ACCURACY PROBLEMS WHEN COMBINING TERRESTRIAL AND SATELLITE OBSERVATIONS

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ABSTRACT

The combination of terrestrial and satellite aided network observations requires hybrid models of two kinds: a hybrid functional model and a stochastic model being able to process the heterogeneous observations appropriately. An appropriate combination of stochastic information means to adjoin proper weights to the various types of observations on one hand and on the other hand to separate the relevant internal accuracy from that accuracy which is affected by a number of network external influences. The first task can be overcome by the technique of estimating variance components of the individual observation groups, the second one can be handled by applying S-transformations to covariance matrices. The paper deals with problems of the hybrid stochastic model. Examples are given.

1. INTRODUCTION

The combination of classically performed terrestrial and satellite aided geodetic network observations rank among the many tasks being attended with the introduction of the new observation techniques. Doing so, several aspects are important. One of them is the functional connection of the heterogeneous observations. This is the problem of knotting together different reference systems whose interrelations are not clearly defined. As a rule the task is mastered by similarity transformations. It has often been treated, as an extensive bibliography proves.

The stochastic hybrid model, however, has been hardly ever dealt with. Therefore, the following representation intends to make a contribution, first by analyzing the accuracies of individually adjusted terrestrial and satellite networks. These accuracies are of two kinds: accuracies which are influenced by numerous network external effects, and internal accuracies which are relevant for relative accuracy statements within the network. The combination of those internal and external accuracies in a hybrid stochastic model has to be treated with care. This is one problem. The other problem is harmonizing the stochastic model by adjoining proper weights to the individual and heterogeneous groups of terrestrial and satellite aided observations.

Therefore, when comparing or combining terrestrial and satellite aided network observations the first task is to get a deeper insight into the accuracies of the individual and heterogeneous observations.
2. **The Accuracy of Satellite Aided Network Observations**

First information about the accuracies of satellite aided network observations can be taken from the various data processing software packages. The covariance matrices provided are, however, affected with all kinds of network external effects such as instrumental and propagation medium influences, ephemeris errors etc. In many cases, especially in cases of small networks and short observation periods with multi-station instrumentations, those effects exercise a systematic influence by shifting, rotating and scaling the network - both the coordinates and the elements of the covariance matrix $C_x$. In case only the mutual or relative positions of the network points are of interest, this "external" covariance matrix $C_x$ does not provide an appropriate measure of the relative or internal accuracies of the point positions. To achieve such a measure the external effects have to be split off. According to the "Internal Error Theory of Networks" (Ref. 1; Appendix A) this can be done by translations and rotations and a change of scale common to all network points. That means the covariance matrix $C_x$ of the network points provided by those data processing software packages and representing externally affected accuracies has to undergo a similarity transformation (S-transformation) in order to reveal the "real" internal accuracy $C_x^i$ of the network point determination.

The diagonal block structure of the covariance matrix of the GPS measurements of the INN VALLEY Network (Appendix C) shows the following characteristics:

$$ s_{C_x} : s_x = \pm 1.5 \text{ cm}, \ s_y = \pm 1.3 \text{ cm}, \ s_h = \pm 2.0 \text{ cm} \quad (1) $$

(subscript $s_*$ for satellite). These values contain external effects. After their removal by S-transformation the internal accuracies are

$$ s_{C_x^i} : s_x = \pm 1.3 \text{ cm}, \ s_y = \pm 1.2 \text{ cm}, \ s_h = \pm 1.6 \text{ cm} \quad (2) $$

3. **The Accuracy of Terrestrial Network Observation Results**

If, for instance, horizontal directions, spatial distances and zenith angles are introduced into a free three-dimensional network adjustment, freedom of the geodetic datum exists only with respect to 3 translations and the rotation around the $h$-axis. The remaining 3 datum dispositions are established by the scale of distances and the orientation of the zenith angles (plumbline or normal of the ellipsoid). Due to these dispositions of 3 datum constraints which are network external effects the network is somehow restrained from being absolutely free. The restraints are reflected in the covariance matrix of the adjusted coordinates, too. Thus the covariance matrix provided by a "free" network adjustment of this kind provides external accuracies which are, in the case of the terrestrial INN VALLEY Network,
\[ \tau^e_x : s_x = \pm 0.4 \, \text{cm}, \ s_y = \pm 0.4 \, \text{cm}, \ s_h = \pm 3.6 \, \text{cm} \]  

(subscript \( \tau^* \) for terrestrial). After S-transformation the corresponding results of the internal accuracies are

\[ \tau^i_x : s_x = \pm 0.4 \, \text{cm}, \ s_y = \pm 0.4 \, \text{cm}, \ s_h = \pm 2.1 \, \text{cm} . \]  

As expected, the S-transformation influences especially the accuracy in heights by rotations around the x- and y-axes since strong datum effects are filtered off.

4. **COMPARISON OF SATELLITE AND TERRESTRIAL OBSERVATION RESULTS**

The first possibility to achieve assertions on accuracies of satellite aided network observations in comparison to terrestrial network observations is the simple comparison of specific results obtained from the individual adjustments of the terrestrial and satellite observations, respectively.

The coordinates of those adjustments can be compared with each other after a 7-parameter similarity transformation which splits off network external effects caused for instance by a different geodetic datum.

\[ \tau^x - m R \tau^s \]

\[ \tau^x \]

coordinate differences to be investigated after transformation

\[ \tau^s \]

centre of gravity related terrestrial coordinates

\[ s^x \]

centre of gravity related GPS coordinates

\[ m \]

scale factor

\[ R \]

rotation matrix.

Applied to the example of the INN VALLEY Network the following results have been obtained (Ref. 2):

The coordinate differences in \( dx \) range from \(-2.4 \, \text{cm} \) to \(+3.0 \, \text{cm} \), in \( dy \) from \(-3.2 \, \text{cm} \) to \(+2.8 \, \text{cm} \), in height \( dh \) from \(-12.8 \, \text{cm} \) to \(+12.6 \, \text{cm} \). The r.m.s. error in position is \( s_p = \pm 2.6 \, \text{cm} \), while the r.m.s. error in height is \( s_h = \pm 8.0 \, \text{cm} \). \( s_p \) is an excellent result, it corresponds quite nicely to what can be derived from Eq. 1. The uncertainty in height \( s_h \) can be questioned, it is much higher than the r.m.s. value \( s_h = \pm 2.0 \, \text{cm} \) of the GPS derived heights and also higher than \( s_h = \pm 3.6 \, \text{cm} \) of the terrestrial heights. The conclusion may be drawn that the GPS heights cannot be burdened with the discrepancy. It is most likely that the refraction influenced terrestrial heights are much weaker and not so consistent; therefore, they cannot compete with the satellite aided height observations.
Most suitable for comparison purposes are also those elements which are by themselves independent of network external influences and geodetic datum invariant, i.e. spatial distances.

The differences

$$T_S - S^\delta = T^\delta - S^\delta$$  \hspace{1cm} (6)

of the terrestrial and satellite network distances are, with only one exception, positive, ranging from +0.2 cm to +9.8 cm; related to the length of the distances a scale factor of \(m_S = +2.47 \text{ ppm}\) can be derived. The differences of the scaled distances range from -2.6 cm to +6.6 cm having a relative accuracy of \(\pm 2.0 \text{ ppm}\).

The above-mentioned statements are the usual ones when terrestrial and satellite observations are compared. However, the rigorous combination of both the observation types yields accuracy statements which go much further.

The first step is some considerations on the functional and stochastic hybrid network models.

5. THE COMBINATION OF TERRESTRIAL AND SATELLITE AIDED NETWORK OBSERVATIONS
   - THE FUNCTIONAL MODEL

The observation equations of a combined or hybrid adjustment model have to be formulated with respect to a common reference system. For the following models the "satellite" coordinates, i.e. the network point coordinates determined by satellite aided observations, and their covariance matrix were transformed onto the terrestrial coordinate system referring to the national geodetic datum.

5.1 Gauss-Markov model

A relatively simple model regards the satellite aided observations as direct coordinate observations which are added to the observation equations of the terrestrial measurements. Therefore, the combined system of observation equations is

\[
\begin{align*}
T^\ell + T^\nu &= A x, \\
T^\delta &= T^C = T^C \delta \\
S^\ell + S^\nu &= I x, \\
S^\delta &= S^C = S^C \delta
\end{align*}
\]

\[
C = \begin{bmatrix}
T^C & 0 \\
0 & S^C
\end{bmatrix}
\]

\hspace{1cm} (\ell \text{ terrestrial observations, } \nu \text{ satellite coordinates, } T^\delta, T^C \text{ and } S^\delta, S^C \text{ weight and covariance matrices respectively of the observations}).
The vector \( x \) of unknowns contains, among others, the point coordinates \( x, y, h \) common to both types of networks.

In case the satellite coordinates are used as approximate coordinates, \( s^T \ell = 0 \). Then the normal equations are

\[
(A^T T^P A + s^P x)x = A^T T^P \ell ,
\]

(8)

where the satellite coordinates are only represented by their weight matrix \( s^P x \). The cofactor matrix of the adjusted coordinates

\[
Q_x = (A^T T^P A + s^P x)^+ = (T^P + s^P x)^+ = T + s Q_x
\]

(9)

demonstrates that the weights of the terrestrial and satellite coordinates are directly added. The mutual interaction will depend on the size and on the "nature" of the two partners. With point coordinates observed by satellite aided techniques, \( Q_x \) is regular in general.

5.2 Gauss-Helmert model

Both the sets of terrestrial and satellite coordinates are considered observations within a system of condition equations with unknowns. The unknowns are the transformation parameters \( p \) (Eq. A.1). The system of equations is

\[
T^x + T^v - (s^x + s^v) + G \cdot p = 0
\]

(10a)

or

\[
8v + G \cdot p + \omega = 0
\]

(10b)

with

\[
S = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 \\
\end{bmatrix}
\]

\[
v^T = \begin{bmatrix}
T^x & T^y & T^h & s^x & s^y & s^h & \ldots
\end{bmatrix}
\]

\[
\omega^T = \begin{bmatrix}
T^x - s^x & T^y - s^y & T^h - s^h & \ldots
\end{bmatrix}
\]

The weight matrix of the observations results from the individually adjusted terrestrial and satellite networks in
\( \mathbf{P}_x = \begin{bmatrix} \mathbf{A}^T \mathbf{T} \mathbf{A} & 0 \\ 0 & \mathbf{S}_x \end{bmatrix} \mathbf{P}_x = \begin{bmatrix} \mathbf{T}_x & 0 \\ 0 & \mathbf{S}_x \end{bmatrix} \). \hspace{1cm} (11)

The solution is according to the well-known algorithms. Attention has to be paid to the inversion of \( \mathbf{P}_x \) which is possibly positive semi-definit.

If one transforms model Eq. 10 owing to

\[ \mathbf{B} \mathbf{u} = \mathbf{v} \] \hspace{1cm} (12)

into the Gauss-Markov model

\[ \mathbf{w} + \mathbf{u} + \mathbf{G} \mathbf{p} = 0 , \] \hspace{1cm} (13)

the weight matrix \( \mathbf{P}_x \) results from

\[ \mathbf{P}_x = \mathbf{T}_x \mathbf{P}_x \mathbf{S}_x \] \hspace{1cm} (14)

if the \( Q \)-matrices are the inverses of the corresponding \( P \)-matrices.

6. THE COMBINATION OF TERRESTRIAL AND SATELLITE AIDED NETWORK OBSERVATIONS
   - THE STOCHASTIC MODEL

There are two major problems involved in the proper combination of the stochastic information of terrestrial and satellite aided network observation.

6.1 The combination of heterogeneous observations

The stochastic information contained in geodetic observations can often not be utilized in its entirety in "usual" adjustments especially in cases of heterogeneous observation material since with the unit variance \( \mathbf{S}_0 \) only one single quantity of the stochastic model is estimated on a global scale. If, however, some knowledge about the stochastic structure of the observations is pre-given, the estimation of the variance components \( \mathbf{S}_i \) of individual groups of observations is made possible. This a posteriori estimation technique (Appendix B) is suitable to adjoin proper weights to the various observation groups.

Due to the linear structure of the stochastic model in Eq. 7

\[ \mathbf{C}_1 = \mathbf{\sigma}_0^2 \mathbf{Q}_1 = \mathbf{\sigma}_0^2 \mathbf{Q}_1 + \mathbf{\sigma}_0^2 \mathbf{Q}_x \]

\[ \mathbf{\sigma}_0^2 \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_x \end{bmatrix} = \mathbf{\sigma}_0^2 \begin{bmatrix} \mathbf{T}_1 & 0 \\ 0 & 0 \\ 0 & \mathbf{S}_x \end{bmatrix} + \mathbf{\sigma}_0^2 \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_x \end{bmatrix} \] \hspace{1cm} (15)
the unit variance \( \sigma_0^2 \) as well as the variance components \( \sigma_T^2 \) and \( \sigma_S^2 \) can be estimated (Eq. B.16). Even a refinement is possible if one splits \( \sigma_T^2 \) into its components

\[
\sigma_T^2 = \sigma_{\text{dir}}^2 \sigma_{\text{dir}} + \sigma_{\text{dis}}^2 \sigma_{\text{dis}} + \sigma_{\text{zen}}^2 \sigma_{\text{zen}}
\]

(dir directions, dis distances, zen zenith angles)

and estimates the respective variance components.

For the combination of the terrestrial observations with the internally highly accurate GPS measurements of the INN VALLEY Network there were 6 separate groups of observations established for the VCE: horizontal directions, zenith angles, 3 groups of spatial distances and the GPS observations. By this means the interaction of the satellite measurements and the individual groups of terrestrial observations can be studied thoroughly (Ref. 3).

Applying VCE to the Gauss-Markov model (Eq. 13) the linear structure (Eq. 14) to be considered is

\[
\begin{align*}
\tilde{c}_1 &= \sigma_0^2 \sigma_{\text{dir}}^2 = \sigma_0^2 \sigma_{\text{dis}}^2 = \sigma_T^2 \sigma_{\text{zen}}^2 \sigma_{\text{dis}}^2 + \sigma_S^2 \sigma_{\text{zen}}^2 \\
\sigma_0^2 \sigma_{\text{dir}}^2 &= \sigma_T^2 \sigma_{\text{zen}}^2 \sigma_{\text{dis}}^2 + \sigma_S^2 \sigma_{\text{zen}}^2 \sigma_{\text{dis}}^2 
\end{align*}
\]

(17)

The estimation procedure is according to Eq. B.13.

6.2 Combination of internal and external stochastic information

There are several possibilities of combining the stochastic information in terms of internal and external accuracies.

<table>
<thead>
<tr>
<th>terrestrial accuracies</th>
<th></th>
</tr>
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<tbody>
<tr>
<td>internal</td>
<td>external</td>
</tr>
</tbody>
</table>

Combination of stochastic information
The combination of only internal accuracies provides, after proper weighting by VCE, internal accuracies within the combined network. An analogous conclusion can be drawn if exclusively external accuracies are combined. It can be questioned, of course, whether such a combination is reasonable in this case - are apples and oranges assembled? The same question arises if internal accuracies are combined with external ones. However, after S-transformation in all cases the same internal accuracy of the combined network can be revealed. This is illustrated in the following (see Appendix A):

\[
\begin{align*}
\hat{\sigma}_{T}^{2} & = H_{T}O_{H}, \quad \bar{\sigma}_{S}^{2} = H_{S}O_{H} \\
T + S \hat{\sigma}_{T}^{2} & = H_{T}O_{H} + H_{S}O_{H} \\
& = H_{(T+S)}O_{H}.
\end{align*}
\]  

Equation 19 shows that the internal covariance matrix of the combined network can be achieved either by the sum of the internal covariance matrices of the terrestrial and satellite networks or by the S-transformed sum of the external covariance matrices. Due to the idempotence of \( H \) the same is valid if an external covariance matrix is combined with an internal one.

Applied to Eq. 9

\[
T + S \hat{\sigma}_{T}^{2} = H_{T}O_{H} = (H_{T}P_{H} + P_{H}H)^{+} = (H_{T}P_{H} + H_{S}P_{H})^{+},
\]

and applied to Eq. 14

\[
\bar{\sigma}_{S}^{2} = \bar{H}_{O_{H}} = H_{(T+S)}O_{H} = H_{T}O_{H} + H_{S}O_{H}
\]

the same statement holds (Eq. A.11).

7. THE COMBINATION OF TERRESTRIAL AND SATELLITE AIDED NETWORK OBSERVATIONS

- RESULTS

The INN VALLEY Network serves again as an example for numerical experiments (Appendix C).

7.1 The accuracy of coordinate observations

As the first measure the accuracy of the satellite aided coordinate observations as obtained from a hybrid network adjustment is investigated.

Introducing the external GPS covariance matrix \( \sigma_{S}^{C} \) (Tab. 1, row 2) leads to coordinate accuracies of the combined network which cover the terrestrial accuracies (Tab. 1,
row 1) totally. After applying VCE the a priori weights of the GPS coordinate observations turn out to be by a factor 4.5 less accurate than before (Tab. 1, row 3 vs. 2). The accuracy situation of adjusted coordinates of the combined network (Tab. 1, row 4) which represents at the same time the accuracy of the adjusted coordinate observations is governed by the external accuracy of the GPS coordinates (Eq. 1). For further processing, see next paragraph.

<table>
<thead>
<tr>
<th>ROW</th>
<th>status</th>
<th>$s_x$</th>
<th>$s_y$</th>
<th>$s_h$</th>
</tr>
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<tbody>
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<td>terrestrial network</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.4</td>
<td>3.6</td>
<td></td>
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<tr>
<td>combined terrestrial and GPS ($s_x^{C_i}$) network</td>
<td></td>
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<td>3</td>
<td>0.4</td>
<td>0.4</td>
<td>3.1</td>
</tr>
</tbody>
</table>

**Table 1**
Accuracies of coordinate observations

Status: 1 = a priori input standard deviations of coordinate observations; 2 = a priori output standard deviations after VCE; 3 = standard deviations of the adjusted coordinate observations after VCE; they represent also the standard deviations of the adjusted net point coordinates. All standard deviations are r.m.s. values.

Using the internal GPS covariance matrix (Eq. 2; Tab. 1, row 5) one obtains after VCE the internal observational accuracies of the GPS measurements (Tab. 1, row 6) which lead to standard deviations of adjusted coordinate observations and coordinates (Tab. 1, row 7) being the same as the terrestrial ones. An exception is the standard deviation $s_h$ demonstrating the good influence of GPS measurements to an increased (external) accuracy of heights in this combined network.
7.2 Accuracy of adjusted coordinates

The results of the combined terrestrial and GPS-network are:

\[ s_{T_x}^C \ (\text{Eq. 1}) \text{ and } s_{T_x}^C \ (\text{Eq. 3}) \text{ lead to } s_{T_x}^{C+T_x} \text{ after VCE (Tab. 1, row 4)}: \]

\[ s_{T_x}^{C+T_x} : s_x = \pm 1.7 \text{ cm}, \ s_y = \pm 1.5 \text{ cm}, \ s_n = \pm 3.4 \text{ cm}. \]  \hspace{1cm} (22)

\[ s_{T_x}^{C+T_x} : s_x = \pm 0.4 \text{ cm}, \ s_y = \pm 0.4 \text{ cm}, \ s_n = \pm 1.6 \text{ cm}. \]  \hspace{1cm} (23)

S-transformation of Eq. 22 leads to Eq. 23 as does S-transformation of any other combination of external and internal accuracies, for instance of the results as given in Tab. 1, row 7. Note that the internal accuracy of heights has been improved by the introduction of GPS-measurements.

A more detailed investigation of the behaviour of the accuracies of the original and adjusted terrestrial observations and the coordinate observations is given by Ref. 3.

8. CHANGES IN POSITIONS AND HEIGHTS DUE TO THE NETWORK COMBINATION

The coordinates based on satellite aided observations were transformed onto the adjusted coordinates of the terrestrial network prior to the network combination. The terrestrial network coordinates were serving as approximate coordinates in all cases of hybrid adjustments. However, in contrast to the terrestrial adjustment with a datum definition based on the national geodetic datum, the satellite coordinates and their covariance matrices determine the geodetic reference system within the combined adjustment models.

If one considers the mean values of coordinate unknowns resulting from the various combinations of stochastic information, one can realize that systematic changes in positions and heights result from the application of the external covariance matrices. This effect is due to the external influences which govern the external covariance matrices; in contrast, the use of the internal covariance matrices leaves on an average the hybrid network coordinates in the position of the terrestrial network.

This effect is the clearest with the height coordinates. Therefore, the behaviour of the changes in heights will be discussed in the following.

The observed GPS heights differ from the terrestrial heights between -12.9 cm ... +12.3 cm (mean ± 0 cm) before the combination. After combination with \( s_{T_x}^C \) (Eq. 1) and VCE, height changes occur from -7.4 cm ... +3.6 cm (mean -2.5 cm). Particularly interesting are the results of the combination with the internal GPS covariance matrix \( s_{T_x}^{C+T_x} \) (Eq. 2): they reach from -11.0 cm ... +6.5 cm (mean ± 0 cm).
Considering the essential results of the combination of terrestrial with GPS heights, the distinct influence of the GPS heights to the height determination of the network points becomes clear. Especially the extreme values -11.0 cm and +5.5 cm are highly significant improvements of two, terrestrially only weakly determined heights.

9. **CONCLUSIONS**

An appropriate combination of the stochastic information obtained from terrestrial and satellite aided network observation requires

- a hybrid functional and stochastic model of adjustment
- variance component estimation techniques
- S-transformation of covariance matrices.

Some results are

- various combinations of external and internal covariance matrices can be applied;
- after variance component estimation an identical internal covariance matrix of the adjusted coordinates of the hybrid network can be obtained from different combinations by S-transformations;
- the covariance matrices of satellite aided coordinate observations as obtained from various standard software packages are strongly affected by external influences;
- for comparison of accuracies only internal covariance matrices (or invariant functions) are suitable;
- GPS observations can contribute to an improvement of the internal accuracy of even very precise local terrestrial networks.
Appendix A

S-TRANSFORMATION

The S-transformation parameters \( p \) are 3 translations, 3 rotations and 1 scale factor for a three-dimensional network

\[
p^T = \begin{vmatrix} t_x & t_y & t_z & e_x & e_y & e_z & m \end{vmatrix},
\]

(A.1)

the coefficient matrix \( G \) is for \( n \) points

\[
G^T = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\
0 & -z_1 & y_1 & 0 & -z_2 & y_2 & \cdots & 0 & -z_n & y_n \\
z_1 & 0 & -x_1 & z_2 & 0 & -x_2 & \cdots & z_n & 0 & -x_n \\
-x_1 & 0 & -y_1 & x_2 & 0 & -y_2 & \cdots & -y_n & x_n & 0 \\
x_1 & y_1 & z_1 & x_2 & y_2 & z_2 & \cdots & x_n & y_n & z_n
\end{bmatrix}.
\]

(A.2)

The transformation of a \( C_x \) covariance matrix is carried out with

\[
H = \left( I - G(G^T G)^{-1} G^T \right)
\]

(A.3)

according to Ref. 1 to

\[
C_x^I = H C_x H^T
\]

(A.4)

The matrices \( G \) and \( H \) have the following important properties:

\[
\text{r}(G) = d
\]

(A.5)

\((r(*) \text{ rank}), \text{ if the number of transformation parameters applied is } d \).

\[
\text{r}(G^T G) = d,
\]

(A.6)

so that \((G^T G)\) is regular and \((G^T G)^{-1}\) exists.

\[
\tilde{G} = G(G^T G)^{-1} G^T = G^T
\]

(A.7)

is symmetric and idempotent.
\[ \tilde{G} \bar{G} = \tilde{G}, \ r(\tilde{G}) = \text{tr}(\tilde{G}) = d \]  \hspace{1cm} (A.8)

\( \text{(tr(*) trace)} \).

The transformation matrix \( H \) is symmetric and idempotent

\[ H = H^T, \ HH = H, \ r(H) = \text{tr}(H) = \text{tr}(I) - \text{tr}(\tilde{G}) = 3n-d, \]  \hspace{1cm} (A.9)

and the Moore-Penrose generalized inverse \( H^+ \) is the matrix itself

\[ H^+ = H. \]  \hspace{1cm} (A.10)

From the idempotence (Eq. A.9) it follows that a S-transformation can only once be applied to a covariance matrix \( C_x \); a second S-transformation of the same matrix would not change it.

The rank defect of the resulting matrix \( C^{-1}_x \) is \( d \), if \( r(H) = 3n-d \leq r(C_x) \); \( d = 7 \) at the most in case of a three-dimensional network. On this condition, with \( P_x \) as the inverse of \( C_x \) and with Eq. A.10 also

\[ C^{-1}_x = HC_x H = (HP_x H)^+ \]  \hspace{1cm} (A.11)

holds (Ref. 4).
VARIANCE COMPONENT ESTIMATION

In the following only a short review is given. For detailed studies see for instance Ref. 5 - 7.

The stochastic part of the Gauss-Markov model

\[ \ell + v = Ax, \quad C_1 \]  \hspace{1cm} (B.1)

\( \ell \) vector of observations \((n \times 1)\)
\( v \) vector of residuals \((n \times 1)\)
\( A \) design matrix \((n \times u)\)
\( x \) vector of unknowns \((u \times 1)\)

Consists of the covariance matrix \( C_1 \) of the observations which can be divided in two

\[ C_1 = \sigma_0^2 Q_1 \]  \hspace{1cm} (B.2)

\( \sigma_0^2 \) theoretical unit variance
\( Q_1 = P_1^{-1} \) cofactor matrix of the observations.

\( Q_1 \) is considered pre-given, \( \sigma_0^2 \) is after the adjustment (a posteriori) unbiasedly estimated by the empirical unit variance \( s_0^2 \), i.e.

\[ s_0^2 = \frac{v^T P_1 v}{r} \]  \hspace{1cm} (B.3)

\( r \) redundancy of the observations

\[ r = \text{tr}(P_1 Q^{vv}) = n - u \]  \hspace{1cm} (B.4)

\( \text{tr}(\cdot) \) trace operator,

\[ Q^{vv} = Q_1 - A(A^T P_1 A)^+ A^T = Q_1 - A Q_x A^T \]  \hspace{1cm} (B.5)

\( Q^{vv} \) cofactor matrix of the residuals \((n \times n)\),

\[ v = (A Q_x A^T P_1 - I) \ell = - Q^{vv} P_1 \ell \]  \hspace{1cm} (B.6)

and

\[ V = V_1 + V_2 \]  \hspace{1cm} (B.7)

\( V \) variance of the estimates
\( V_1 \) variance of the regression
\( V_2 \) variance of the regression coefficient
\[ v_i^T P_1 v = \ell_i^T P_1 Q_{0i} Q_{0i}^T P_1 \ell = s^2 \text{tr}(P_1 Q_{0i}^T P_1 Q_{0i}) \]  

(B.7)

Frequently observations of different types have to be adjusted in one hybrid model, i.e. horizontal directions, spatial distances, zenith angles drop in the adjustment model of a terrestrial network, or even terrestrial and satellite aided network observations have to be combined. In such a case the original Gauss-Markov model is split into several parts corresponding to the \( c \) different groups of observations

\[
\begin{bmatrix}
\ell_1 \\
\vdots \\
\ell_c 
\end{bmatrix} + \begin{bmatrix}
v_1 \\
\vdots \\
v_c 
\end{bmatrix} = \begin{bmatrix} A_1 \\
\vdots \\
A_c 
\end{bmatrix} \cdot x \quad .
\]

(B.8)

Also the stochastic model falls into groups

\[
C_1 = \sigma_1^2 Q_1 = \sigma_1^2 Q_1 + \ldots + \sigma_c^2 Q_c = \sigma_1^2 p_{11} + \ldots + \sigma_c^2 p_{cc} \quad .
\]

(B.9)

While the model was homogeneous, only the estimation of the unit variance \( \sigma_0^2 \) was of interest. In the inhomogeneous case, however, the wish may exist to estimate the variance components \( \sigma_1^2 \ldots \sigma_c^2 \) as well.

This can be achieved by dissecting Eqs. B.5 and B.7 according to

\[
Q_{0i} = \begin{bmatrix} Q_{01} & \ldots & Q_{0c} \\
\vdots & \ddots & \vdots \\
Q_{ci} & \ldots & Q_{cc} 
\end{bmatrix}
\]

(B.10)

with

\[
Q_{0i} = Q_1 - A_1 Q_1 A_1^T \quad i = 1 \ldots c \\
Q_{0i} = - A_1 Q_1 A_1^T \quad i,j = 1 \ldots c , i \neq j
\]

and

\[
v_i^T P_1 v = \sum_{i=1}^{c} v_i^T P_1 v_i
\]

(B.11)

with

\[
v_i^T P_1 v_i = \sum_{j=1}^{c} \text{tr}(P_1 Q_{0j} Q_{0j}^T P_1) s_j^2
\]

\[
= \sum_{j=1}^{c} t_{ij} s_j^2 \quad .
\]

(B.12)
After composing to one system of equations

\[
\begin{bmatrix}
\mathbf{v}_1^T \mathbf{p}_1 \mathbf{v}_1 \\
\vdots \\
\mathbf{v}_c^T \mathbf{p}_c \mathbf{v}_c
\end{bmatrix}
\begin{bmatrix}
t_{11} & \cdots & t_{1c} \\
\vdots & \ddots & \vdots \\
t_{c1} & \cdots & t_{cc}
\end{bmatrix}
\begin{bmatrix}
s_1^2 \\
\vdots \\
s_c^2
\end{bmatrix}
\]

(B.13)

the variance components \( s_i^2 \) can be solved for by inversion. The solution has to be found iteratively until all components converge \( s_i^2 \rightarrow s_j^2 \). The final estimation result is unbiased.

Eq. B.13 is the general formula which has to be applied if the linear structure of the \( Q_1 \) matrix is "overlapping", i.e.

\[
\sigma_0^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \sigma_1^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \sigma_2^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(B.14)

or "sequential"

\[
\sigma_0^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \sigma_1^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \sigma_2^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(B.15)

In this case the equation system B.13 can be simplified. The estimation formula is

\[
\mathbf{v}_1^T \mathbf{p}_1 \mathbf{v}_1 = s_1^2 \text{tr}(\mathbf{q}_{1,1}^N)
\]

for each variance component \( s_i^2 \) independent of the others. The estimation process is iterative, too.

One very important fact of the estimation of variance components is its independence of the geodetic datum or other network external effects.
Appendix C

THE INN VALLEY NETWORK

Terrestrial observations

The INN VALLEY Network of the Institute of Geodesy of the Bundeswehr University Munich covers a space of 25 km $\times$ 15 km $\times$ 1.3 km. All points but one are monumented by concrete pillars. The terrestrial coordinates were determined by 48 spatial distances (three classes of accuracies), 44 horizontal directions, 44 zenith distances and 8 $\times$ 2 components of the deflections of the vertical.

![Diagram of INN VALLEY Network]

$\bar{m}_H = 36 \text{mm}$  $\bar{m}_P = 5.9 \text{mm}$

Fig. 1 The INN VALLEY Network: terrestrial observations
GPS observations

In November 1984 coordinate differences of the INN VALLEY Network points were determined by Macrometer V 1000 measurements (Ref. 2).

Fig. 2 The INN VALLEY Network: Macrometer measurements (1 simultaneous observations)

* * *
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3) W. Welsch, W. Oswald, The hybrid adjustment of terrestrial and satellite aided network observations - investigation of accuracies, XVIII. International FIG Congress, invited paper 503.1, Toronto 1986


