Large-N limit of non-local 2D generalized Yang-Mills theories on non-orientable surface

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Abstract

The large-group behavior of the non-local two dimensional generalized Yang-Mills theories (nlgYM₂’s) on arbitrary closed non-orientable surfaces is investigated. It is shown that all order of φ²ᵏ model of these theories have thired order phase transition only on projective plane (RP²). Also the phase structure of φ² + γφ⁴ model of nlgYM₂ is studied and it is found that for γ > 0, this model has third order phase transition only on RP² and for γ < 0 it has third order phase transition on any closed non-orientable surfaces except RP² and Klein bottle.

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1 Introduction

In recent years there have been much effort to analyze the different aspects of two dimensional Yang-Mills (YM\(_2\)) theory [1-9]. The YM\(_2\) theory is defined by the Lagrangian \(\text{tr}(\frac{1}{4} F^2)\) on a Riemann surface, where \(F\) is the field strength tensor. This theory is equivalent to the so-called B- F theory, which characterized by the Lagrangian \(i\text{tr}(BF) + \text{tr}(B^2)\), such as invariance under area preserving diffeomorphisms and lack of any propagating degrees of freedom [4]. These properties are also shared by a large class of theories, called the generalized two dimensional Yang-Mills (gYM\(_2\)'s) theories. These theories, however, are defined by replacing an arbitrary class function of \(B\) instead of \(\text{tr}(B^2)\) [10]. Several aspect of this theories such as, partition function, generating functional and large -N limit on an arbitrary two dimensional orientable and non-orientable surfaces has been discussed in [16-22]. There is another way to generalize YM\(_2\) and gYM\(_2\) and that is to use a nonlocal action for the auxiliary field, leading to the so-called nonlocal YM\(_2\) (nlYM\(_2\)) and nonlocal gYM\(_2\)(nlgYM\(_2\)) theories, respectively [12]. Several aspects of nlYM\(_2\) nlgYM\(_2\), such as, classical behavior, wave function, partition function, generating functional, and also large-\(N\) limit of it, have been studied on orientable surfaces in [11-15]. In all of these theories, the solution appear as some infinite summations over the irreducible representations of the gauge group. In the large - \(N\) limit, however, these summations are replaced by suitable path integrals over continuous parameters characterizing the Young tableaux, and saddle-point analysis shows that the only significant representation is the classical one, which minimizes some effective action. This continuous parameters characterizing the representation is a constrained, as the length of the rows of the Young tableau is non-increasing. So for small values of the surface area, the classical solution satisfies the constraint; for large values of the surface area, it dose not. Therefore the dominating representation is not the one, which minimizes the effective action. This introduces a phase transition between these two regime. Some problem has been studied for special cases of YM\(_2\)[17, 23], gYM\(_2\) [20, 21,
In [12, 14], nYLM in [13, 15] on sphere. In this paper I investigate this problem (large-N limit) of nYLM theories on arbitrary non-orientable surface.

The scheme of this paper is the following. In sect. 2, I briefly review the large-N limit of nYLM theories of $U(N)$ gauge group on any arbitrary non-orientable surface and obtain effective action of theory at large-N limit. In sect. 3, I study the phase structure of nYLM for $\phi^{2k}$ model in all order on an arbitrary surface. In sect. 4, I study the phase structure of theory for $\phi^2 + \frac{1}{2}\phi^4$ models on any arbitrary non-orientable surface.

2 Preliminaries

In [12], a non-local generalized two dimensional Yang-Mills (nYLM’s) theories was defined as:

$$e^{S} := \int DB \exp \left\{ \int \text{tr}(BF) d\mu + \omega \left[ \int \Lambda(B) d\mu \right] \right\}, \quad (1)$$

where $B$ is an auxiliary field at the adjoint representation of gauge group, $F$ is the field strength, $\Lambda$ is a similarity-invariant function, $d\mu$ is the invariant measure of the surface; $d\mu := \frac{1}{2} \epsilon_{\mu \nu} dx^\mu dx^\nu$. It was further shown that the partition function for this theory on a closed non-orientable manifold, $\Sigma_{g,s,r}$, with area $A$, genus $g$, $s$ copies of Klein bottle, and $r$ copies of projective plane ($RP^2$), is given by the exact formula [12, 13] as:

$$Z_{\Sigma_{g,s,r}}(A) = \sum_{R=R} d^{2g+2s+2r-p} \exp \left\{ \omega[-A\Lambda(R)] \right\}, \quad (2)$$

where $R$’s label the irreducible representation of the gauge group, which the sum is only over self-conjugate representation, and also $\Lambda(R)$ is:

$$\Lambda(R) = \sum_{k=1}^{p} \frac{\alpha_k}{N^{k-1}} C_k(R). \quad (3)$$

Here $C_k$ is the k’th Casimir of gauge group, $\alpha_k$’s are arbitrary constant. The representation of the $U(N)$ gauge group are labelled by $N$ integers $n_i$ satisfying $n_i \geq n_j \quad (i \leq j)$ and it is
found that:

\[
d_R = \prod_{1 \leq i \leq j \leq N} \left(1 + \frac{n_i - n_j}{j - i}\right),
\]

(4)

\[
C_k(R) = \sum_{i=1}^{N} [(n_i + N - i)^k - (N - i)^k] .
\]

(5)

One defines a function \(V\) by

\[
-N^2 V \left[ A \sum_{k=1}^{p} \alpha_k \hat{C}_k(R) \right] := \omega[-A\Lambda(R)],
\]

(6)

where

\[
\hat{C}_k(R) = \frac{1}{N^{k+1}} \sum_{i=1}^{N} (n_i + N - i)^k.
\]

(7)

At the large-N limit of the gauge group, I use the following definitions \([23]\)

\[
\phi(x) = \lim_{N \to \infty} \frac{1}{N} (i - n_i - \frac{N}{2}),
\]

(8)

where

\[
0 \leq x := \frac{i}{N} \leq 1 \quad \text{and} \quad n(x) := \frac{n_i}{N}.
\]

(9)

Then, apart some unimportant constant, the partition function takes the form:

\[
Z_{\Sigma_0, s, r}(A) = \int D\phi(x) \exp \left\{-N^2 S(\phi)\right\},
\]

(10)

where

\[
S[\phi] = V \left( A \int_0^1 W[\phi(x)] dx \right) - \left(1 - (g + s + r/2)\right) \int_0^1 dx \int_0^1 dy \log |\phi(x) - \phi(y)|,
\]

(11)

and

\[
W[\phi] := \sum_{k=1}^{p} (-1)^k \alpha_k \phi^k(x).
\]

(12)

Note that the sum in (2) is only over self-conjugate representations, which \(\int'\) in (10) shows this constraint also. This requirement in \(U(N)\) means that there is the additional constraint to the sum as:

\[
n_i = -n_{N-i+1}.
\]

(13)
In the large-N limit, this implies that the continuum variables, \( \phi(x) \), satisfy:

\[
\phi(x) = -\phi(1 - x). \tag{14}
\]

So one can define a new function such as:

\[
\phi(x) = \begin{cases} 
\psi(x) & 0 \leq x \leq 1/2 \\
-\psi(1 - x) & 1/2 \leq x \leq 1
\end{cases} \tag{15}
\]

Here the function \( \psi(x) \) being defined on the interval \([0, 1/2]\), in which \( \psi(1/2) = 0 \). Then, by institute (15) in (11), it is seen that these models have interesting solution, only for even values of \( k \) and therefore one can arrive at:

\[
S[\psi] = V \left( 2A \int_0^{1/2} W[\psi(x)]dx \right) - 2(1 - (g + s + r/2)) \int_0^{1/2} dx \int_0^{1/2} dy \log |\psi^2(x) - \psi^2(y)|. \tag{16}
\]

Introducing the density function as \( u(\psi) := \frac{dx(\psi)}{d\psi} \) [19]. Thereof \( W[\psi] \) is an even function of \( \psi \), thus \( u[\psi] \) is even, then the interval corresponding to values of \( Z[\psi] \ (u(\psi)) \) is \([-a, a]\). It is seen that the condition \( n_i \geq n_j \) demands \( u(\psi) \leq 1 \), and also

\[
\int_{-a}^{a} u(z)dz = 1. \tag{17}
\]

The saddle point that dominates this path integral is given by the equation of motion. It is found that:

\[
h(z) = P \int_{-a}^{a} \frac{z u(z')dz'}{z^2 - z'^2}, \tag{18}
\]

where \( P \) indicates the principal value of integral and

\[
h(z) = \eta W'(z) \tag{19}
\]

\[
\eta = \frac{A}{4(1 - g - s - r/2)} V' \left\{ A \int_0^{1/2} W[\psi(x)]dx \right\}, \tag{20}
\]

\[
= \frac{A}{4(1 - g - s - r/2)} V' \left\{ A \int_{-a}^{a} W(z)u(z)dz \right\}.
\]
A part from some coefficient, this equation is the same as that obtained in [12, 13], which obtained for this theory on orientable surface, and can be solved in the same manner. the density function, \( u(z) \), found from (18) depend on the modified area \( \eta \) and therefore \( A \), in which as \( A \) increases, a situation is encountered where \( u \) exceeds 1. So the solution of (18) is valid only for \( A \) less than some critical value \( A_c \). \( A_c \) is the value of \( A \) at which the maximum of \( u \) becomes 1, \( u_{\text{max}}(A_c) = 1 \). The region \( A < A_c \) is called the weak coupling phase (WCP) regime and the region \( A > A_c \) is called the strong coupling phase (SCP) regime. By the same procedure which used in [21], I can expand the density function in WCP regime, \( u_w(z) \), around absolute maxima, \( z_0 \), and it is found that for the points which are adjacent of critical point, \( A_c \), the difference of free energy in SCP and WCP regime is:

\[
F_s - F_w \simeq \xi^3, \tag{21}
\]

where the free energy of the theory is defined as:

\[
F := S|_{\phi_{\text{cla}}}, \tag{22}
\]

and

\[
\xi = u_w(z_0) - 1. \tag{23}
\]

By considering \( \xi \) as a function of \( A \) and expand it around \( A = A_c \), one can arrive at:

\[
\xi(A) = \xi'(A)(A - A_c) + \ldots \tag{24}
\]

where

\[
\xi' = \left( \frac{\partial \xi}{\partial \eta} \right)_{\eta = \eta_c} \frac{d\eta}{dA}|_{A = A_c} \tag{25}
\]

So for the case which \( \frac{d\xi}{d\eta}|_{\eta = \eta_c} \) and \( \frac{d\eta}{dA}|_{A = A_c} \) are nonzero, we have

\[
F_s - F_w \simeq \beta(A - A_c)^3 + \ldots \tag{26}
\]

Here \( \beta \) is a constant which is independent of modified area of manifold, \( \eta \) (or \( A \)). Thus, almost all models which, the density function of that have some absolute maxima, and
exist the proper quantity $\xi(A)$ in which, $\frac{d\xi}{dA} \neq 0$, has a third order phase transition on any arbitrary non-orientable surface.

3 $W(\phi) = \phi^{2k}(k > 1)$ models.

In this case $W(\phi)$ is an even function of $\phi$, then we can use of (18). So by solving (18), it is found that:

$$u_w(z) = \frac{k\eta}{\pi} \sqrt{a^2 - z^2} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} a_k^{2n} z^{2k-2n-2},$$

where $a_k$ is obtained from (17) as:

$$a_k = \left[ \frac{2^{k}(k-1)!!}{(2k-1)!!\eta} \right]^{\frac{1}{2k}}.$$

It has been shown that (27) has three extremum points at $z = 0$ and $z_{1,2} = \pm \alpha_k \sqrt{\xi_{k}}$ [22], in which $\xi_{k}$ is independent of $a_k$ and is determined from

$$\sum_{n=0}^{k-2} \frac{(2n-1)!!}{2^{n+1}(n+1)!} \xi_{k}^{-(n+1)} = 1.$$ (29)

Using $u_w'(z_k) = 0$, (where $z_k$'s are extremum points), one can see:

$$u_w''(z_0) = -\frac{k\eta a^{2k-2}(2k-1) \sum_{n=0}^{k-2} \frac{(2n-1)!!z_{0}^{2k-2n-4}}{2^n n!a^{2k-2n-4}}}{\pi \sqrt{a^2 - z_0^2}}.$$ (30)

So when $z_k = z_0 = 0$, then $u_w''(z_0) = 0$, so that for $\psi^{2k}(\phi^{2k})$ models the density function $u_w(z)$ is minimum at $z_0 = 0$ but for $z_k = z_{1,2}$ the density function is maximum only for cases which $\eta > 0$ or

$$1 > g + s + r/2,$$ (31)

therefore , we see that the density function of these theories is maxima at $z_k = z_{1,2}$, when (31) has been satisfied. By substitute (28) in (27), we obtain:

$$u_w(z_{1,2}) = \eta^{\frac{1}{2k}} f(k)$$ (32)
where $f(k)$ is independent of $\eta$ (or $A$). Using (24) and (25), we have

$$\xi(A) = \left\{ \frac{d}{dA} \ln \eta^{\frac{1}{2}} \right\}_{A=A_c} (A - A_c) + \ldots$$  \hspace{1cm} (33)$$

where, we use of $\xi(A_c) = 0$. Thus if $\left. \frac{d \ln \eta^{\frac{1}{2}}}{dA} \right|_{A=A_c}$ is nonzero, these theories has a third order phase transition. Note that for the case which $g = s = r = 0$, the relation (31) is satisfied and this means that these theories has third order phase transition on sphere, also if $g = s = 0$ and $r = 1$, the order of phase transition of theory on projective plane ($\mathbb{R}P^2$) is 3 for all order of $\psi^{2k}$, but if $g = r = 0$ and $s = 1$, the condition (31) is not satisfied, then the theory has no phase transition on Klien bottle and other non-orientable surfaces.

4 \quad \textbf{the $\psi^2 + \frac{\gamma}{4}\psi^4$ models.}

I now consider $\psi^2 + \frac{\gamma}{4}\psi^4$ with arbitrary $\gamma$. At the first I assume that $\gamma > 0$. So by solving (18), one can obtain:

$$u_w(z) = \frac{\eta}{4\pi} \sqrt{a^2 - z^2} (4 + \gamma a^2 + 2\gamma z^2),$$  \hspace{1cm} (34)$$

and from (17),

$$\eta a^2 (8 + 3\gamma a^2) = 16.$$  \hspace{1cm} (35)$$

The shape of $u_w(z)$ depends on $\eta$. It is seen that

$$u''_w(0) = \frac{\eta}{4\pi} (3\gamma a^2 - 4).$$  \hspace{1cm} (36)$$

So for $3\gamma a^2 < 4$, $u_w(z)$ has only one extremum point at $z = 0$, if $\eta > 0$, and also

$$\xi'(A_c) = \frac{d(u_w(0) - 1)}{dA} |_{A=A_c},$$

$$= \frac{1}{\sqrt{108\pi^2 \gamma}} \left\{ \frac{12\gamma/\eta_c + \sqrt{1 + 48\gamma/\eta_c} - 1}{\sqrt{1 + 48\gamma/\eta_c} - 1} \right\}.$$  \hspace{1cm} (37)$$

Thus, the density function is maximum at $z = 0$ ($u''_w(0) < 0$) for the case which $\eta > 0$ ($1 - g - s - r/2 > 0$), and this is satisfied only on sphere and projective plane. However the
order of phase transition of this model is three on sphere and projective plain and on other orientable and non-orientable surface has no phase transition. For $3\gamma a^2 > 4$ this model has three extremum point at $z = 0$ and $z = \pm z_0$, where

$$z_0 = \sqrt{\frac{2}{3\gamma} \left( \sqrt{1 + \frac{48\gamma}{\eta}} - 2 \right)}.$$  

(38)

From this for the case which $\eta > 0$, one can obtain that $u_w(z_0)$ is minimum at $z = 0$ and has two maxima at $\pm z_0$. So by obtaining $u_w(z_0)$ and use of (23-25), one can arrive at:

$$\xi'(A_c) = \sqrt{\frac{2}{3\gamma}} \times \frac{4\eta_c + 3\gamma}{6\pi\eta^{3/4}(\eta_c + 3\gamma)^{1/4}} \times \frac{d\eta}{dA}|_{A=A_c} > 0.$$  

(39)

So according with (25) and (26), this theory has phase transition only on $S^2$ and $RP^2$ and the order of it is 3. For the case which $\gamma < 0$ (for all value of $\gamma$), the density function $u_w(z)$ has three extremum point at $z = 0$ and $\pm z_0$, where

$$z_0 = \sqrt{\frac{2}{3|\gamma|} \left( 2 + \sqrt{1 + 3|\gamma|} \right)}.$$  

(40)

It is clearly seen that this model of theory has phase transition on all surfaces which $\eta < 0$. Indeed, it is not difficult to check that $u_w''(z_0)$ is negative, and also, from (40), one can obtain $u_w(z_0)$, and then:

$$\frac{d\xi}{dA}|_{A=A_c} = \frac{d}{dA}(u_w(z_0) - 1)|_{A=A_c},$$  

$$= \frac{2}{3\pi} \sqrt{\frac{2}{3|\gamma|}} \times \frac{d\eta}{dA}|_{A=A_c},$$  

(41)

so according with (25) and (26), it is found that if $\frac{d\eta}{dA}|_{A=A_c} \neq 0$, then the $\psi^2 + \frac{\gamma}{4}\psi^4$ model whit $\gamma < 0$ has a third order phase transition on all closed orientable and non-orientable manifold except on sphere, torus, projective plain and Klein bottle.

### 5 conclusion

I study the large-N limit of nlgYM$_2$ theories on arbitrary non-orientable surface. By obtaining the effective action of these theories at the large-N limit, it is shown that apart
some coefficient, the saddle point equation is the same as that obtain for these theories on orientable surface. I study the phase structure of $\phi^{2k}$ model and it is seen that all order of this model has third order phase transition only on projective plane. Also, by considering the $\phi^2 + \frac{\gamma}{4} \phi^4$ model of nlgYM$_2$, I found that, for the case $\gamma > 0$, this theory has third order phase transition on $\mathbb{RP}^2$, and on other non-orientable surface has no phase transition, whereas for the case $\gamma < 0$ this model has phase transition on all closed non-orientable surfaces except projective plane and Klein bottle, and the order of phase transition is 3. It is clearly seen that all order of $\phi^2 + \alpha \phi^{2k+1}$ model on any arbitrary closed non-orientable surfaces has the same phase structure of $\phi^2$ model of nlYM$_2$.

Note that the whole reasoning is independent of the number of points as $u_w$ attains its absolute maximum. It is clear that a similar situation prevails for the cases which $u_w$ has many absolute maximum, and also in this case, one can realize the WCP regime and obtain the phase structure of theory in the multi-critical points. It is remarkable that there might be cases which $u'_w(z_0)$ is zero (not negative). At this state the theory has phase transition, but the order of phase transition is not 3.

References


