \hat{\phi} \times \nabla_{\perp} \phi$ and the magnetic field is $\mathbf{B} = B_0 \hat{\phi} + \hat{\psi} \times \nabla_{\perp} (\psi - r^2/2)$.

We consider MHD equilibria given by $\phi_0 = 0$ and by an helical magnetic flux $\psi_0 (r)$, related to the safety profile $q(r)$ through $d_r \psi_0 = r [1 - 1/q(r)]$, such that $q = 1$ for an internal radius $r = r_s$. Thus $d_r \psi_0 (r_s) = 0$. This means that the low-frequency ideal linear equations associated to are singular at $r = r_s$, with a formally diverging current density. This marks the presence of a critical layer in which the dynamics differs considerably from the outer one and where resistivity enters to cure the singularity.

We wish to analyse perturbatively the time evolution of the $m = 1$ mode. For this, we assume that only the $m = 1$ mode is destabilized initially with an amplitude $A_0$, neglect all ideal MHD transients and restrict to the linear resistive timescale $\tau \equiv \eta^{1/3} t$. We do not consider the somehow ill-posed, singular limit $\eta \rightarrow 0$, but instead
realize that two small parameters are indeed competing in this problem, namely the small \( \eta \) and the time-dependent amplitude \( A(\tau) \) of the linear \( m = 1 \) mode. This introduces some subtleties in the amplitude expansion. The order one solution is given by linear theory using an asymptotic analysis to match inner and outer solutions. Excitation of the \( m = 1 \) mode leads to a self-consistent correction to the location of the critical layer.

One estimates the amplitude threshold, scaling with \( \eta \), at which next order solution is required and the procedure iterated. Separability in time and space propagates at each order resulting in an amplitude expansion in \( \eta \). As in any perturbative approach, the solution is formally known when the order one solution is. This is given by the linear theory reviewed now.

Let \( f^{(m)} \) be the projection on \( \exp(i\alpha \tau) \) of any function \( f \) at order \( n \). In the inner resistive layer, Eqs. 1-2 read

\[
\begin{align*}
\left[ \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2} \right] \psi_1^{(1)}(x, \alpha, \tau) &+ i \kappa_0 x \frac{\partial^2}{\partial x^2} \psi_1^{(1)}(x, \alpha, \tau) = 0 \\
\left[ \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2} \right] \phi_1^{(1)}(x, \alpha, \tau) &+ i \kappa_0 x \phi_1^{(1)}(x, \alpha, \tau) = 0
\end{align*}
\]

where we define \( \kappa_0 \equiv \psi_0''(r_{s\theta})/r_{s\theta} \). In these equations, \( x \) is the stretched coordinate \( x = (r - r_{s\theta})/w \) and \( w \equiv \eta^{1/3} \) the magnitude of the width of the critical layer giving the maximal resistive ordering in \( x \). In the layer, radial derivatives are large, since \( \partial_r = w^{-1} \partial_x \) and \( \partial_x = \partial_\alpha - w^{-1} (\partial_{r_\alpha}/\partial_\alpha) \partial_x \) are then replaced

\[\kappa_0 \equiv \psi_0''(r_{s\theta})/r_{s\theta} \]

The second order critical layer equations involve an inhomogeneous part composed of quadratic terms in the order one solutions \( A(\tau) \) and \( A(\tau) \). This acts to force the growth \( \kappa_0 \), \( \kappa_2 \) and \( 1 \) solutions \( \psi_1^{(1)}, \phi_1^{(1)} \). The validity threshold of the second order solution is reached when the instantaneous critical line moves out of the critical layer of width \( w \) centered on \( r_{s\theta} \). This corresponds to \( r_{s\theta} \) when \( \eta = \eta^{1/3} \), which gives \( A(\tau) = O(\eta^{1/3}) \). This threshold in the amplitude of the linear \( m = 1 \) mode marks the onset of third order terms, that will contribute again to the \( m = 1 \) dynamics. Its brutal manifestation is visible on Aydemir’s plots. They clearly report a transition in the \( m = 1 \) kinetic energy when this becomes of order \( \eta^{1/2} \), namely around \( 5 \times 10^{-8} \) for \( \eta = 10^{-7} \) and around \( 5 \times 10^{-8} \) for \( \eta = 10^{-5} \).
where the initial value of the growth rate $\gamma(0) = \gamma_L$ and where the early time dependence of $\gamma$ comes from the motion of the critical layer and is computed quantitatively below. In Eq. (11), $c$ is a constant of order one and $t_{NL}$ denotes the (magnitude of the) time at which $A$ becomes of order $\eta^{1/2}$. Eq. (10) describes effectively the transition between two (almost) exponential stages. Because $\phi_3^{(1)}$ and $\psi_3^{(1)}$ are zero at the onset of the third order regime, Eq. (10) remains valid during some stage even if the structure and scaling of the critical layer should substantially change as the generalized linear stage is left.

For the convective exponential stage to be fully valid, the overlap between the linear and instantaneous critical layers should be large enough. One expects then a qualitatively different late behavior of the $m$ = 1 dynamics if the $x$-point region is far away from the linear layer when $A(\tau) = O(\eta^{1/2})$, that is, due to (3), if $\eta^{-1/6} \gg 1$. This regime is extremely challenging to reach numerically but may be satisfied in tokamak plasmas.

We finally examine the early nonlinear effects on the growth rate of the $m$ = 1 mode due to the motion of the critical layer. This amounts to solve the system of differential equations (3)-(4) for $A(\alpha, \tau)$ and modify the magnitude of linear terms is $\sim \eta L$, equal, by continuity, to the growth rate of the linear terms, there is one condition shared with the linear derivation: for a solution in separate variables $x(\alpha, \tau)$, it is that $\gamma(\tau) / \kappa(\tau)$ be constant. This constant is then fixed by continuity with the linear solution at time zero giving

\begin{equation}
\frac{\gamma(\tau)}{\kappa(0)}(\tau) = \frac{\gamma_L}{\kappa_0} = \kappa_0^{-1/3}.
\end{equation}

Here one implicitly assumes that the spatial part of the linear eigenfunctions remains valid [14]. The instantaneous critical radius is $r_s(\alpha, \tau) = r_s(0) + \eta^{1/3} x_s(\alpha, \tau)$ where $x_s(\alpha, \tau)$ is given by the approximate expression

\begin{equation}
x_s(\alpha, \tau) = H^{-1} \frac{A(\tau) \kappa_0^{1/3} \cos \alpha}{\eta^{1/3} \sqrt{2} \psi_0(r_s(0))}.
\end{equation}

$H^{-1}$ denotes the inverse of the monotonously growing function defined by $H(x) \equiv x / g_L(\kappa_0^{1/3} x / \sqrt{2})$. Due to the asymmetric nature of the $m$ = 1 resistive eigenfunctions (12), $H^{-1}(x)$ is very asymmetric, grossly equal to $x$ below $x = 0$ and exponentially small above. This confers a much more important weight on negative arguments of $H^{-1}$ than on positive ones in the averaging (12). The magnetic island has thus a higher effective contribution to the early nonlinear correction of the growth rate than the region of x-point. A rough estimate of the angular average of $x_s$ is given by $x_s(0)(\tau) \approx$
Eq. (11) defines a first order differential equation in \( A(\tau) \) that admits then the approximate form 
\[
\dot{\gamma}(\tau) \simeq \gamma_L + \eta^{1/3} d_r [r^{-1} q''_0(r)] (r_s \sigma_0) x_{s0}^{(0)}(\tau).
\]

Going back to time \( t \) and to \( \gamma_L \equiv \eta^{1/3} \gamma_L \), this gives
\[
\frac{dA}{dt} \simeq \gamma_L A(t) - C_0 A(t)^2 \tag{13}
\]
where \( C_0 = (q'_0 + r_s \sigma_0 q''_0 - 2 r_s \sigma_0 q''_0^2) / (\pi \sqrt{2} \sigma_0 q''_0^{1/3}) \) and the index 0 denotes an evaluation at \( r_s \sigma_0 \). Eq. (13) shows the first nonlinear contribution to the \( m = 1 \) evolution. The early behavior of the \( m = 1 \) growth rate is thus \( \gamma(t) \simeq \gamma_L - C_0 A_0 \exp(\gamma_L t) \). In order to check numerically these analytic predictions for the generalized linear stage, that brings the first nonlinear contributions to the growth rate, we used Aydemir’s initial conditions. The safety profile is \( q(r) = q_m \left\{ 1 + r^4 \left[ (q_a/q_m)^2 - 1 \right] \right\}^{1/2} \) with \( q_m = 0.9, q_a = 3 \), giving \( C_0 > 0 \). The differential equation (11) was integrated numerically for \( A_0 = \sqrt{2} \times 10^{-5.5} \) corresponding to an initial kinetic energy in the \( m = 1 \) mode of the order \( 10^{-11} \). The nonlinear growth rate \( \gamma(t) \equiv \eta^{1/3} \gamma_L(\tau) \) is plotted on Fig. 2 for \( S = 10^7 \). This curve appears to be in fine agreement with the Fig. 1 of Ref. [4] for times \( t \) roughly below 1000 Alfvèn times.

Fig. 3 illustrates the influence of the \( q \)-profile around \( r_s \sigma_0 \) on the time evolution of \( \gamma \) due to [9]. A sudden bump in the nonlinear growth could thus even be observed, before the onset of convective effects, for the special shape of \( q \) chosen in Fig. 3. Moreover, some \( q \)-profile may induce a saturation of \( A \) below the convective threshold and lead to partial reconnection. Most importantly, the approach described here may be transposed to model the early nonlinear behavior of a variety of internal kinks such as two-fluid [12, 16, 17] and/or collisionless [18].

Discussions with L. Sugiyama are gratefully acknowledged. MCF thanks A. Aydemir for several communications on his simulations. This work was supported in part by the U.S. Department of Energy.

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[12] It can be checked that angular contributions in Laplacians are negligible in the critical layer for \( A(\tau) \ll \eta^{1/3} \).
[13] The factor \( 1/2 \) in the kinetic energy of the \( m = 1 \) mode comes from the expression of the linear solution \( \hat{\phi}_1 \) in [3].
[14] This is partly justified by the matching to the vanishing order one outer solution for \( r \geq r_s \sigma_0 \).