Cardy-Verlinde Formula and Achücarro-Ortiz Black Hole

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Abstract

In this paper it is shown that the entropy of the black hole horizon in the Achücarro-Ortiz spacetime, which is the most general two-dimensional black hole derived from the three-dimensional rotating BTZ black hole, can be described by the Cardy-Verlinde formula. The latter is supposed to be an entropy formula of conformal field theory in any dimension.
Introduction

Holography is believed to be one of the fundamental principles of the true quantum theory of gravity [1, 2]. An explicitly calculable example of holography is the much-studied anti-de Sitter (AdS)/Conformal Field Theory (CFT) correspondence. More recently, it has been proposed that defined in a manner analogous to the AdS$_d$/CFT$_{d-1}$ correspondence, quantum gravity in a de Sitter (dS) space is dual to a certain Euclidean CFT living on a spacelike boundary of the dS space [3] (see also earlier works [4–6]). Following the proposal, some investigations on the dS space have been carried out recently [5–28].

The Cardy-Verlinde formula recently proposed by E. Verlinde [29], relates the entropy of a certain CFT to its total energy and Casimir energy in arbitrary dimensions. In the spirit of AdS$_d$/CFT$_{d-1}$ and dS$_d$/CFT$_{d-1}$ correspondence, this formula has been shown to hold exactly for the cases of dS Schwarzschild, dS topological, dS Reissner-Nordström, dS Kerr, and dS Kerr-Newman black holes. In this paper we have further checked the Cardy-Verlinde formula with the two-dimensional Achúcarro-Ortiz black hole which is derived from the three-dimensional rotating BTZ black hole. In 1992 Bañados, Teitelboim and Zanelli (BTZ) [30, 31] showed that (2 + 1)-dimensional gravity has a black hole solution. This black hole is described by two parameters, its mass $M$ and its angular momentum (spin) $J$. It is locally anti-de-Sitter space and thus it differs from Schwarzschild and Kerr solutions in that it is asymptotically anti-de-Sitter instead of flat. Additionally, it has no curvature singularity at the origin. AdS black holes, are members of the two-parametric family of BTZ black holes, Specifically AdS(2) black hole is most interesting in the context of string theory and black hole physics [32, 33].

For two-dimensional (2D) gravitational systems (more in general systems that admit 2D CFTs as duals) one can make use, directly, of the Cardy formula [34] that gives the entropy of a CFT in terms of the central charge $c$ and the eigenvalue of the Virasoro operator $l_0$. However, this is possible only for 2D systems for which one can explicitly show (e.g using the AdS$_d$/CFT$_{d-1}$ correspondence) that they are in correspondence with a 2D CFT [35,36]. Even in this most favorable case the use of the Cardy formula for the computation of the entropy of the gravitational system is far from trivial. The central charge $c$ and the eigenvalue $l_0$ of the Virasoro operator have to be expressed in terms of the gravitational parameters, an operation that sometimes turns out to be very hard [37]. The two-dimensional (2D) limit of the Cardy-Verlinde proposal is interesting for various reasons. From investigations of the AdS$_d$/CFT$_{d-1}$ correspondence, we know that there are 2D gravitational systems that admit 2D CFTs as duals [35,38]. In this case one can make direct use of the original Cardy formula [34] to compute the entropy [35,38]. A comparison of these results with a 2D generalization of the Cardy-Verlinde formula could be very useful in particular for the understanding of the puzzling features of the AdS$_d$/CFT$_{d-1}$
correspondence in two dimensions [39]. Another point of interest in extending the Cardy-Verlinde formula to \( d = 2 \) is the clarification of the meaning of the holographic principle for 2D spacetimes. The boundaries of spacelike regions of 2D spacetimes are points, so that even the notion of holographic bound is far from trivial. A generalization of the work of Verlinde to two spacetime dimensions presents several difficulties, essentially for dimensional reasons. First of all, in two dimensions one cannot establish a area law, since black hole horizons are isolated points. Moreover, the spatial coordinate is not a “radial” coordinate and hence one cannot impose a natural normalization on it.

1 Achúcarro-Ortiz Black Hole

The black hole solutions of Bañados, Teitelboim and Zanelli [30,31] in \((2 + 1)\) spacetime dimensions are derived from a three dimensional theory of gravity

\[
S = \int dx^3 \sqrt{-g} \left( R + 2\Lambda \right)
\]

with a negative cosmological constant \((\Lambda > 0)\).

The corresponding line element is

\[
ds^2 = -\left( -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right) dt^2 + \frac{dr^2}{\left( -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right)} + r^2 \left( d\theta - \frac{J}{2r^2} dt \right)^2
\]

There are many ways to reduce the three dimensional BTZ black hole solutions to the two dimensional charged and uncharged dilatonic black holes [40,41]. The Kaluza-Klein reduction of the \((2 + 1)\)-dimensional metric (2) yields a two-dimensional line element:

\[
ds^2 = -g(r) dt^2 + g(r)^{-1} dr^2
\]

where

\[
g(r) = \left( -M + \Lambda r^2 + \frac{J^2}{4r^2} \right)
\]

with \(M\) the ADM mass, \(J\) the angular momentum (spin) of the BTZ black hole and \(-\infty < t < +\infty, \ 0 \leq r < +\infty, \ 0 \leq \theta < 2\pi\).

The outer and inner horizons, i.e. \(r_+\) (henceforth simply black hole horizon) and \(r_-\) respectively, concerning the positive mass black hole spectrum with spin \((J \neq 0)\) of the line element (2) are given as

\[
r^2_{\pm} = \frac{l^2}{2} \left( M \pm \sqrt{M^2 - \frac{J^2}{l^2}} \right)
\]
and therefore, in terms of the inner and outer horizons, the black hole mass and the angular momentum are given, respectively, by

\[ M = \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2} \]  

(6)

and

\[ J = \frac{2r_+r_-}{l^2} \]  

(7)

with the corresponding to the angular momentum angular velocity to be

\[ \Omega = \frac{J}{2r^2} . \]  

(8)

The Hawking temperature \( T_H \) of the black hole horizon is \([42]\)

\[ T_H = \frac{1}{2\pi r_+} \left( \frac{r_+^2}{l^2} + \frac{J^2}{4r_+^2} \right)^{1/2} - \frac{J^2}{l^2} \]

\[ = \frac{1}{2\pi r_+} \left( \frac{r_+^2}{l^2} - \frac{J^2}{4r_+^2} \right) . \]  

(9)

The area \( A_H \) of the black hole horizon is

\[ A_H = 2\pi l \left( \frac{M + \sqrt{M^2 - J^2}}{2} \right)^{1/2} \]

\[ = 2\pi r_+ \]  

(10)

(11)

thus the entropy of the two-dimensional Achúcarro-Ortiz black hole, if we employ the well-known Bekenstein-Hawking area formula \( S_{BH} \) for the entropy \([43–45]\), is given as

\[ S_{bh} = \frac{1}{4\hbar G} A_H = S_{BH} \]  

(12)

and using the BTZ units where \( 8\hbar G = 1 \) takes the form

\[ S_{bh} = 4\pi r_+ . \]  

(13)

2  Cardy-Verlinde Formula

In a recent paper, E. Verlinde \([29]\) propound a generalization of the Cardy formula which holds for \((1 + 1)\) dimensional Conformal Field Theory (CFT) to \((n + 1)\)-dimensional spacetime described by the metric

\[ ds^2 = -dt^2 + R^2 d\Omega_n \]  

(14)

where \( R \) is the radius of a \( n \)-dimensional sphere.
The generalized Cardy formula (hereafter named Cardy-Verlinde formula) is given by

\[ S_{\text{CFT}} = \frac{2\pi R}{\sqrt{ab}} \sqrt{E_C (2E - E_C)} \]  

(15)

where \( E \) is the total energy and \( E_C \) is the Casimir energy. The definition of the Casimir energy is derived by the violation of the Euler relation as

\[ E_C \equiv n (E + pV - TS - \Phi Q) \]  

(16)

where the pressure of the CFT is defined \( p = E/nV \). The total energy may be written as the sum of two terms

\[ E(S, V) = E_E(S, V) + \frac{1}{2} E_C(S, V) \]  

(17)

where \( E_E \) is the purely extensive part of the total energy \( E \). The Casimir energy \( E_C \) as well as the purely extensive part of energy \( E_E \) expressed in terms of the radius \( R \) and the entropy \( S \) are written as

\[ E_C = \frac{b}{2\pi R} S^{1-\frac{4}{n}} \]  

(18)

\[ E_E = \frac{a}{4\pi R} S^{1+\frac{4}{n}} . \]  

(19)

After the work of Witten on AdS\(_d/CFT\(_{d-1}\) correspondence, E. Verlinde proved that the Cardy-Verlinde formula (15) can be derived using the thermodynamics of an AdS Schwarzschild black holes in arbitrary dimension.

### 3 Entropy of Achúcarro-Ortiz black hole in Cardy-Verlinde Formula

We would like to derive the entropy of the two-dimensional Achúcarro-Ortiz black hole (13) from the Cardy-Verlinde formula (15). First, we evaluate the Casimir energy \( E_C \) using (16). It is easily seen from (9) and (13) that

\[ T_H S_{\text{bh}} = 2 \left( \frac{r_+^2}{l^2} - \frac{J^2}{4r_+^2} \right) \]  

(20)

while from (7) and (8) we have

\[ \Omega_+ J = \frac{J^2}{2r_+^2} . \]  

(21)

Since the two-dimensional Achúcarro-Ortiz black hole is asymptotically anti-de-Sitter, the total energy is \( E = M \) and thus the Casimir energy, substituting (6), (20) and (21) in (16), is given as

\[ E_C = \frac{J^2}{2r_+^2} \]  

(22)
where in our analysis the charge $Q$ is the angular momentum $J$ of the two-dimensional Achúcarro-Ortiz black hole, the corresponding electric potential $\Phi$ is the angular velocity $\Omega$ and $n = 1$. Making use of expression (18), Casimir energy $E_C$ can also be written as

$$E_C = \frac{b}{2\pi R}. \quad (23)$$

Additionally, it is obvious that the quantity $2E - E_C$ is given, by substituting (20) and (21) in (16), as

$$2E - E_C = 2\frac{r^2}{l^2}. \quad (24)$$

The purely extensive part of the total energy $E_E$ by substituting (24) in (17) is

$$E_E = \frac{r^2}{l^2} \quad (25)$$

whilst by substituting (13) in (19) is

$$E_E = \frac{4\pi a}{R}r^2. \quad (26)$$

At this point it is useful to evaluate the radius $R$. By equating the right hand sides of (22) and (23) the radius is written as

$$R = \frac{br^2}{\pi J^2} \quad (27)$$

while by equating the right hand sides (25) and (26) it can also be written as

$$R = 4\pi al^2. \quad (28)$$

Therefore, the radius expressed in terms of the arbitrary positive coefficients $a$ and $b$ is

$$R = 2r_+ \left( \frac{1}{J} \right) \sqrt{ab}. \quad (29)$$

Finally, we can substitute the expressions (22), (24) and (29) which were derived in the context of thermodynamics of the two-dimensional Achúcarro-Ortiz black hole black hole, in the Cardy-Verlinde formula (15) which in turn was derived in the context of CFT

$$S_{CFT} = 2\pi \sqrt{ab} 2r_+ \left( \frac{1}{J} \sqrt{ab} \right) \sqrt{\frac{J^2}{2r_+^2} \frac{r_+^2}{l^2}} \quad (30)$$

and we get

$$S_{CFT} = S_{bh}. \quad (31)$$

It has been proven that the entropy of the the two-dimensional Achúcarro-Ortiz black hole can be expressed in the form of Cardy-Verlinde formula.
4 Conclusions

Among the family of $AdS_d/CFT_{d-1}$ dualities, the pure gravity case $AdS_3/CFT_2$ is the best understood. In contrast, the AdS/CFT correspondence in two space-time dimensions is quite enigmatic. Some progress has been made in [35,39]. The aim of this paper is to further investigate the $AdS_2/CFT_1$ correspondence in terms of Cardy-Verlinde entropy formula. Naively, one might expect that holographic dualities in a two-dimensional bulk context would be the simplest cases of all. This may certainly be true on a calculational level; however, one finds such two-dimensional dualities to be plagued by conceptually ambiguous features. One of the remarkable outcomes of the AdS/CFT and dS/CFT correspondence has been the generalization of Cardy’s formula (Cardy-Verlinde formula) for arbitrary dimensionality as well as for a variety of AdS and dS backgrounds. In this paper, we have shown that the entropy of the black hole horizon of Achúcarro-Ortiz spacetime can also be rewritten in the form of Cardy-Verlinde formula.

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References


