Effective Actions, Boundaries and Precision Calculations of Casimir Energies

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We perform the matching required to compute the leading effective boundary contribution to the QED lagrangian in the presence of a conducting surface, once the electron is integrated out. Our result resolves a confusion in the literature concerning the interpretation of the leading such correction to the Casimir energy. It also provides a useful theoretical laboratory for brane-world calculations in which kinetic terms are generated on the brane, since a lot is known about QED near boundaries.

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I. INTRODUCTION

More than 50 years ago Casimir [1] forced physicists to recognize the reality of quantum vacuum fluctuations by showing that those of the electromagnetic field can mediate physical effects, such as causing a force between two parallel plates. For instance, for plane parallel metallic plates separated by a distance $a$ the energy per unit area associated with this force is

$$\epsilon_0 = \frac{E_0}{A} = -\frac{\pi^2}{720 a^3}. \quad (1)$$

The study of this effect has experienced a recent revival, largely due to prospects for its improved measurement. (For a modern review see, for example, ref. [2].) With this recent attention has come more detailed calculations, including the one-loop corrections within Quantum Electrodynamics (QED) due to virtual photons and electrons [3]. For $ma \gg 1$ the leading correction found in this way is

$$\epsilon_1 = \frac{\pi^2 \alpha}{2560 m a^3}, \quad (2)$$

where $m$ is the electron mass and $\alpha = e^2 / 4\pi$ is the usual fine-structure constant.

Surprisingly, a whiff of controversy has lingered over the physical interpretation of eq. (2), a controversy which can be traced to its dependence on the electron mass. The controversial issue has been cast most sharply when the calculation is formulated within an effective-lagrangian framework, as might be expected to be appropriate given that the electron mass is much higher than the energies, $E \sim 1/a$, of the photon modes whose contributions dominate in the Casimir effect. Early workers [4] found a much smaller contribution than eq. (2), with the leading effect instead found to arise from the Euler-Heisenberg effective interaction obtained by integrating out the electron, giving:

$$\epsilon_1' = \frac{11 \pi^4 \alpha^2}{3,888,000 m^4 a^7}. \quad (3)$$

Subsequent workers have verified the calculation leading to the result (2), and have attributed the discrepancy either to a failure of effective field theory itself [5] or to a misidentification of the most important effective interaction which is relevant [6,7]. There remains a disagreement about which effective interaction is most relevant, with ref. [6] arguing that eq. (2) can be reproduced by an interaction localized on the plates (the ‘boundary’), with an effective coupling which is of order $\alpha / m$. By contrast ref. [7] argues that the required effective interaction arises in the space between the plates (the ‘bulk’), with a coupling which is of order $\alpha / (ma)$. Unfortunately, neither reference resolved the discrepancy by performing the matching calculation which is required in order to properly identify which effective interactions actually arise in the low-energy theory. (See refs. [8,9] for other discussions of effective lagrangians in the Casimir energy problem.)

It is our purpose with this paper to settle the issue of which electron-mass corrections are dominant, and how they arise within an explicitly-constructed effective field theory. In order to do so, we set up the relevant effective lagrangian, and perform the requisite matching calculation which determines the size of the effective couplings.

We draw the following conclusions:

- We conclude that the correct effective operator is the local boundary operator of ref. [6], which describes how the vacuum polarization alters the interaction between a test charge and the surface charges which it induces on the conductor.

- As the matching calculation shows explicitly, this effect relies on polarizations of the vacuum charge over distances of order $1/m$, and so presupposes that the conductor’s boundaries are sharp on these scales. As such they are likely to be dominated by surface effects for real conductors, for which the scale of penetration depths for electromagnetic fields are set by much larger interatomic separations, of order $1/(\alpha m)$.

*We use units for which $\hbar = c = 1$. 
We present our results in more detail in the next sections, starting with a brief summary of some general features which all effective field theories must share. We then identify the dominant effective operator for the Casimir effect which arises when the electron is integrated out. Finally we perform the matching calculations which are required to identify the dominant electron contributions to the Casimir energy in both of these cases. Our conclusions are briefly summarized at the end.

II. THE EFFECTIVE FIELD THEORY

Although their use has been largely systematized only during the past 20 years, the main ideas of effective field theories go back to the much earlier Born-Oppenheimer approximation used in atomic physics decades earlier. The idea is to exploit the way physical systems simplify when they are probed only at very low energies, \( E \), compared with some intrinsic energy scale, \( M \). Expressions for general low-energy observables usually greatly simplify once they are expanded in powers of \( E/M \), and so it pays to take advantage of this expansion as early as possible in a calculation.

The contribution of field theory to this process comes with the recognition that all of the low-energy effects of virtual high-energy states, \( h \), on low-energy degrees of freedom, \( \ell \), can always be expressed in terms of local operators involving only the light states \( \ell \). That is, the physics of the full hamiltonian

\[
H_{\text{full}}(\ell, h) = H_{\text{h.e.}}(\ell) + H_{\text{h.e.}}(h) + H_{\text{int}}(\ell, h),
\]

is indistinguishable from the physics of the effective hamiltonian

\[
H_{\text{eff}}(\ell) = H_{\text{h.e.}}(\ell) + \sum_{k \geq k_0} \frac{1}{M^k} \sum_{\ell} c_{kl} O_{kl}(\ell),
\]

to any fixed order in powers of \( 1/M \), for some choice for the effective interactions, \( O_{kl} \), and effective couplings, \( c_{kl} \). Furthermore, since the uncertainty principle only permits energy and momentum conservation to fail over very short times and short distances, the effective interactions, \( O_{kl} \), which are required are all local in space and time — i.e. involves products of fields and their derivatives at fixed positions in space and instants of time.

The utility of this observation is that the required effective interactions and couplings can be computed once and for all by comparing to simple observables, and once obtained may be used for any calculations in the low-energy theory. Such a determination of the effective interactions and couplings is called a ‘matching’ calculation, because the effective interactions of \( H_{\text{eff}} \) are matched onto what is required by the full microscopic theory, \( H_{\text{full}} \). (See refs. [10] for reviews on effective field theories.)

A. Effective Field Theories and QED

Quantum electrodynamics in particular lends itself to this kind of effective analysis, because of the huge hierarchy in scales between the electron mass, \( m \), and the other scales of usual interest such as those appropriate to the propagation of light or to atomic energy levels. Two kinds of effective field theories have been explicitly treated in this way. One corresponds to integrating out electrons and positrons and high-energy photons to describe the interactions of low-energy photons [11], and the other involves integrating out positrons and high-energy electrons and photons to describe the low-energy interactions of nonrelativistic electrons and low-energy photons [12].

For electromagnetism it is more convenient to work with an effective action or lagrangian than with an effective hamiltonian. In the absence of all boundaries and charge distributions, integrating out the electron leads to the following Maxwell plus Euler-Heisenberg [11] effective interactions for low-energy photons

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) + \frac{2\alpha^2}{45m^4} \left[ (\mathbf{E}^2 - \mathbf{B}^2)^2 + 7 (\mathbf{E} \cdot \mathbf{B})^2 \right] + \cdots
\]

In this expression, corrections to the first (Maxwell) term in this lagrangian have been removed using an appropriate rescaling of the electromagnetic field. Similarly, a possible \( O(\alpha/m^2) \) (Uehling) term, of the form \( F^\mu\nu \Box F_{\mu\nu} \) arising from the vacuum polarization, is not written here because it can be removed to this order by performing a field redefinition of the form \( A_\mu \rightarrow A_\mu + \frac{c_1 \alpha}{m^2} \Box A_\mu + \frac{c_2 \alpha^2}{m^4} \Box^2 A_\mu \), for an appropriate choice of coefficients, \( c_1 \) and \( c_2 \). Any effective interaction which can be removed in this way is called ‘redundant’, since it cannot contribute to physical quantities.

The numerical coefficient of the quartic (Euler-Heisenberg) term may be found by comparing the amplitude for light-by-light scattering as computed to order \( \alpha^2/m^4 \) using eq. (6) and with the full QED lagrangian, including electrons. Once this is done, the result captures the influence, to order \( \alpha^2/m^4 \), of electrons on any low-energy observable (in the absence of boundaries) because eq. (6) completely exhausts the possible local, Lorentz-invariant and gauge invariant effective interactions which may be constructed to this order.

B. Boundary Charges and Screening

For applications to the Casimir effect we must ask how the presence of the conducting boundaries can affect the matching process just described. For simplicity we examine the case considered by refs. [3-7] and restrict our
discussion to the case where the electrons do not ‘see’ the boundaries, and only the boundary conditions of the electromagnetic field are changed. For conducting plates we take the electromagnetic boundary conditions to be
\[ e^{\mu \lambda \rho} n_\nu F_{\mu \rho} |_{\partial M} = 0, \]
where \( n_\nu = \{0, n\} \) is a normal vector on the surface which points into the bulk. This captures the usual conducting boundary conditions, that \( B_n = n \cdot B \) and \( n \times E \) both vanish. \( \partial M \) here denotes the surfaces of the plates, considered as the boundaries of the intervening bulk, \( M \).

Our starting point is then the QED action, with boundary terms
\[ S_{\text{full}} = -\int_M d^4 x \left( \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \overline{\psi} (i \gamma^\mu D^\mu + m) \psi \right) - \int_M d^4 x J^\mu A_\mu - \int_{\partial M} d^3 x j^\mu A_\mu, \]
where \( D_\mu = \partial_\mu + ie A_\mu \) is the usual covariant derivative for the electron field. Here \( J^\mu = \{ \rho, J \} \) denotes any classical test charges and currents with which we choose to probe the system, and \( j^\mu = \{ \sigma, j \} \) denotes the surface charge and current densities whose presence enforces the boundary condition, eq. (7). The classical Maxwell equations obtained by varying eq. (8) with respect to \( A_\mu \) in the bulk and on the boundaries is:
\[ \partial_\mu F^{\mu \nu} - J^\nu = 0 \quad \text{in } M; \]
\[ n_\mu F^{\mu \nu} - j^\nu = 0 \quad \text{on } \partial M. \]

The boundary part of these equations relate the surface charges and currents to the boundary electric and magnetic fields and are more familiar once written explicitly in terms of \( E \) and \( B \). For the surface charge distribution they imply, for instance,
\[ n \cdot E|_{\partial M} = E_n|_{\partial M} = \sigma. \]

The presence of these boundary charges plays a crucial role in constructing the effective theory, because of its interplay with the photon’s vacuum polarization. To see this imagine we now compute the electron’s contribution to the vacuum polarization
\[ \Pi_{\mu \nu}(q^2) = \left( g^2 \eta_{\mu \nu} - q_\mu q_\nu \right) \Pi(q^2) \]
\[ = -ie^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ i \gamma^\nu \frac{1}{p^2 + m^2 - ie} \right] \]
with \( m^2 = m^2 - ie \). After renormalization, \( \Pi(q^2) \) is given by
\[ \Pi(q^2) = \frac{2\alpha}{\pi} \int_0^1 d\zeta \zeta (1 - \zeta) \log \left[ 1 + \zeta (1 - \zeta) \frac{q^2}{m^2} \right]. \]

\( \Pi(q^2) \) describes the polarization of the vacuum about any given charge distribution, effectively smearing it over a distance of order \( 1/m \). Given a point charge source, \( Q \delta^3(\mathbf{r}) \), the position-space charge density which the electron vacuum polarization produces is (to leading order)
\[ \rho_{\text{eff}} = Q \left[ \delta^3(\mathbf{r}) + \eta(\mathbf{r}) \right], \]
where \[ \eta(\mathbf{r}) = \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} e^{i \mathbf{q} \cdot \mathbf{r}} \Pi(q^2) \]
\[ = \frac{\alpha}{\pi} \int_0^1 d\zeta \zeta (1 - \zeta) \log \left[ 1 + \frac{q^2}{m^2} \zeta (1 - \zeta) \right] \]
\[ = N \delta^3(\mathbf{r}) - \frac{\alpha}{2\pi^2 r^2} \int_0^1 d\zeta \left( 1 + \frac{m^2 r^2}{\zeta (1 - \zeta)} \right) \exp \left[ -\frac{m^2 r^2}{\zeta (1 - \zeta)} \right]. \]

Here \( r = |\mathbf{r}| \) and \( N \) is a constant which renormalizes the bare charge \( Q \), determined by the condition
\[ \int d^3 r \eta(\mathbf{r}) = \frac{1}{2} \Pi(q^2) = 0. \]

For our purposes what is important is that virtual electrons also act to screen the surface charges which are required at the boundary \( \partial M \) to enforce conducting boundary conditions there. The resulting charge distribution may be obtained by integrating eq. (13) over a planar sheet of charge, \( \sigma \delta_+(z) \), where \( \delta_+(z) \) is normalized so it integrates to unity on one side of the boundary:
\[ \int_0^\infty dz \delta_+(z) = 1. \]
(For instance, this can be represented by \( \delta_+(z) = \lim_{\lambda \to -\infty} (\lambda e^{-\lambda |z|} = 2 \delta(z)) \). We write, then
\[ \sigma \delta_+(z) = 2\sigma \delta(z) = 2\sigma \int d^2 a \delta^3(\mathbf{r} - \mathbf{a}), \]
where we take \( \mathbf{a} \) to be a vector lying on the conducting surface and \( z \) to be the coordinate in the direction perpendicular to this surface. Using this with eq. (13) gives the result for the resulting polarized charge distribution around a surface-charge sheet positioned at \( z = 0 \)
\[ \rho(z) = \sigma \left\{ (1 + N) \delta_+(z) + (1 + N) \delta_+(z) - 2 \alpha \int_0^1 d\zeta \zeta (1 - \zeta) \log \left[ 1 + \zeta (1 - \zeta) \frac{q^2}{m^2} \right] \right\} \]
\[ = \sigma \left\{ (1 + N) \delta_+(z) - \frac{2\alpha}{\pi |z|} \int_0^1 d\zeta \zeta (1 - \zeta) \exp \left[ -\frac{m^2 |z|}{\sqrt{\zeta (1 - \zeta)}} \right] \right\}. \]

The first term in eq. (16), involving the delta function, expresses how virtual electrons renormalize the bare surface charge distribution. Of more interest for the present purposes is the second, position-dependent component.
This polarization charge distribution has the physical effect of generating multipole moments around the uniform surface charge at \( z = 0 \), which are detectable (in principle) through their interactions with probe charges in the bulk. In the effective theory obtained when the electron is integrated out, the effect of these multipole moments must be replaced by effective interactions which are localized on the surface of the conducting plates.

The particular moment of interest in what follows is the electric dipole moment density, \( p \), which this charge density defines

\[
p(z) = \rho(z) z e_z = -\frac{2 \omega \sigma e_z}{\pi} \int_0^1 d\zeta \zeta (1 - \zeta) \exp \left[ -\frac{m|z|}{\sqrt{\zeta(1-\zeta)}} \right],
\]

and which for large \( m \) becomes

\[
p(z) \rightarrow -\frac{2 \omega \sigma e_z}{\pi m} \delta_+(z) \int_0^1 d\zeta \left[ \zeta (1 - \zeta) \right]^{3/2}
\]

\[
= -\frac{3 \omega \sigma e_z}{64 m} \delta_+(z).
\]

This polarization distribution introduces a surface contribution to the field energy, \( U \), when a test charge, \( Q \), is placed in the vicinity of a conducting plate. In the absence of vacuum polarization we have seen that any test charge induces a nonzero charge density, \( \sigma \), on the conductor’s surface, as given by eq. (10). Virtual electrons then polarize the vacuum within a distance \( 1/m \) of both the test charge and this surface charge, leading to a change in the field energy. To leading order this change is the sum of the interaction of the induced charge density \( \sigma \) with the polarization around the test charge, plus the interaction of the test charge \( Q \) with the induced polarization near the surface charge. Each of these effects has precisely the same size,\(^\dagger\) leading to a correction of the interaction energy given by

\[
\Delta U = 2 \times \left( \frac{1}{2} \right) \int d^3 x \mathbf{E} \cdot \mathbf{p}
\]

\[
= -\frac{3 \omega}{64 m} \int d^2 r \sigma E_n
\]

\[
= -\frac{3 \omega}{64 m} \int d^2 r E_n^2,
\]

where we use the lowest-order result, eq. (10), to write \( \sigma = E_n \). Here \( \mathbf{E} \) is the lowest-order electric field (or electric displacement) not including the vacuum-polarization corrections.

\[^\dagger\]The equality of the two contributions is most easily seen if the surface charge is instead represented as an equivalent image charge, on the opposite side of the boundary.

\[\text{C. Matching Conditions}\]

We now consider integrating out the electron, and ask what effective interaction in a low-energy theory without electrons describes their effects to leading order in \( 1/m \). As we have seen above, in the absence of the charge densities on the surfaces of the conducting plates, in the bulk the vacuum polarization provides only interactions of the form \( F_{\mu\nu} F_{\mu\nu} \) or \( F_{\mu\nu} \square F_{\mu\nu} \), which are redundant interactions with no physical consequences. The vacuum polarization does give nontrivial contributions once the surface charges are considered, however. As discussed above, it generates multipole moments along all of the conducting surfaces due to the induced bulk charge redistribution it implies.

For plane parallel conducting plates the symmetries of the problem require that the operator obtained must be translation- and Lorentz-invariant within the dimensions parallel to the plates, as well as being parity and time-reversal invariant. Keeping in mind that \( \mathbf{E}_\parallel \) and \( B_n \) both vanish on the conductors, the lowest-dimension operator which is possible involving the electromagnetic field is

\[
\Delta S = \frac{1}{2} \int_{\partial M} d^3 x \left[ c_1 E_n^2 + c_2 B_n^2 \right],
\]

where \( c_1 \) and \( c_2 \) are constants to be determined. Since the electrons do not themselves see the boundary, their lowest-order contribution is actually Lorentz-invariant in all 4 dimensions, allowing the simplification \( c_2 = -c_1 \).

The coefficient \( c_1 \) may be determined by computing the contribution this operator makes to the field energy density, \( u(r) \), of a classical static test charge, giving

\[
\Delta u = \frac{c_1}{2} E_n^2
\]

which when compared with eq. (19) (using \( U = \int d^3 x \ u \)) gives the result

\[
c_1 = -\frac{3 \omega}{32 m}.
\]

We see that the coefficient of this operator may be obtained for any conductor independent of the presence of any other conductors, because it describes a local condition — the vacuum polarization — near the conductor’s surface. Once it is determined, its physical origin may be forgotten and its influence in any other low-energy process may be obtained perturbatively in the coefficient \( c_1 \). In particular, the effective operator with the coefficient \( c_1 \) given in (22) may be used to calculate the leading correction to the Casimir energy. This has been performed in ref. [6] and agrees with the calculation in the full theory [3], as it must.
III. CONCLUSIONS

The general utility of effective field theories lies in their efficient identification of the scales which are relevant to any particular physical system, and we see that this is also true for the Casimir energy. The novel feature which conducting boundaries introduce into the low-energy QED effective theory is the presence of surface charges and currents, whose presence provides a way for the vacuum polarization to have physical implications which it would not otherwise have. Its effects are described by local boundary interactions because the vacuum polarization only extends over distances of order \(1/m\), it so cannot reach very far into the bulk. Of course, it should be emphasized that for real systems the effective interactions computed in this article are not the most important, since other microscopic physical length scales arise which are much larger than the electron’s Compton wavelength.

Because effective field theories are so easy to use, they normally really come into their own once one proceeds beyond leading order in small quantities like \(\alpha\) or \(1/m\), since they make possible calculations which would otherwise be impractical. This is likely also to be true for precision calculations of Casimir energies, for which there is now considerable practical interest in understanding the dependence of the effect on the geometry and physical make-up of the conductors and dielectrics involved.

For real systems there are a number of effects whose contributions to the Casimir energy must be disentangled, and since each comes with an associated length scale which is short compared with the inter-plate separation it is likely most efficient to do so within an effective-field-theory analysis. These effects include thermal fluctuations; shape effects due to the curvature or roughness of the conducting surface; and effects due to imperfections in the ideal-conductor boundary conditions. Conductivity effects can be associated with the fact that static external electric fields have a finite penetration length into the conductor, and that conduction electrons cannot adjust quickly enough to respond perfectly to fields which oscillate sufficiently rapidly. Each of these should correct the Casimir energy by amounts proportional to powers of small ratios of the form \(\lambda/a\), where \(\lambda\) is the length scale relevant to the microscopic physics of interest, and their dominant effects can be parameterized in terms of local \(\lambda\)-dependent effective interactions.

In general the dominant corrections must correspond to the lowest-dimension effective interactions, some of which we now display. Effective interactions which describe some conductivity effects take a form similar to those considered here, eq (20),

\[
\Delta S = \frac{1}{2} \int_{\partial M} d^2 r \left[ c_1 E_n^2 + c_2 B_n^2 + c_3 E_n^2 + c_4 B_n^2 \right].
\]  

On dimensional grounds one expects the coefficients of these operators to receive effects of order \(c_i \sim \lambda\). Notice that the fields \(E_n\) and \(B_n\) can now appear here because in real systems the conducting boundary conditions are only imperfectly imposed.

Geometrical effects associated with the curvature of the conducting surface may be similarly parameterized in terms of a general effective lagrangian built from the surface’s intrinsic and extrinsic curvature tensors, \(R^{\alpha\beta\gamma\delta}\) and \(K_{\alpha\beta}\). Sample low-dimension terms include

\[
\Delta S = \int_{\partial M} d^2 r \sqrt{|g|} \left[ s_0 + s_1 K + s_2 R + \cdots \right],
\]  

where \(R = g^{\alpha\beta} R_{\alpha\beta}\), \(K = g^{\alpha\beta} K_{\alpha\beta}\) and \(|g| = \det(g_{\alpha\beta})\), for \(g_{\alpha\beta}\) the induced metric on the surface. On dimensional grounds one also expects the effective couplings, \(s_k\), to be proportional to powers of the relevant microscopic length scale, \(\lambda\).

Besides its practical applications, the Casimir energy system also provides a useful theoretical framework in which to test some of the more speculative theoretical ideas which have gained currency of late. Similar issues to those considered here arise in brane-world scenarios, wherein ordinary particles are confined to surfaces (branes), with various interactions probing the bulk. Renormalization issues can also have practical implications in this case [15], and QED provides a useful benchmark against which these more speculative calculations can be tested.

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